Design and implementation of ARL-unbiased CCC$_r$-chart for monitoring high-yield processes

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Abstract: In this paper, we consider the -charts for monitoring the high-quality processes considering the cumulative counts of conforming (CCC) items up to the r^{th} non-conforming one. But the charts perform poorly in detection of small downward shifts in the fraction non-conforming because of their undesirable ARL-biased property. In this paper, we eliminate the ARL-biasedness property and propose the ARL-unbiased charts using the notion of uniformly most powerful unbiased (UMPU) test to ensure that a user will get an OOC signal more quickly than a false alarm for the shifts in both upward and downward directions. The performance of the proposed chart is also compared with the existing ARL-unbiased CCC chart and it is found that the former has an improved ability of detecting shifts in the fraction non-conforming over the latter. An illustrative example is given and a summary and conclusions are offered.

Keywords: average run length; ARL; ARL-unbiased; control chart; high-yield processes; in-control performance; out-of-control performance; geometric distribution; fraction non-conforming; uniformly most powerful unbiased; UMPU; UMPU test.

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1 Introduction

The attribute control charts play a significant role in monitoring processes with several advantages over the variable control charts. For example, in the most manufacturing and service industries, numerous quality characteristics need to be analysed simultaneously and the attribute control charts could monitor them all together at a fraction of the cost (see, Chapter 8, Mitra, 2016). Among them, the *p*, *c*, *np*-charts are widely used and consider the number of non-conforming items in a sample or non-conformities in the inspection unit as the quality characteristic. However, in monitoring high-yield processes (also known as 'nearly' zero-defect processes), they are found less useful due to several reasons, for example, increased false alarm rate (FAR), physically meaningless control limits and failure in detecting decrements in fraction non-conforming (Kaminsky et al., 1992; Xie and Goh, 1993; Xie et al., 1999, 2000). The production or transactional processes which produce non-conforming items or defectives at a very low rate, say, parts per millions or billions are termed as high-yield processes (Xie et al., 2002; Montgomery, 2019), which could ultimately approach the states of zero defect (Golbafian et al., 2017). Such processes are encountered, for example, in manufacturing of printed circuits (Wang, 2009), filling process in the manufacture of low voltage liquid crystal display units (Chan et al., 2003), automated manufacturing processes, and in healthcare surveillance where infrequent events such as medical errors occur (Acosta-Mejia, 2012).

For efficient monitoring of the high-yield processes, it has been suggested to consider the count of conforming items until the occurrence of non-conforming one as the quality characteristic (Calvin, 1983). Based on this idea, the CCC-chart is proposed to monitor the high-yield processes (Goh, 1987). Since then, several authors including Bourke (1991), Xie and Goh (1992), Chang and Gan (2001), and Chen (2009) have used the CCC-chart in monitoring high-yield processes. It is worth to note that Xie et al. (2010) used the same idea to monitor the continuous production processes. Due to its worthwhile contribution in monitoring high-yield processes, the efforts are made to enhance its ability of early detection of an out-of-control (OOC) signal, thus the CCC$_r$-charts ($r \in$ 1, 2, …) are proposed (see Ohta et al., 2001; Kudo et al., 2004; Schwertman, 2005; Albers, 2010; Zhang et al., 2019). The CCC$_r$-charts monitor the cumulative count of conforming items until the r^{th} non-conforming item is observed. The CCC$_r$-charts are also known as the negative binomial charts because the quality characteristic follows a negative binomial distribution (Albers, 2010). Note that the CCC-chart (also known as geometric chart) is a particular case of the CCC$_r$-chart when $r = 1$. The CCC$_r$-charts have been well studied by several authors including Ohta et al. (2001), Kudo et al. (2004), Schwertman (2005) and Albers (2010) and they found that the CCC$_r$-charts ($r > 1$) outperform the CCC-chart. Recently, Joekes et al. (2016) proposed two kinds of CCC$_r$-chart for monitoring an injection moulding process and compared their performances. We should mention here that the choice of r is crucial in designing the CCC$_r$ because on one hand it requires to wait for a longer time to observe the r^{th} non-conforming item, but on the other hand the sensitivity of the chart increases as r increases (see Chan et al., 2003). Many authors including Schwertman (2005) and Albers (2010) have suggested to take the value of r up to 4 or 5.

Very often, the control chart's performance is assessed by the most popular metric which is the average run length (ARL). It is defined as the expected number of charting points plotted on the control chart until an OOC signal is observed. Importantly, two properties are desirable for any control chart:

1 the OOC ARL values should be less than the in-control (IC) ARL value for every shift in the process so that the chart can produce an OOC signal more quickly than it raises a false alarm

2 the IC ARL value should achieve exactly a pre-specified value of IC ARL.

The control chart possessing these properties is termed as the ARL-unbiased chart, otherwise it is called the ARL-biased. The ARL-biased chart is usually not welcomed in the SPC literature because the corresponding ARL function does not achieve its maximum at the IC parameter value. This implies that the ARL-biased chart takes a longer time to detect the changes in the process parameter than it takes to give a false alarm. Therefore, it is worth to eliminate the bias in the ARL-function to ensure a more balanced guard against both decrease and increase in the process parameter.

Note that the ARL-biased charts are encountered in practice when the quality characteristic has skewed distribution (Lowry at al., 1995). For the quality characteristic of continuous nature, the chart can be designed easily using the methods of calculus. Therefore, most of the associated work has been focused on the control chart where the quality characteristic is absolutely continuous in nature. These include the ARL-unbiased exponential and t_r-charts for monitoring times to events (Zhang et al., 2006; Cheng and Chen, 2011; Kumar et al., 2017; Kumar and Baranwal, 2019; Kumar, 2020), the ARL-unbiased S^2-charts for monitoring the dispersion in normal processes (Pignatiello et al., 1995; Acosta-Mejia and Pignatiello, 2000; Knoth, 2010; Uhlmann, 2013; Guo and Wang, 2015). Unlike the ARL-unbiased chart for the continuous quality characteristic, to design the ARL-unbiased chart is rather intricate when the quality characteristic is of discrete nature. Hence, the initial work was focused on designing the near-maximal and nearly ARL-unbiased control charts such as *np*-chart proposed by Acosta-Mejia (1999), geometric chart designed by Zhang et al. (2004) and CCC-chart under the group inspection by Zhang et al. (2012). Recently, Knoth and Morais (2015) established a relationship between the ARL-unbiased chart and the notion of uniformly most powerful unbiased (UMPU) tests (see also, Cox and Hinkley, 1979; Zhang et al., 2004). Given this relationship, Paulino et al. (2016) and Morais (2016) proposed the ARL-unbiased design for the *c*-chart and *np*-chart, respectively. Following the same line of arguments, Morais (2017) proposed the ARL-unbiased CCC-chart and CCC_G-chart (the CCC-chart under group inspection) respectively.

As it was stated earlier that the charting statistic for the CCC_r chart follows a negative binomial distribution which is skewed in nature. Thus, the existing CCC_r chart based on an equal-tail probability approach is ARL-biased. The objective of this paper is to eliminate the bias in the ARL function of the CCC_r chart and compare its performance with the existing ARL-unbiased CCC chart. In order to design the ARL-unbiased CCC_r, we follow Morais (2017) approach based on the UMPU test with the randomisation probabilities.

Rest of the paper is organised as follows. Section 2 provides an overview of the CCC_r-chart. Section 3 discusses the ARL-unbiased design of CCC_r-chart. In Section 4, the performance of the proposed ARL-unbiased CCC_r-chart is examined. Numerical comparisons with the ARL-unbiased CCC-chart are also provided in the same section. An example is given illustrating the practical implementation of the proposed chart in Section 5, and concluding remarks are provided in Section 6.

2 The CCC_r control chart

Recall that the CCC_r-chart detects a change in the fraction non-conforming, say, p by monitoring a decrease or an increase in the number of conforming items until the r^{th} non-conforming item is produced. Let X denote the number of cumulative count of the conforming items until the process produces the r^{th} non-conforming item. Clearly, the random variable X follows a negative binomial distribution with parameters $r \in \{1, 2, \ldots\}$ and $p \in (0, 1)$.

The probability mass function of X is given by

$$P_p(x) = P_p[X = x] = \binom{x-1}{r-1} p^r (1-p)^{x-r} \quad x = r, r+1, r+2, \ldots; \; p \in (0,1), \tag{1}$$

where $\binom{x-1}{r-1}$ is defined as all possible ways of occurrences of the $(r - 1)$ non-conforming items out of x inspected items when x^{th} inspected item is the r^{th} non-conforming one. Clearly, when $r = 1$, the quality characteristic X follows a geometric distribution and the corresponding control chart reduces to the CCC-chart.

Let us consider the target value of p is known and is denoted by $p_0 \in (0, 1)$. Thus, for given nominal FAR, α and $p = p_0$, the lower control limit (LCL) and upper control limit (UCL) of a two-sided CCC_r-chart based on equal-tail probability approach are respectively given by (see, Xie and Goh, 1997)

$$P[X < \text{LCL}] = \sum_{i=r}^{\text{LCL}-1} \binom{i-1}{r-1} p_0^r (1-p_0)^{i-r} = \frac{\alpha}{2} \tag{2}$$

$$P[X < \text{UCL}] = \sum_{i=\text{UCL}+1}^{\infty} \binom{i-1}{r-1} p_0^r (1-p_0)^{i-r} = \frac{\alpha}{2} \tag{3}$$

Clearly, the chart triggers an OOC signal when the charting point is plotted either above the UCL or below the LCL. The charting point plotted above the UCL indicates a decrement in the fraction non-conforming which results in the larger number of conforming items before the occurrence of the r^{th} non-conforming item than what is normally expected. This corresponds to the improvement case. On the other hand, an increase in the fraction non-conforming produces the smaller values of X and it is expected that the charting point will be plotted below the LCL. This is known as deterioration case which is considered as more serious in practical situations than the improvement one in the SPC literature (Ohta et al., 2001; Chan et al., 2003).

Let us define $\rho = \frac{p_1}{p_0}$ where $\rho \in \left(0, \frac{1}{p_0}\right)$ quantifies the change in process parameter when the fraction non-conforming p has been shifted from $p = p_0$ to the new value $p = p_1$. Clearly, the process is IC if $\rho = 1$ (i.e., $p = p_0$) and OOC otherwise. Thus, the probability of signal which is defined as the probability that a charting point lies below the LCL or above the UCL is given by

$$\beta(\rho) = P\left[X < \text{LCL or } X > \text{UCL} \mid p = p_1\right] = 1 - \sum_{x=\text{LCL}}^{\text{UCL}} P_{\rho p_0}(x) \tag{4}$$

Note that the run length of a Shewhart-type control chart follows a geometric distribution with parameter $\beta(\rho)$, therefore, the ARL is the reciprocal of $\beta(\rho)$. Thus, the ARL function for the CCC$_r$-chart is given by

$$\text{ARL}(\rho) = \frac{1}{\beta(\rho)} = \frac{1}{1 - \sum_{x=\text{LCL}}^{\text{UCL}} P_{\rho p_0}(x)} \tag{5}$$

As we have already mentioned that the ARL function of the CCC$_r$-chart in equation (5) does not achieve its maximum at the IC situation, i.e., when $\rho = 1$. Pignatiello et al. (1995) called such charts as the ARL-biased. Moreover, due to discrete nature of the charting statistic, the IC ARL value, i.e., ARL(1) does not coincide with the pre-specified value $\text{ARL}_0 = \frac{1}{\alpha}$ (Morais, 2017). The charts with the ARL-biased property are discouraged in practice and hence, several attempts have been made to eliminate the bias in the ARL function. In the next section, we eliminate the bias in the ARL function of the CCC$_r$-chart and construct the ARL-unbiased CCC$_r$-chart.

3 ARL-unbiased CCC$_r$ control chart

As stated earlier that when the quality characteristic is of discrete in nature and its distribution is skewed, then neither the ARL function of the control chart attains its maximum at $\rho = 1$ nor the IC ARL value, i.e., ARL(1) is exactly equal to the pre-assigned value, $1 / \alpha$. Knoth and Morais (2015) showed that the designed control limits of the ARL-unbiased chart with a pre-specified FAR, α is equivalent to a size α UMPU test, say, H_0: $p = p_0$ against H_1: $p \neq p_0$ with two boundaries, say LCL and UCL and randomisation probabilities, say, γ_L and γ_U on these boundaries (Paulino et al., 2016; Morais, 2016). Following the same line of arguments, we extend the works of Morais (2017) on geometric chart to construct the control limits of the ARL-unbiased CCC$_r$-chart. See also Kumar and Singh (2020). Now, we define the control limits of the CCC$_r$-chart via the critical function $\varphi(X)$ as follows.

$$\varphi(X)=\begin{cases}1 & X<\text{LCL or } X>\text{UCL}\\ \gamma_{\text{L}} & X=\text{LCL}\\ \gamma_{\text{U}} & X=\text{UCL}\\ 0 & \text{LCL}<X<\text{UCL}\end{cases}\tag{6}$$

Clearly, the CCC$_r$-chart gives an OOC signal with probability one if a realisation of X is beyond the control limits LCL and UCL. If the charting point lies on the LCL (respectively UCL), the process is declared OOC with probability γ_{L} (respectively γ_{U}). Two sets of the randomisation probabilities (γ_{L}, γ_{U}) and the control limits (LCL, UCL) are obtained so that the CCC$_r$-chart is ARL-unbiased with the nominal ARL value, $1 / \alpha$. Please note that the probability of a signal for the CCC$_r$-chart with control limits defined by equation (6) is equivalent to the power function of a size α UMPU test. Thus, the probability of signal of the CCC$_r$-chart, defined by equation (6), is given by

$$\begin{aligned}\beta(\rho)&=E_{p_1}\left[\varphi(X)\right]\\ &=P_{\rho p_0}[X<\text{LCL}]+P_{\rho p_0}[X>\text{UCL}]+\gamma_{\text{L}}P_{\rho p_0}[X=\text{LCL}]\\ &\quad+\gamma_{\text{U}}P_{\rho p_0}[X=\text{UCL}]\\ &=1-\sum_{x=\text{LCL}}^{\text{UCL}}P_{\rho p_0}(x)+\gamma_{\text{L}}P_{\rho p_0}(\text{LCL})+\gamma_{\text{U}}P_{\rho p_0}(\text{UCL})\end{aligned}\tag{7}$$

The corresponding ARL function is given by

$$\text{ARL}(\rho)=\frac{1}{1-\sum_{x=\text{LCL}}^{\text{UCL}}P_{\rho p_0}(x)+\gamma_{\text{L}}P_{\rho p_0}(\text{LCL})+\gamma_{\text{U}}P_{\rho p_0}(\text{UCL})}\tag{8}$$

For the IC process, i.e., $\rho = 1$, equation (7) gives the FAR value whereas equation (8) provides the IC ARL value. Please note that for a two-sided testing problem H_0: $p = p_0$ against the alternative H_1: $p \neq p_0$, there exists a UMPU test given by equation (6) with size $E_{p_0}(\varphi(X))=\alpha.$ Furthermore, the power function $E_p(\varphi(X))$ is differentiable and must have a minimum at $p = p_0$. Please see Lehmann and Romano (2006, pp.111–112) for a detail. As a result, the probability of signal $\beta(\rho)$ and the ARL function ARL(ρ) have their respective minimum and maximum at $p = p_0$, or equivalently, $\rho = 1$ and they are also differentiable. Thus, the ARL-unbiased CCC$_r$-chart with the specified ARL value, $1 / \alpha$, must satisfy the following conditions.

$$\text{ARL}(1)=\frac{1}{\alpha}\tag{9}$$

$$\frac{d}{d\rho}\text{ARL}(\rho)\,|_{\rho=1}=0\tag{10}$$

Clearly, eequation (9) guarantees that the IC ARL is equal to the nominal $1 / \alpha$, whereas equation (10) ensures that the CCC$_r$-chart is indeed ARL-unbiased. Conditions in equations (9)–(10) reduce to the following system of linear equations that can be written as follows.

$$\gamma_{\mathrm{L}}.P_{p_0}(X=\mathrm{LCL})+\gamma_{\mathrm{U}}.P_{p_0}(X=\mathrm{UCL})=\alpha-1+\sum_{\mathrm{LCL}}^{\mathrm{UCL}}P_{p_0}[X=x] \tag{11}$$

$$\gamma_{\mathrm{L}}.\mathrm{LCL}.P_{p_0}(X=\mathrm{LCL})+\gamma_{\mathrm{U}}.\mathrm{UCL}.P_{p_0}(X=\mathrm{UCL})=\frac{(\alpha-1)}{p_0}+\sum_{x=\mathrm{LCL}}^{\mathrm{UCL}}x.P_{p_0}[X=x] \tag{12}$$

The randomisation probabilities γ_{L} and γ_{U} can be obtained explicitly from the system of linear equations for the pair of control limits (LCL, UCL) as follows (see also, Morais, 2017).

$$\gamma_{\mathrm{L}}=\frac{de-bf}{ad-bc} \text{ and } \gamma_{\mathrm{U}}=\frac{af-ce}{ad-bc} \tag{13}$$

where $a=P_{p_0}(\mathrm{LCL})$, $b=P_{p_0}(\mathrm{UCL})$, $c=\mathrm{LCL}.P_{p_0}(\mathrm{LCL})$, $d=\mathrm{UCL}.P_{p_0}(\mathrm{UCL})$, $e=\alpha-1+\sum_{\mathrm{LCL}}^{\mathrm{UCL}}P_{p_0}(x)$, $f=\frac{(\alpha-1)}{p_0}+\sum_{x=\mathrm{LCL}}^{\mathrm{UCL}}xP_{p_0}(x)$ provided that $ad-bc\neq 0$ and LCL < UCL.

Noting that the values of γ_{L} and γ_{U} need not be necessarily in unit square $(0, 1)^2$ for a given pair of control limits (LCL, UCL). Because γ_{L} and γ_{U} are the probabilities, hence, in order to obtain these values in unit square, Paulino et al. (2016) obtained a set of pairs of the lower and upper control limits, i.e., (LCL, UCL) so that for each pair in this set, the corresponding γ_{L} and γ_{U} values in equation (13) fall in the unit square. The required set of pairs (LCL, UCL) of the ARL-unbiased CCC$_r$-chart is provided by the following lemma.

Lemma 1: For $X\in\mathbb{N}_0$, we define

$$F(x)=P_{p_0}(X\le x)$$

$$G(x)=\frac{1}{E_{p_0}(X)}\sum_{1=0}^{x}i.P_{p_0}(X=i),\ x=0,1,2,\ldots \tag{14}$$

Defining the (pseudo) inverse function for any $\alpha\in(0, 1)$ as

$$F^{-1}(\alpha)=\min\{x:F(x)\ge\alpha\}$$

$$\tilde{F}^{-1}(\alpha)=\min\{x:F(x)>\alpha\}$$

$$G^{-1}(\alpha)=\min\{x:G(x)\ge\alpha\}$$

$$\tilde{G}^{-1}(\alpha)=\min\{x:G(x)>\alpha\}$$

Thus, the set of pairs of the control limits that may provide the admissible solutions in equation (13) lying between 0 and 1 is given by

$$C=\{(\mathrm{LCL},\mathrm{UCL}):\mathrm{LCL}_{\min}\le\mathrm{LCL}\le\mathrm{LCL}_{\max},\ \mathrm{UCL}_{\min}\le\mathrm{UCL}\le\mathrm{UCL}_{\max}\}$$

where

$$\mathrm{LCL}_{\max}=\min\{\tilde{F}^{-1}(\alpha),\tilde{G}^{-1}(\alpha)\}$$

$$\mathrm{UCL_{min}} = \max\left\{F^{-1}(1-\alpha), F^{-1}(1-\alpha)\right\}$$

$$\mathrm{LCL_{min}} = \max\left\{F^{-1}\left(\max\left\{0, F\left(\mathrm{UCL_{min}}-1\right)-1\right.\right.\right.$$
$$\left.\left.\left.+\alpha\right\}\right), G^{-1}\left(\max\left\{0, G\left(\mathrm{UCL_{min}}-1\right)-1+\alpha\right\}\right)\right\}$$

$$\mathrm{UCL_{max}} = \min\left\{\tilde{F}^{-1}\left(\min\left\{1, F\left(\mathrm{LCL_{max}}\right)+1\right.\right.\right.$$
$$\left.\left.\left.-\alpha\right\}\right), \tilde{G}^{-1}\left(\min\left\{1, G\left(\mathrm{LCL_{max}}\right)+1-\alpha\right\}\right)\right\}$$

The control limits of the ARL-unbiased CCC$_r$-chart can be obtained by searching the required pair of control limits in the set C which gives the admissible solution of γ_L and γ_U. To search, we start with the pair (LCL, UCL) = (LCL$_{min}$, UCL$_{min}$) and stop as soon as an admissible solution is found. The readers are referred to Paulino et al. (2016) for detailed derivation of the search grid, i.e., the set C. We obtained the control limits of the ARL-unbiased CCC$_r$-charts ($r \in$ 1, 2, 3, 4) for fixed FAR = 0.0027. The control limits with randomisation probabilities and corresponding search grids are provided in Table 1 for p_0 = 0.01, 0.005, 0.001, 0.0005, 0.0001, 0.00005, 0.00001. The control limits for the CCC-chart can also be found in Morais (2017).

Table 1 Limits of the search grid, control limits and randomised probabilities of the ARL-unbiased CCC$_r$-chart for FAR, α = 0.0027 and r = 1, 2, 3, 4

p_0	LCL_{min}	LCL_{max}	LCL	UCL_{min}	UCL_{max}	UCL	γ_L	γ_U
				$r = 1$				
0.00001	240	271	241	812,555	812,708	812,575	0.736799	0.103699
0.00005	48	55	49	162,508	162,540	162,533	0.146400	0.270193
0.0001	24	28	25	81,252	81,269	81,265	0.072600	0.166091
0.0005	4	6	5	16,248	16,252	16,250	0.813599	0.468725
0.001	2	3	3	8,122	8,125	8,123	0.406312	0.224264
0.005	0	1	1	1,622	1,624	1,622	0.480974	0.448242
0.01	0	1	1	809	813	809	0.240561	0.010422
				$r = 2$				
0.00001	6,807	7,536	6,824	1,003,091	1,006,173	1,005,384	0.509382	0.926526
0.00005	1,361	1,508	1,366	200,615	201,232	2,001,073	0.074652	0.006722
0.0001	681	754	683	100,305	100,614	100,535	0.770301	0.766718
0.0005	136	152	137	20,058	20,121	20,104	0.927463	0.774723
0.001	68	76	69	10,027	10,059	10,038	0.696759	0.649456
0.005	14	16	15	2,002	2,010	2,007	0.117833	0.748246
0.01	7	9	8	999	1,004	1,001	0.293658	0.124661

Table 1 Limits of the search grid, control limits and randomised probabilities of the ARL-unbiased CCC_r-chart for FAR, $\alpha = 0.0027$ and $r = 1, 2, 3, 4$ (continued)

p_0	LCL_{min}	LCL_{max}	LCL	UCL_{min}	UCL_{max}	UCL	γ_L	γ_U
				$r = 3$				
0.00001	24,639	27,063	24,778	1,178,715	1,187,693	1,185,076	0.119800	0.485258
0.00005	4,928	5,414	4,957	235,740	237,537	2,370,112	0.124897	0.837215
0.0001	2,464	2,708	2,479	117,868	118,767	118,504	0.500536	0.881300
0.0005	493	543	497	23,570	23,751	23,697	0.401279	0.316564
0.001	247	272	249	11,783	11,874	11,846	0.639165	0.121017
0.005	50	55	51	2,353	2,372	2,366	0.431401	0.764347
0.01	25	28	26	1,174	1,184	1,181	0.658710	0.845310
				$r = 4$				
0.00001	51,620	56,420	52,065	1,345,041	1,361,119	1,355,995	0.095457	0.362816
0.00005	10,325	11,285	10,414	269,004	272,221	271,195	0.810868	0.296360
0.0001	5,163	5,644	5,208	134,500	136,109	135,595	0.525324	0.288207
0.0005	1,033	1,130	1,043	26,896	27,219	27,115	0.497152	0.281832
0.001	517	566	522	13,446	13,608	13,555	0.869268	0.281351
0.005	104	115	106	2,685	2,719	2,707	0.369131	0.282239
0.01	52	58	54	1,340	1,358	1,351	0.310461	0.284412

4 Performance evaluation

In this section, we evaluate the ARL-unbiased CCC_r-charts (r = 2, 3, 4), for $\alpha = 0.0027$ and p_0 = 0.00001, 0.0001, 0.001 and then compare their performances with that of the existing ARL-unbiased CCC-chart, proposed by Morais (2017). In order to examine the OOC performance, we considered the ρ = 0.5(0.1)1.5 which reflect the deviations in IC parameter p_0 so that the true value of the parameter (OOC parameter value) becomes $p = \rho p_0$. Recall that the values of $\rho > 1$ reflect the deterioration in the process while the values $\rho < 1$ show the process improvement. In Table 2, we can find the ARL values of the ARL-unbiased CCC_r-charts corresponding to the ρ values. Few observations can be made from Table 2. Firstly, the proposed CCC_r-charts are ARL-unbiased which implies that the charts take less time to detect (increases or decreases) deviations in p_0 than to raise the false alarm. Secondly, the ARL-unbiased CCC_r-charts (r = 2, 3, 4) provide better performance than the corresponding CCC-chart in both improvement and deterioration cases. For example, with p_0 = 0.0001 and ρ = 0.7, the ARL values of CCC_1, CCC_2, CCC_3, and CCC_4-charts 197.24, 122.50, 85.67 and 64.53 respectively. In other words, to detect this shift, the CCC_4-chart requires nearly 64 charting points to be plotted on the control chart, whereas CCC_1-chart takes about 197 charting points to detect same shift size. Thirdly, the performance of the charts is not much affected by the change in the IC fraction non-conforming, i.e., p_0 value.

The study further investigates the relative gain in the performance in terms of the ARL values if the ARL-unbiased CCC_r-chart (r = 2, 3, 4) is used instead of the ARL-unbiased CCC-chart. Table 3 reports the associated relative gain in the ARL values corresponding to ρ values, i.e., $\left(1-\dfrac{\text{ARL}(\text{CCR}_r)}{\text{ARL}(\text{CCC})}\right)\times 100\%$ when the ARL-unbiased CCC-chart is replaced by the ARL-unbiased CCC_r-chart (r = 2, 3, 4). Table 3 suggests that the relative gain in the ARL values increases as r increases but at a slow pace. For example, when the ARL-unbiased CCC_2-chart is used in place of the ARL-unbiased CCC-chart, then the relative gain is 54% corresponding to ρ = 0.5 whereas it is about 71% when the ARL-unbiased CCC_3-chart is used and 80% when the ARL-unbiased CCC_4-chart is used. Therefore, when one is more interested in detecting the large shifts, the CCC_2 or CCC_3-chart may be recommended, whereas, for detecting small shifts, the CCC_4-chart is more useful. For example, for the shift ρ = 1.2, the relative gains of the CCC_2 and CCC_3-charts over the CCC-chart are only 9% and 18% respectively. Thus, to increase the ability of detecting this shift, the CCC_4-chart is recommended with the relative gain 25%.

Table 2 OOC ARL values of the ARL-unbiased CCC_r-chart for r = 1, 2, 3, 4 and p_0 = 0.00001, 0.0001, 0.001

$\rho \rightarrow$	*0.00001*	*0.0001*	*0.001*	*0.00001*	*0.0001*	*0.001*
$p_0 \downarrow$		*r = 1*			*r = 2*	
0.5	54.34	54.36	54.32	24.94	24.94	24.79
0.6	110.19	110.25	110.17	56.52	56.52	56.13
0.7	197.24	197.33	197.23	122.50	122.49	121.63
0.8	291.84	291.95	291.88	230.29	230.29	228.89
0.9	353.24	353.32	353.29	334.29	334.29	332.97
1	370.33	370.37	370.37	370.37	370.37	369.66
1.1	360.25	360.27	360.26	348.52	348.52	348.22
1.2	339.80	339.80	339.78	307.72	307.72	307.58
1.3	317.41	317.40	317.37	267.49	267.49	267.39
1.4	296.20	296.19	296.15	232.91	232.90	232.81
1.5	277.03	277.03	276.97	204.17	204.16	204.07
		r = 3			*r = 4*	
0.5	15.23	15.23	15.21	10.62	10.62	10.61
0.6	36.03	36.03	35.99	25.67	25.67	25.64
0.7	85.67	85.66	85.59	64.53	64.53	64.47
0.8	186.95	186.94	186.85	155.96	155.95	155.86
0.9	316.05	316.05	316.00	299.23	299.23	299.17
1	370.37	370.37	370.37	370.37	370.37	370.37
1.1	336.70	336.70	336.67	325.33	325.33	325.29
1.2	278.69	278.68	278.62	253.41	253.40	253.33
1.3	226.75	226.74	226.66	194.39	194.39	194.30
1.4	185.72	185.71	185.62	151.05	151.04	150.95
1.5	153.97	153.96	153.87	119.47	119.46	119.37

Table 3 The relative gain in OOC performance of the ARL-unbiased CCC_r (r = 2, 3, 4)-charts with respect to the ARL-unbiased CCC-chart

ρ	$r = 2$	$r = 3$	$r = 4$
0.5	54.11	71.98	80.45
0.6	48.71	67.30	76.71
0.7	37.89	56.57	67.28
0.8	21.09	35.94	46.56
0.9	5.36	10.53	15.29
1	0.00	0.00	0.00
1.1	3.26	6.54	9.69
1.2	9.44	17.98	25.42
1.3	15.72	28.56	38.75
1.4	21.37	37.30	49.00
1.5	26.30	44.42	56.88

5 An example

In this section, we use the simulated data provided by Chen (2013) in order to illustrate an application of the proposed ARL-unbiased CCC_r-charts ($r > 1$). It was assumed that the process produces the non-conforming items at a rate $p_0 = 0.005$ under the IC situation. Further, it was assumed that the process had been shifted to the fraction non-conforming $p = 0.0025$ which represents the improvement case. Table 4 presents 100 simulated data, provided by Chen (2013), which were generated from geometric distribution with $p = 0.0025$. In order to implement the CCC_r-chart ($r > 1$), the original data is converted into a new set of data by aggregating the $r > 1$ consecutive observations. The aggregated observations are reported in Table 5 for implementing the CCC_2, CCC_3 and CCC_4-chart respectively. The pair of control limits and randomisation probabilities for the ARL-unbiased CCC_r-chart ($r \in \{1, 2, 3, 4\}$) can be taken from Table 1 for FAR, $\alpha = 0.0027$ and $p_0 = 0.0005$. The signalling points can be traced in Table 4 for the CCC_1-chart and Table 5 for the CCC_r, r = 2, 3, 4 charts respectively which are represented in bold and underline. Please read Tables 4 and 5 from top to bottom.

It can be observed from Tables 4 and 5 that all four ARL-unbiased CCC_r charts raise an alarm. The ARL-unbiased CCC_1-chart gives an OOC signal at 87th point which is 16,814 whereas the ARL-unbiased CCC_2-chart gives a signal at 44th point. Recall that 44th charting point which is 21,617 is the sum of 87th and 88th observations of original data.

The CCC_3-chart triggers an OOC signal at 28th charting point which is the sum of 82nd, 83rd and 84th observations in original data. Moreover, theARL-unbiased CCC_4-chart gives signal at 21st charting point which is the sum of 81st, 82nd, 83rd and 84th observations in original data. Clearly, the CCC_3 and CCC_4-charts detects OOC signal earlier and hence have ability of detecting signals more quickly than the CCC_1 and CCC_2-charts. This example supports the findings that the ability of detecting OOC signal of the ARL-unbiased CCC_r-chart can be improved with an increase in r, i.e., by taking into considerations more failures r to which conforming items are accumulated.

Table 4 The simulated data provided by Chen (2013) following geometric distribution with $p_0 = 0.00025$

1,948	8,088	12,743	4,449	293	1,503	5,450	2098	301	8,779
1,245	4,985	5,549	3,526	1,607	1,014	14,544	10,333	11,690	2,698
2,330	1,824	656	2,133	4,234	1,678	1,020	566	9,308	5,753
3,144	2,881	1,785	15,108	3,892	1,664	2,999	562	6,350	3,025
5,588	1,711	1,258	1,502	2,217	2,139	5,506	6,964	1,597	6,442
4,168	566	4,082	315	11,657	1,128	8,615	1,010	2,068	2,964
2,999	109	99	1,246	3,641	14,833	923	11,188	**16,814**	4,492
88	13,054	12,430	7,469	1,020	79	4,620	737	4,860	1,487
4,140	5,804	1,140	296	5,181	2,593	1,253	606	7,405	4,757
136	392	4,670	2,344	4,572	4,628	5,780	263	7,732	881

Table 5 Aggregate data for implementing the ARL-unbiased CCC_r-chart ($r > 1$)

Aggregate data for the CCC_2-chart									
3,193	13,073	18,292	7,975	1,900	2517	19,994	12,431	11,991	11,477
5,474	4,705	2,441	17,241	8,126	3342	4,019	1,128	15,658	8,778
9,756	2,277	5,340	1,817	13,874	3267	14,121	7,974	3,665	9,406
3,087	13,163	12,529	8,715	4,661	14912	5,543	11,925	**21,674**	5,979
4,276	6,196	5,810	2,640	9,753	7221	7,033	869	15,137	5,638
Aggregate data for the CCC_3-chart aggregate data for the CCC_3-chart									
5,523	6,416	5,439	9,011	11,256	21,014	8,092	20,479	10,736	
12,900	13,729	18,240	4,244	4,356	17,120	12,935	19,997		
7,227	18,939	10,108	10,343	18,100	6,796	1,170	17,230		
13,209	7,990	16,925	16,318	7,300	18,211	**27,348**	12,431		
Aggregate data for the CCC_4-chart aggregate data for the CCC_4-chart aggregate data for the CCC_4-chart									
8,667	6,982	17,869	11,355	12,270	24,013	9,102	25,339	11,617	
12,843	19,359	13,785	10,026	6,609	19,664	12,794	26,614		
17,349	20,733	19,058	18,535	22,133	19,464	**27,649**	18,184		

6 Conclusions

In this paper, we considered the CCC_r-charts for monitoring the cumulative count of conforming items up to the occurrence of r^{th} non-conforming item in the high-yield processes. The CCC_r-charts ($r > 1$) have ability of early detection of an OOC signal than the CCC or geometric chart. However, the existing CCC_r chart based on equal-tail probability approach is an ARL-biased which is undesirable in practical situation. Because a user needs to wait for a longer to get an OOC signal than a false alarm, especially when the process deteriorates, i.e., in the case of increased fraction non-conforming. Thus, we designed the control limits of the CCC_r-chart so that the chart achieves a specified nominal IC ARL value and is ARL-unbiased and hence, it gives an

OOC signal more quickly than it raises false alarm. The performance study reveals that the ARL-unbiased CCC_r-chart (r = 2, 3, 4) outperforms the existing ARL-unbiased CCC-chart. Moreover, as r increases, the proposed ARL-unbiased CCC_r-chart becomes more sensitive to detect OOC signals. Thus, we recommend the CCC_4-chart when the aim is to detect small changes in the process and the CCC_2 or CCC_3-chart otherwise. The choice of r is purely subjective and depends on how long a practitioner is willing to wait until the r^{th} non-conforming item.

Several future research topics may be of interest as a follow-up. First, the proposed charts can be studied in the estimated parameter case. Recently, an examination of the effects of parameter estimation on the chart's performance has drawn a lot of interest in the literature because the assumption of known IC parameter values is rarely met in practice. Further, in the reliability and survival analysis contexts where geometric distribution as a life time model has a significant role, we encounter the censored data which are obtained either intentionally or naturally. Thus, another future area of work might be using the proposed CCC_r-charts with censored data. Finally, all the calculations were performed in R Core Team (2017) and the programs are available from the authors on request.

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