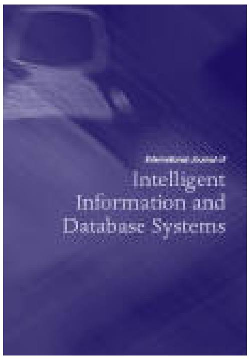

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Abstract: This paper aims to provide a method for solving multi-criteria group decision making problems with the evaluation information in the type of intuitionistic triangular fuzzy numbers (ITFNs). To this end, a new score function of ITFNs is proposed. In order to depict the similarity between two intuitionistic triangular fuzzy sets (ITFSs), the cosine similarity measure of ITFSs is defined. Based on the similarity measure, the consensus measures on three levels are defined, and a programming model-based approach is introduced to deal with situations where the group consensus level dose not reach the given threshold. By transforming the reference points of classical prospect theory (PT) into reference intervals, a method combining extended PT and PROMETHEE is presented to obtain the ranking results of alternatives. Then, an intuitionistic triangular fuzzy multi-criteria GDM (ITFMCGDM) method is developed. Finally, an example is given to illustrate the feasibility and effectiveness of the proposed method.

Keywords: multi-criteria group decision making; consensus; extended prospect theory; intuitionistic triangular fuzzy number; similarity measure.

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1 Introduction

According to the type of decision information, fuzzy multi-criteria decision making (FMCDM) can be divided into different types. Among them, intuitionistic triangular FMCDM (ITFMCDM) with the decision information in the type of intuitionistic triangular fuzzy numbers (ITFNs) can more accurately depict the uncertain preferred and non-preferred information of decision makers (DMs). In addition, to avoid the one-sidedness and limitation of single DM, group decision making (GDM) method is usually adopted in practice (Wu and Zhang, 2024). Since Liu and Yuan (2007) first came up with the definition of intuitionistic triangular fuzzy sets (ITFSs), many fruitful achievements about intuitionistic triangular fuzzy multi-criteria group decision making (ITFMCGDM) have been extracted.

By taking the intuitionistic triangular fuzzy weighted geometric and intuitionistic triangular fuzzy ordered weighted geometric operators, the comprehensive evaluation information of all the alternatives is aggregated in (Chen et al., 2010), then the score function and variation function based on the mean and standard deviation of triangular fuzzy number are defined for the ranking of ITFNs. In view of the prioritisation relationship among the DMs or the criteria, Yu (2013) proposed intuitionistic triangular fuzzy prioritised weighted average and the intuitionistic triangular fuzzy prioritised weighted geometric operators. By extending the Bonferroni mean operators introduced in (Yager, 2009), Zhou et al. (2015) defined intuitionistic triangular fuzzy weighted Bonferroni mean operator to reflect the interrelationship between the criteria. For situations where the interaction exists among the decision makers or the criteria, the fuzzy measure and Choquet integral are introduced by Liu et al. (2015) to develop some aggregation operators with interactions, such as intuitionistic triangular fuzzy Choquet geometric operator and the induced intuitionistic triangular fuzzy Choquet geometric operators. Moreover, based on the integration of ITFNs, ITFNs compound weight Bonferroni hybrid geometric operator and multi-attribute GDM theory, Zhang and Qi (2021) proposed a system evaluation method to evaluate the safety input of coal enterprises. By aggregating the heterogeneous information into ITFNs, Xu et al. (2019) presented a method to solve heterogeneous MCGDM problems. Based on the pairwise comparison of alternatives, Li et al. (2022) studied the GDM with intuitionistic triangular fuzzy preference relations.

The existing ITFMCGDM methods are mainly based on aggregation operators to obtain the comprehensive evaluation information of the alternatives, and then the ranking results of alternatives can be obtained by the score function and the accuracy function. However, the existing ranking methods for ITFNs still have some limitations in discriminability. Although the ranking method proposed by Zhang et al. (2023) can provide a total order on the ITFNs, the problem is that the ranking results lack robustness. In addition, all the ITFMCGDM methods do not explore the group consensus, in order to ensure the DMs to form a higher consistency level on the decision results, the consensus analysis should be incorporated into the GDM process.

Among the traditional multi-criteria decision-making methods, the preference ranking organisation method for enrichment evaluations (PROMETHEE) method proposed by Brans et al. (1986) is an outranking method, which can avoid the complete compensability among the criteria and has been widely used in practice (Bakshi et al., 2025; Szaja and Ziemba, 2025). On the other hand, the prospect theory (PT) proposed by Kahneman and Tversky (1979) is an effective tool to reflect the influence of subjective psychological characteristics of DMs. In exploring the combination of PT and PROMETHEE method, some scholars take the prospect value function as PROMETHEE preference function directly (Chang and Liu, 2021). In this way, the reference points of PT are different under each comparison. However, in line with the principle of uniformity, all the alternatives should have a common reference point under a criterion. Based on this perspective, Chen et al. (2020) first compare the evaluation information of each alternative with the common reference point, and then compare the alternatives with PROMETHEE preference function based on the obtained prospect value matrix.

Classical PT is based on the comparison of each alternative with single reference point to calculate the prospect values. In MCDM problem, the reference point is usually not a crisp number, but an interval. For example, when a DM evaluates the fuel consumption per 100 kilometres of a family car, the expected fuel consumption corresponds to a range, such as 7–8 litres. When the fuel consumption of alternative models changes within this range, the additional attention of buyers will not be aroused. However, when the fuel consumption of a certain vehicle is below or above the expected range, a strong sense of satisfaction or dissatisfaction will be aroused. Therefore, it is necessary to discuss the prospect theory with the reference interval, which is called the extended prospect theory (EPT), and to study the corresponding decision-making method.

Based on the analysis of the above two aspects, this paper intends to carry out research on ITFMCGDM method based on consensus analysis and EPT. The main contributions include the following aspects:

1. a new score function for the ranking of ITFNs is defined based on the mean and stand deviation of triangular fuzzy numbers
2. the cosine similarity measure of ITFSs is proposed and its properties are proved
3. the consensus measures on three levels are defined, and a programming model-based method is introduced into the consensus reaching process
4. a decision-making method combining EPT and PROMETHEE is introduced to obtain the ranking results of alternatives.

The main research contents of the rest sections of this paper are arranged as follows: Section 2 introduces some basic concepts as the basis for the follow-up research. Section 3 provides a novel score function for the ranking of ITFNs. Section 4 defines the cosine similarity measure for ITFNs, and the corresponding properties are proved. Section 5 conducts the consensus analysis for ITFMCGDM, introduces the definition of consensus measure and the improvement method of group consensus level. Section 6 proposes an EPT and PROMETHEE combined method to get the prioritisation of alternatives. Section 7 illustrates the feasibility and effectiveness of the proposed GDM method by taking the selection of social capital parties in pension institutions as an example. Section 8 completes the conclusion of this paper.

2 Preliminaries

In the application of fuzzy set theory, it is found that only considering the membership information cannot reflect the DMs' hesitation caused by subjective uncertain cognitive. To overcome this problem, Atanassov (1999) proposed the concept of intuitionistic fuzzy sets (IFSs), which can reflect the information of membership, non-membership and hesitation at the same time. In order to better reflect the uncertainty of membership and non-membership information in IFSs, Liu and Yuan (2007) further introduced triangular fuzzy numbers into IFSs, and put forward the definition of ITFSs.

Definition 2.1 (See Liu and Yuan, 2007): Let X be a universe of discourse. An ITFS $\tilde{A}$ over X is an object having the form:

$$\tilde{A} = \left\{ \langle x, \tilde{\mu}_A(x), \tilde{v}_A(x) \rangle \middle| x \in X \right\} \tag{1}$$

where $\tilde{\mu}_A(x) = (\mu_A^L(x), \mu_A^M(x), \mu_A^U(x))$, $\tilde{v}_A(x) = (v_A^L(x), v_A^M(x), v_A^U(x))$ are two triangular fuzzy numbers in [0, 1], which correspond to the membership and non-membership degrees of the element $x \in A$, and $0 \le \mu_A^U(x) + v_A^U(x) \le 1$, $\forall x \in X$.

For convenience, we denote an ITFN by $\tilde{\alpha} = (\tilde{\mu}, \tilde{v}) = ((\mu^L, \mu^M, \mu^U), (v^L, v^M, v^U))$.

Definition 2.2: Let $\tilde{\alpha}_i = ((\mu_i^L, \mu_i^M, \mu_i^U), (v_i^L, v_i^M, v_i^U))$ $(i = 1, 2, \ldots, n)$ be a collection of ITFNs, and w_i $(i = 1, 2, \ldots, n)$ be a weights vector such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. The arithmetic intuitionistic triangular fuzzy weighted aggregation (AITFWA) operator is defined by

$$\begin{aligned} AITFWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \Big(&\Big(\sum_{i=1}^{n} w_i \mu_i^L, \sum_{i=1}^{n} w_i \mu_i^M, \sum_{i=1}^{n} w_i \mu_i^U \Big), \\ &\Big(\sum_{i=1}^{n} w_i v_i^L, \sum_{i=1}^{n} w_i v_i^M, \sum_{i=1}^{n} w_i v_i^U \Big) \Big). \end{aligned} \tag{2}$$

The PROMETHEE method was first developed by Brans et al. (1986), which can obtain a partial ranking (PROMETHEE I) or complete ranking (PROMETHEE II) based on the pairwise comparison of alternatives. It is an effective outranking method to solve MCDM problems. The main steps of the classical PROMETHEE method can be summarised as follows:

Step 1 For the given alternatives set $A = \{a_1, a_2, \ldots, a_m\}$ and criteria set $C = \{c_1, c_2, \ldots, c_n\}$, we can calculate the preferred value $P_j(a_i, a_s)$ of the alternative a_i over a_s,

$$P_j(a_i, a_s) = f_j\left[d_j(a_i, a_s)\right] \tag{3}$$

where f_j is a preferred function with the range of [0, 1], $d_j(a_i, a_s)$ is the difference between the assessments of the alternatives a_i and a_s under the criterion c_j for $i, s = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$.

Step 2 Calculate the weighted preferred degree $\Gamma(a_i, a_s)$ of the alternative a_i over a_s for all criteria, expressed as

$$\Gamma(a_i, a_s) = \sum_{j=1}^{n} w_j P_j(a_i, a_s) \tag{4}$$

where w_j is the weight of criterion c_j, $i, s = 1, 2, \ldots, m$.

Step 3 Calculate the positive outranking flow $\phi^+(a_i)$ and the negative outranking flow $\phi^-(a_i)$ of the alternative a_i,

$$\begin{cases} \phi^+(a_i) = \dfrac{1}{m-1} \sum_{s=1, s \ne i}^{m} \Gamma(a_i, a_s) \\ \phi^-(a_i) = \dfrac{1}{m-1} \sum_{s=1, s \ne i}^{m} \Gamma(a_s, a_i) \end{cases} \tag{5}$$

for $i = 1, 2, \ldots, m$.

Step 4 Calculate the net flow of the alternative a_i

$$\phi(a_i) = \phi^+(a_i) - \phi^-(a_i) \tag{6}$$

for $i = 1, 2, \ldots, m$.

Step 5 Rank alternatives a_i, $i = 1, 2, \ldots, m$, according to their net flows.

3 A new score function of ITFNs

The existing ranking of ITFNs are mainly based on the generalisation of score function and accuracy function for the ranking of intuitionistic fuzzy numbers (Li and Sun, 2023). For ITFN $\tilde{\alpha} = (\tilde{\mu}, \tilde{v}) = ((\mu^L, \mu^M, \mu^U), (v^L, v^M, v^U))$, Wang (2008) proposed the following score function $S_1(\tilde{\alpha})$ and extended score function $H_1(\tilde{\alpha})$:

$$\begin{cases} S_1(\tilde{\alpha}) = \dfrac{\mu^L + 2\mu^M + \mu^U}{4} - \dfrac{v^L + 2v^M + v^U}{4}, \\ H_1(\tilde{\alpha}) = \dfrac{\mu^L + 2\mu^M + \mu^U}{4}\left(2 - \dfrac{\mu^L + 2\mu^M + \mu^U}{4} - \dfrac{v^L + 2v^M + v^U}{4}\right). \end{cases} \tag{7}$$

For ITFNs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, the comparison rules are as below:

1 If $S_1(\tilde{\alpha}_1) < S_1(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 < \tilde{\alpha}_2$

2 If $S_1(\tilde{\alpha}_1) = S_1(\tilde{\alpha}_2)$ and $H_1(\tilde{\alpha}_1) = H_1(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 = \tilde{\alpha}_2$; if $S_1(\tilde{\alpha}_1) = S_1(\tilde{\alpha}_2)$, but $H_1(\tilde{\alpha}_1) < H_1(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 < \tilde{\alpha}_2$.

However, take ITFNs $\tilde{\alpha}_1 = ((0.1, 0.3, 0.5), (0.2, 0.3, 0.4))$ and $\tilde{\alpha}_2 = ((0.2, 0.3, 0.4), (0.1, 0.3, 0.5))$ for example, we have $S_1(\tilde{\alpha}_1) = S_1(\tilde{\alpha}_2) = 0$, $H_1(\tilde{\alpha}_1) = H_1(\tilde{\alpha}_2) = 0.42$. Therefore, ITFNs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are indistinguishable under these two functions.

Based on the mean $m(\tilde{y}) = (a + 2b + c)/4$ and the variance $v(\tilde{y}) = (3a^2 + 4b^2 + 3c^2 - 4ab - 2ac - 4bc)/80$ of a triangular fuzzy variable $\tilde{y} = (a, b, c)$, Chen et al. (2010) proposed a ranking method for ITFN $\tilde{\alpha} = (\tilde{\mu}, \tilde{v}) = ((\mu^L, \mu^M, \mu^U), (v^L, v^M, v^U))$ by the new score function and variation function:

$$\begin{cases} S_1(\tilde{\alpha}) = \dfrac{\mu^L + 2\mu^M + \mu^U}{4} - \dfrac{v^L + 2v^M + v^U}{4}, \\ H_1(\tilde{\alpha}) = \dfrac{\mu^L + 2\mu^M + \mu^U}{4}\left(2 - \dfrac{\mu^L + 2\mu^M + \mu^U}{4} - \dfrac{v^L + 2v^M + v^U}{4}\right). \end{cases} \tag{8}$$

Then, two ITFNs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are compared by the rules:

1 if $S_3(\tilde{\alpha}_1) < S_3(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 < \tilde{\alpha}_2$

2 if $S_3(\tilde{\alpha}_1) = S_3(\tilde{\alpha}_2)$ and $V(\tilde{\alpha}_1) = V(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 = \tilde{\alpha}_2$; if $S_3(\tilde{\alpha}_1) = S_3(\tilde{\alpha}_2)$, but $V(\tilde{\alpha}_1) > V(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 < \tilde{\alpha}_2$.

Here, we take the example as $\tilde{\alpha}_1 = ((0.2, 0.25, 0.3), (0.2, 0.25, 0.3))$ and $\tilde{\alpha}_2 = ((0.4, 0.45, 0.5), (0.4, 0.45, 0.5))$. By formula (8) we have $S_3(\tilde{\alpha}_1) = 0.0625$, $S_3(\tilde{\alpha}_2) = 0.0225$, which shows that $\tilde{\alpha}_1$ is superior to $\tilde{\alpha}_2$. However, since the membership and non-membership are equal both in $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, $\tilde{\alpha}_1$ has more hesitation than $\tilde{\alpha}_2$, so $\tilde{\alpha}_2$ should be superior to $\tilde{\alpha}_1$. Therefore, the comparison result of this method is counter-intuitive. Moreover, this method still cannot distinguish ITFNs $\tilde{\alpha}_1 = ((0.1, 0.3, 0.5), (0.2, 0.3, 0.4))$ and $\tilde{\alpha}_2 = ((0.2, 0.3, 0.4), (0.1, 0.3, 0.5))$.

By taking account of the DM's attitudinal character, Liu et al. (2015) defined an attitudinal expected score function of ITFNs as follows:

$$AES_\lambda(\tilde{\alpha}) = \frac{(1-\lambda)(\mu^U - \nu^L) + 2(\mu^M - \nu^M) + \lambda(\mu^U - \nu^U) + 3}{6}, \tag{9}$$

where $\lambda \in [0, 1]$.

However, it is not easy for the DM to give an exact value for parameter λ.

In addition, Zhang et al. (2023) constructed six quantitative indices of ITFNs based on credibility theory. The advantage of this method is that it can distinguish all ITFNs, but its limitation is the lack of robustness in ranking results.

Since an ITFN is the fuzzy representation of intuitionistic fuzzy number with the preferred and non-preferred information in the type of triangular fuzzy numbers. The mean and standard deviation are two important indicators to depict the size of a triangular fuzzy number, which respectively reflect the central trend and the deviation trend. Therefore, for a triangular fuzzy number $\tilde{\gamma} = (a, b, c)$, by taking the mean $m(\tilde{\gamma}) = \frac{a + 2b + c}{4}$ and standard deviation $\sigma(\tilde{\gamma}) = \sqrt{\frac{(c-a)^2 + 2(b-a)^2 + 2(c-b)^2}{80}}$ proposed in Lee and Li (1988), we can introduce the quantitative value of $\tilde{\gamma}$ as:

$$q(\tilde{\gamma}) = \frac{m(\tilde{\gamma})}{1 + \sigma(\tilde{\gamma})}. \tag{10}$$

In this way, the ITFN $\tilde{\alpha} = ((\mu^L, \mu^M, \mu^U), (\nu^L, \nu^M, \nu^U))$ can be transformed into intuitionistic fuzzy number $\tilde{\alpha}' = \left(\frac{m(\tilde{\mu})}{1+\sigma(\tilde{\mu})}, \frac{m(\tilde{\nu})}{1+\sigma(\tilde{\nu})}\right)$. Since intuitionistic fuzzy number corresponds to interval number, that is, $\tilde{\alpha}'$ can be further transformed into $\left[\frac{m(\tilde{\mu})}{1+\sigma(\tilde{\mu})}, 1 - \frac{m(\tilde{\nu})}{1+\sigma(\tilde{\nu})}\right]$. Like the above quantisation method of triangular fuzzy number, we still use the mean and interval width to reflect the size of interval numbers. Based on the above analysis, we give a new score function for ITFNs.

Definition 3.1: Let $\tilde{\alpha} = ((\mu^L, \mu^M, \mu^U), (\nu^L, \nu^M, \nu^U))$ be an ITFN. The score function $S(\tilde{\alpha})$ of $\tilde{\alpha}$ is expressed as

$$S(\tilde{\alpha})=\frac{\frac{1}{2}\left(1+\frac{m(\tilde{\mu})}{1+\sigma(\tilde{\mu})}-\frac{m(\tilde{v})}{1+\sigma(\tilde{v})}\right)}{1+\frac{1}{6}\left(1-\frac{m(\tilde{\mu})}{1+\sigma(\tilde{\mu})}-\frac{m(\tilde{v})}{1+\sigma(\tilde{v})}\right)}, \tag{11}$$

where $\begin{cases} m(\tilde{\mu})=\dfrac{\mu^L+2\mu^M+\mu^U}{4} \\ m(\tilde{v})=\dfrac{v^L+2v^M+v^U}{4} \\ \sigma(\tilde{\mu})=\sqrt{\dfrac{(\mu^U-\mu^L)^2+2(\mu^M-\mu^L)^2+2(\mu^U-\mu^M)^2}{80}} \\ \sigma(\tilde{v})=\sqrt{\dfrac{(v^U-v^L)^2+2(v^M-v^L)^2+2(v^U-v^M)^2}{80}} \end{cases}.$

Next, we analysis some properties of the proposed score function.

Theorem 3.1: Let $\tilde{\alpha}=((\mu^L,\mu^M,\mu^U),(v^L,v^M,v^U))$ be an ITFN，the score function $S(\tilde{\alpha})=1$ if and only if $\tilde{\alpha}=((1,1,1),(0,0,0))$.

Proof: For $\tilde{\alpha}=((1,1,1),(0,0,0))$, we substitute it into formula (11), and the corresponding score function $S(\tilde{\alpha})=1$. The sufficiency is proved. Here we prove the necessary. If $S(\tilde{\alpha})=1$, from formula (11), we have $\frac{m(\tilde{\mu})}{1+\sigma(\tilde{\mu})}=1$ and $\frac{m(\tilde{v})}{1+\sigma(\tilde{v})}=0$. It can be concluded that $m(\tilde{\mu})=1$, $\sigma(\tilde{\mu})=0$, $m(\tilde{v})=0$ and $\sigma(\tilde{v})=0$, that is, $\tilde{\mu}=(1,1,1)$, $\tilde{v}=(0,0,0)$. The proof is complete.

Theorem 3.2: Let $\tilde{\alpha}=((\mu^L,\mu^M,\mu^U),(v^L,v^M,v^U))$ be an ITFN，the score function $S(\tilde{\alpha})=0$ if and only if $\tilde{a}=((0,0,0),(1,1,1))$.

Proof: The proof of this theorem is like that of Theorem 3.1.

Theorem 3.3: Let $\tilde{\alpha}=((\mu^L,\mu^M,\mu^U),(v^L,v^M,v^U))$ be an ITFN，the score function $S(\tilde{\alpha})$ complies with $0\le S(\tilde{\alpha})\le 1$.

Proof: From formula (11), we have

$$1+\frac{1}{6}\left(1-\frac{m(\tilde{\mu})}{1+\sigma(\tilde{\mu})}-\frac{m(\tilde{v})}{1+\sigma(\tilde{v})}\right)\ge\frac{1}{2}\left(1+\frac{m(\tilde{\mu})}{1+\sigma(\tilde{\mu})}-\frac{m(\tilde{v})}{1+\sigma(\tilde{v})}\right)\ge 0,$$

then it can be concluded that $0\le S(\tilde{\alpha})\le 1$.

Theorem 3.4: For two *ITFNs* $\tilde{a}_1=(\tilde{\mu}_1,\tilde{v}_1)=((\mu_1^L,\mu_1^M,\mu_1^U),(v_1^L,v_1^M,v_1^U))$ and $\tilde{\alpha}_2=(\tilde{\mu}_2,\tilde{v}_2)=((\mu_2^L,\mu_2^M,\mu_2^U),(v_2^L,v_2^M,v_2^U))$, if $\mu_1^L\le\mu_2^L$, $\mu_1^M\le\mu_2^M$, $\mu_1^U\le\mu_2^U$, and $v_1^L\ge v_2^L$, $v_1^M\ge v_2^M$, $v_1^U\ge v_2^U$, then $S(\tilde{\alpha}_1)\le S(\tilde{\alpha}_2)$.

Proof: Let us first prove that $q(\tilde{\mu}_1)=\frac{m(\tilde{\mu}_1)}{1+\sigma(\tilde{\mu}_1)}\le q(\tilde{\mu}_2)=\frac{m(\tilde{\mu}_2)}{1+\sigma(\tilde{\mu}_2)}$. Since $\mu_1^L\le\mu_2^L$, $\mu_1^M\le\mu_2^M$, $\mu_1^U\le\mu_2^U$, we set $\tilde{\mu}_2=(\mu_1^L+x,\mu_1^M+y,\mu_1^U+z)$, $x, y, z\ge 0$, and assume an auxiliary function that

$$f(x,y,z)=\frac{m(\tilde{\mu}_2)}{1+\sigma(\tilde{\mu}_2)}-\frac{m(\tilde{\mu}_1)}{1+\sigma(\tilde{\mu}_1)}$$

$$=\frac{\dfrac{\mu_1^L+x+2(\mu_1^M+y)+\mu_1^U+z}{4}}{1+\sqrt{\dfrac{(\mu_1^U+z-\mu_1^L-x)^2+2(\mu_1^M+y-\mu_1^L-x)^2+2(\mu_1^U+z-\mu_1^M-y)^2}{80}}}$$

$$-\frac{\dfrac{\mu_1^L+2\mu_1^M+\mu_1^U}{4}}{1+\sqrt{\dfrac{(\mu_1^U-\mu_1^L)^2+2(\mu_1^M-\mu_1^L)^2+2(\mu_1^U-\mu_1^M)^2}{80}}}.$$

In order to prove $f(x, y, z)\ge 0$, that is to prove

$$\begin{aligned}f(x,y,z)&=\left[\mu_1^L+x+2(\mu_1^M+y)+\mu_1^U+z\right]\\&\times\left[\sqrt{80}+\sqrt{(\mu_1^U-\mu_1^L)^2+2(\mu_1^M-\mu_1^L)^2+2(\mu_1^U-\mu_1^M)^2}\right]-(\mu_1^L+2\mu_1^M+\mu_1^U)\\&\times\left[\sqrt{80}+\sqrt{(\mu_1^U+z-\mu_1^L-x)^2+2(\mu_1^M+y-\mu_1^L-x)^2+2(\mu_1^U+z-\mu_1^M-y)^2}\right]\\&\ge 0.\end{aligned}$$

First, we consider

$$\begin{aligned}f(0,0,z)&=\sqrt{80}z+(\mu_1^L+2\mu_1^M+\mu_1^U+z)\sqrt{(\mu_1^U-\mu_1^L)^2+2(\mu_1^M-\mu_1^L)^2+2(\mu_1^U-\mu_1^M)^2}\\&-(\mu_1^L+2\mu_1^M+\mu_1^U)\sqrt{(\mu_1^U+z-\mu_1^L)^2+2(\mu_1^M-\mu_1^L)^2+2(\mu_1^U+z-\mu_1^M)^2}.\end{aligned}$$

To show that $f(0, 0, z)\ge 0$, from $f(0, 0, 0) = 0$, we just need to prove that

$$\begin{aligned}\frac{\partial f}{\partial z}&=\sqrt{80}+\sqrt{(\mu_1^U-\mu_1^L)^2+2(\mu_1^M-\mu_1^L)^2+2(\mu_1^U-\mu_1^M)^2}\\&-\frac{(\mu_1^L+2\mu_1^M+\mu_1^U)(3\mu_1^U+3z-\mu_1^L-2\mu_1^M)}{\sqrt{(\mu_1^U+z-\mu_1^L)^2+2(\mu_1^M-\mu_1^L)^2+2(\mu_1^U+z-\mu_1^M)^2}}\\&\ge 0,\end{aligned}$$

By comparing $\sqrt{80}$ with $\frac{(\mu_1^L+2\mu_1^M+\mu_1^U)(3\mu_1^U+3z-\mu_1^L-2\mu_1^M)}{\sqrt{(\mu_1^U+z-\mu_1^L)^2+2(\mu_1^M-\mu_1^L)^2+2(\mu_1^U+z-\mu_1^M)^2}}$, we have

$$\sqrt{80}\sqrt{\left(\mu_1^U+z-\mu_1^L\right)^2+2\left(\mu_1^M-\mu_1^L\right)^2+2\left(\mu_1^U+z-\mu_1^M\right)^2}$$
$$=\sqrt{16}\left[\sqrt{5}\sqrt{\left(\mu_1^U+z-\mu_1^L\right)^2+2\left(\mu_1^M-\mu_1^L\right)^2+2\left(\mu_1^U+z-\mu_1^M\right)^2}\right].$$

Since $\sqrt{16}\geq\mu_1^L+2\mu_1^M+\mu_1^U$, and $\sqrt{5}\sqrt{(\mu_1^U+z-\mu_1^L)^2+2(\mu_1^M-\mu_1^L)^2+2(\mu_1^U+z-\mu_1^M)^2}$ $\geq 3\mu_1^U+3z-\mu_1^L-2\mu_1^M$, it can be concluded that $\frac{\partial f}{\partial z}\geq 0$, that is $f(0, 0, z)\geq 0$. Similarly, we can prove that $f(x, 0, 0), f(0, y, 0)\geq 0$, thus we have $f(x, y, z)\geq 0$, i.e., $q(\tilde{\mu}_1)\leq q(\tilde{\mu}_2)$.

Similar to the above proof process, we can also derive $q(\tilde{v}_1)\geq q(\tilde{v}_2)$.

Moreover, it is not difficult to obtain from formula (11) that the score function $S(\tilde{\alpha})$ is monotonically increasing with respect to $q(\tilde{\mu})=\frac{m(\tilde{\mu})}{1+\sigma(\tilde{\mu})}$, and monotonically decreasing with respect to $q(\tilde{v})=\frac{m(\tilde{v})}{1+\sigma(\tilde{v})}$.

Based on the above analysis, we can come to the conclusion $S(\tilde{\alpha}_1)\leq S(\tilde{\alpha}_2)$. The proof of Theorem 3.4 is completed.

In addition, for ITFN $\tilde{\alpha}=((\mu^L,\mu^M,\mu^U),(v^L,v^M,v^U))$, when $\mu^L=\mu^M=\mu^U=\mu$ and $v^L=v^M=v^U=v$, then the ITFN degenerates into an intuitionistic fuzzy number $\tilde{\alpha}=(\mu, v)$, and the corresponding score function is $S(\tilde{\alpha})=\frac{\frac{(1+\mu-v)}{2}}{1+\frac{(1-\mu-v)}{6}}$. If further, we have $\mu^L=\mu^M=\mu^U=\mu$, $v^L=v^M=v^U=v$, and $\mu=1-v$, then the ITFN degenerates into usual fuzzy number, and the score function $S(\tilde{\alpha})=\mu$, which means that we just sort the fuzzy number according to the membership.

4 Cosine similarity measures of ITFSs

The similarity measure is an important tool to reflect the relationship between fuzzy information. Different from the ones based on the distance measure (Xu et al., 2019), here we introduce a cosine similarity measure between ITFSs.

Definition 4.1: Let $\tilde{A}$ and $\tilde{B}$ are two ITFSs defined in $X=\{x_1, x_2, \ldots, x_n\}$, expressed as

$$\begin{cases}\tilde{A}=\left\{\left\langle x_i,\left(\mu_A^L(x_i),\mu_A^M(x_i),\mu_A^U(x_i)\right),\left(v_A^L(x_i),v_A^M(x_i),v_A^U(x_i)\right)\right\rangle \middle| x_i\in X\right\}\\ \tilde{B}=\left\{\left\langle x_i,\left(\mu_B^L(x_i),\mu_B^M(x_i),\mu_B^U(x_i)\right),\left(v_B^L(x_i),v_B^M(x_i),v_B^U(x_i)\right)\right\rangle \middle| x_i\in X\right\}\end{cases},$$

then the cosine similarity measure is defined by

$$SI\left(\tilde{A},\tilde{B}\right)=\frac{1}{n}\sum_{i=1}^{n}\frac{K\left(\tilde{A}(x_i),\tilde{B}(x_i)\right)}{\sqrt{E\left(\tilde{A}(x_i)\right)E\left(\tilde{B}(x_i)\right)}} \tag{12}$$

where

$$
\begin{aligned}
K\left(\tilde{A}(x_i), \tilde{B}(x_i)\right) = {} & \mu_A^L(x_i)\mu_B^L(x_i) + 4\mu_A^M(x_i)\mu_B^M(x_i) + \mu_A^U(x_i)\mu_B^U(x_i) \\
& + v_A^L(x_i)v_B^L(x_i) + 4v_A^M(x_i)v_B^M(x_i) + v_A^U(x_i)v_B^U(x_i) \\
& + \left(1-\mu_A^U(x_i)-v_A^U(x_i)\right)\left(1-\mu_B^U(x_i)-v_B^U(x_i)\right) \\
& + 4\left(1-\mu_A^M(x_i)-v_A^M(x_i)\right)\left(1-\mu_B^M(x_i)-v_B^M(x_i)\right) \\
& + \left(1-\mu_A^L(x_i)-v_A^L(x_i)\right)\left(1-\mu_B^L(x_i)-v_B^L(x_i)\right),
\end{aligned}
$$

$$
\begin{aligned}
E\left(\tilde{A}(x_i)\right) = {} & \left(\mu_A^L(x_i)\right)^2 + 4\left(\mu_A^M(x_i)\right)^2 + \left(\mu_A^U(x_i)\right)^2 + \left(v_A^L(x_i)\right)^2 + 4\left(v_A^M(x_i)\right)^2 \\
& + \left(v_A^U(x_i)\right)^2 + \left(1-\mu_A^U(x_i)-v_A^U(x_i)\right)^2 + 4\left(1-\mu_A^M(x_i)-v_A^M(x_i)\right)^2 \\
& + \left(1-\mu_A^L(x_i)-v_A^L(x_i)\right)^2
\end{aligned}
$$

$$
\begin{aligned}
E\left(\tilde{B}(x_i)\right) = {} & \left(\mu_B^L(x_i)\right)^2 + 4\left(\mu_B^M(x_i)\right)^2 + \left(\mu_B^U(x_i)\right)^2 + \left(v_B^L(x_i)\right)^2 + 4\left(v_B^M(x_i)\right)^2 \\
& + \left(v_B^U(x_i)\right)^2 + \left(1-\mu_B^U(x_i)-v_B^U(x_i)\right)^2 + 4\left(1-\mu_B^M(x_i)-v_B^M(x_i)\right)^2 \\
& + \left(1-\mu_B^L(x_i)-v_B^L(x_i)\right)^2.
\end{aligned}
$$

We can prove that the proposed cosine similarity measure satisfies the following properties:

1 $0 \le SI(\tilde{A}, \tilde{B}) \le 1$

2 $SI(\tilde{A}, \tilde{B}) = SI(\tilde{B}, \tilde{A})$

3 $SI(\tilde{A}, \tilde{B}) = 1$ if and only if $\tilde{A} = \tilde{B}$

4 if $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, then $SI(\tilde{A}, \tilde{C}) \le SI(\tilde{A}, \tilde{B})$, and $SI(\tilde{A}, \tilde{C}) \le SI(\tilde{B}, \tilde{C})$.

Proof:

1 By the Cauchy-Schwarz inequality and the non-negative components, we know that $0 \le \frac{K(\tilde{A}(x_i), \tilde{B}(x_i))}{\sqrt{E(\tilde{A}(x_i))E(\tilde{B}(x_i))}} \le 1$, thus $0 \le SI(\tilde{A}, \tilde{B}) \le \frac{1}{n}\sum_{i=1}^{n} 1 = 1$.

2 Since the similarity measure is symmetric with respect to ITFSs $\tilde{A}$ and $\tilde{B}$, $SI(\tilde{A}, \tilde{B}) = SI(\tilde{B}, \tilde{A})$ is clearly true.

3 When $\tilde{A} = \tilde{B}$, there are $\mu_A^L(x_i) = \mu_B^L(x_i)$, $\mu_A^M(x_i) = \mu_B^M(x_i)$, $\mu_A^U(x_i) = \mu_B^U(x_i)$, $v_A^L(x_i) = v_B^L(x_i)$, $v_A^M(x_i) = v_B^M(x_i)$ and $v_A^U(x_i) = v_B^U(x_i)$ for $i = 1, 2, \ldots, n$. This implies that $SI(\tilde{A}, \tilde{B}) = 1$.

If $SI(\tilde{A}, \tilde{B}) = 1$, we have $\frac{K(\tilde{A}(x_i), \tilde{B}(x_i))}{\sqrt{E(\tilde{A}(x_i))E(\tilde{B}(x_i))}} = 1$ for $i = 1, 2, \ldots, n$. Let

$$A_i = \left(\mu_A^L(x_i), 2\mu_A^M(x_i), \mu_A^U(x_i), v_A^L(x_i), 2v_A^M(x_i), v_A^U(x_i), 1-\mu_A^U(x_i)-v_A^U(x_i),\right.$$
$$\left. 2\left(1-\mu_A^M(x_i)-v_A^M(x_i)\right), 1-\mu_A^L(x_i)-v_A^L(x_i)\right)$$

and

$$B_i = \left(\mu_B^L(x_i), 2\mu_B^M(x_i), \mu_B^U(x_i), v_B^L(x_i), 2v_B^M(x_i), v_B^U(x_i), 1-\mu_B^U(x_i)\right.$$
$$\left. -v_B^U(x_i), 2\left(1-\mu_B^M(x_i)-v_B^M(x_i)\right), 1-\mu_B^L(x_i)-v_B^L(x_i)\right).$$

By Cauchy-Schwarz inequality, we have A_i and B_i are parallel, such that there is a non-zero constant l_i with

$$\frac{\mu_A^L(x_i)}{\mu_B^L(x_i)} = \frac{2\mu_A^M(x_i)}{2\mu_B^M(x_i)} = \frac{\mu_A^U(x_i)}{\mu_B^U(x_i)} = \frac{v_A^L(x_i)}{v_B^L(x_i)} = \frac{2v_A^M(x_i)}{2v_B^M(x_i)} = \frac{v_A^U(x_i)}{v_B^U(x_i)}$$
$$= \frac{1-\mu_A^U(x_i)-v_A^U(x_i)}{1-\mu_B^U(x_i)-v_B^U(x_i)} = \frac{2\left(1-\mu_A^M(x_i)-v_A^M(x_i)\right)}{2\left(1-\mu_B^M(x_i)-v_B^M(x_i)\right)}$$
$$= \frac{1-\mu_A^L(x_i)-v_A^L(x_i)}{1-\mu_B^L(x_i)-v_B^L(x_i)} = l_i.$$

Owing to

$$\mu_A^L(x_i)+2\mu_A^M(x_i)+\mu_A^U(x_i)+v_A^L(x_i)+2v_A^M(x_i)+v_A^U(x_i)+1-\mu_A^U(x_i)-v_A^U(x_i)$$
$$+2\left(1-\mu_A^M(x_i)-v_A^M(x_i)\right)+1-\mu_A^L(x_i)-v_A^L(x_i)=4$$

and

$$\mu_B^L(x_i)+2\mu_B^M(x_i)+\mu_B^U(x_i)+v_B^L(x_i)+2v_B^M(x_i)+v_B^U(x_i)+1-\mu_B^U(x_i)-v_B^U(x_i)$$
$$+2\left(1-\mu_B^M(x_i)-v_B^M(x_i)\right)+1-\mu_B^L(x_i)-v_B^L(x_i)=4 \text{ for } i=1,2,\ldots,n,$$

which implies that $l_i = 1$ and $A_i = B_i$, for $i = 1, 2, \ldots, n$. Thus $\tilde{A} = \tilde{B}$.

4 From $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, we set three ITFSs with the following expressions:

$$\begin{cases} \tilde{A} = \left(\left(\mu_B^L - x_1, \mu_B^M - y_1, \mu_B^U - z_1\right), \left(v_B^L + t_1, v_B^M + m_1, v_B^U + n_1\right),\right. \\ \quad \left.\left(1-\mu_B^U - v_B^U + z_1 - n_1, 1-\mu_B^M - v_B^M + y_1 - m_1, 1-\mu_B^L - v_B^L + x_1 - t_1\right)\right) \\ \tilde{B} = \left(\left(\mu_B^L, \mu_B^M, \mu_B^U\right), \left(v_B^L, v_B^M, v_B^U\right), \left(1-\mu_B^U - v_B^U, 1-\mu_B^M - v_B^M, 1-\mu_B^L - v_B^L\right)\right) \\ \tilde{C} = \left(\left(\mu_B^L + x_2, \mu_B^M + y_2, \mu_B^U + z_2\right), \left(v_B^L - t_2, v_B^M - m_2, v_B^U - n_2\right),\right. \\ \quad \left.\left(1-\mu_B^U - v_B^U + n_2 - z_2, 1-\mu_B^M - v_B^M + m_2 - y_2, 1-\mu_B^L - v_B^L + t_2 - x_2\right)\right) \end{cases}$$

where $x_i, y_i, z_i, t_i, m_i, n_i \geq 0$, $i = 1, 2$.

To prove the following inequality:

$$\frac{K\left(\tilde{A}, \tilde{B}\right)}{\sqrt{E\left(\tilde{A}\right)E\left(\tilde{B}\right)}} \geq \frac{K\left(\tilde{A}, \tilde{C}\right)}{\sqrt{E\left(\tilde{A}\right)E\left(\tilde{C}\right)}}, \tag{13}$$

we assume an auxiliary function

$$f\left(x_1, y_1, z_1, t_1, m_1, n_1, x_2, y_2, z_2, t_2, m_2, n_2\right) = K\left(\tilde{A}, \tilde{B}\right)\sqrt{E\left(\tilde{C}\right)} - K\left(\tilde{A}, \tilde{C}\right)\sqrt{E\left(\tilde{B}\right)}.$$

Our goal is to prove that $f(x_1, y_1, z_1, t_1, m_1, n_1, x_2, y_2, z_2, t_2, m_2, n_2) \geq 0$.

First, we consider

$$\begin{aligned}
&f\left(0, \ldots, 0, n_2\right) \\
&= \sqrt{\begin{aligned}&\left(\mu_B^L\right)^2 + 4\left(\mu_B^M\right)^2 + \left(\mu_B^U\right)^2 + \left(v_B^L\right)^2 + 4\left(v_B^M\right)^2 + \left(v_B^U\right)^2 \\ &+ \left(1-\mu_B^U - v_B^U\right)^2 + 4\left(1-\mu_B^M - v_B^M\right)^2 + \left(1-\mu_B^L - v_B^L\right)^2\end{aligned}} \\
&\times \sqrt{\begin{aligned}&\left(\mu_B^L\right)^2 + 4\left(\mu_B^M\right)^2 + \left(\mu_B^U\right)^2 + \left(v_B^L\right)^2 + 4\left(v_B^M\right)^2 + \left(v_B^U - n_2\right)^2 \\ &+ \left(1-\mu_B^U - v_B^U + n_2\right)^2 + 4\left(1-\mu_B^M - v_B^M\right)^2 + \left(1-\mu_B^L - v_B^L\right)^2\end{aligned}} \\
&- \Big[\left(\mu_B^L\right)^2 + 4\left(\mu_B^M\right)^2 + \left(\mu_B^U\right)^2 + \left(v_B^L\right)^2 + 4\left(v_B^M\right)^2 + v_{\tilde{B}}^U\left(v_B^U - n_2\right) \\
&+ \left(1-\mu_B^U - v_B^U\right)\left(1-\mu_B^U - v_B^U + n_2\right) + 4\left(1-\mu_B^M - v_B^M\right)^2 + \left(1-\mu_B^L - v_B^L\right)^2\Big].
\end{aligned}$$

To show that $f(0, \ldots, 0, n_2) \geq 0$, from $f(0, \ldots, 0, 0) = 0$, we just need to prove

$$\begin{aligned}
&\frac{\partial}{\partial n_2} f\left(0,\ldots,0,n_2\right) \\
&= \sqrt{\begin{aligned}&\left(\mu_B^L\right)^2 + 4\left(\mu_B^M\right)^2 + \left(\mu_B^U\right)^2 + \left(v_B^L\right)^2 + 4\left(v_B^M\right)^2 + \left(v_B^U\right)^2 \\ &+ \left(1-\mu_B^U - v_B^U\right)^2 + 4\left(1-\mu_B^M - v_B^M\right)^2 + \left(1-\mu_B^L - v_B^L\right)^2\end{aligned}} \\
&\times \frac{1-\mu_B^U - 2v_B^U + 2n_2}{\sqrt{\begin{aligned}&\left(\mu_B^L\right)^2 + 4\left(\mu_B^M\right)^2 + \left(\mu_B^U\right)^2 + \left(v_B^L\right)^2 + 4\left(v_B^M\right)^2 + \left(v_B^U - n_2\right)^2 \\ &+ \left(1-\mu_B^U - v_B^U + n_2\right)^2 + 4\left(1-\mu_B^M - v_B^M\right)^2 + \left(1-\mu_B^L - v_B^L\right)^2\end{aligned}}} \\
&- \left(1-\mu_B^U - 2v_B^U\right) \geq 0,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\left(\mu_B^U + 2v_B^U - 1\right)\sqrt{\begin{aligned}&\left(\mu_B^L\right)^2 + 4\left(\mu_B^M\right)^2 + \left(\mu_B^U\right)^2 + \left(v_B^L\right)^2 + 4\left(v_B^M\right)^2 + \left(v_B^U - n_2\right)^2 \\ &+ \left(1-\mu_B^U - v_B^U + n_2\right)^2 + 4\left(1-\mu_B^M - v_B^M\right)^2 + \left(1-\mu_B^L - v_B^L\right)^2\end{aligned}} \\
&+ \left(1-\mu_B^U - 2v_B^U + 2n_2\right)\sqrt{\begin{aligned}&\left(\mu_B^L\right)^2 + 4\left(\mu_B^M\right)^2 + \left(\mu_B^U\right)^2 + \left(v_B^L\right)^2 + 4\left(v_B^M\right)^2 + \left(v_B^U\right)^2 \\ &+ \left(1-\mu_B^U - v_B^U\right)^2 + 4\left(1-\mu_B^M - v_B^M\right)^2 + \left(1-\mu_B^L - v_B^L\right)^2\end{aligned}} \\
&\geq 0.
\end{aligned} \tag{14}$$

We divide the problem into two cases:

1 $\mu_B^U + 2v_B^U - 1 \leq 0$

2 $\mu_B^U + 2v_B^U - 1 > 0.$

For case 1, we have

$$\left(1-\mu_B^U - 2v_B^U + 2n_2\right)^2 \Big[\left(\mu_B^L\right)^2 + 4\left(\mu_B^M\right)^2 + \left(\mu_B^U\right)^2 + \left(v_B^L\right)^2 + 4\left(v_B^M\right)^2 + \left(v_B^U\right)^2 + \left(1-\mu_B^U - v_B^U\right)^2 + 4\left(1-\mu_B^M - v_B^M\right)^2 + \left(1-\mu_B^L - v_B^L\right)^2\Big]$$
$$\geq \left(1-\mu_B^U - 2v_B^U\right)^2 \Big[\left(\mu_B^L\right)^2 + 4\left(\mu_B^M\right)^2 + \left(\mu_B^U\right)^2 + \left(v_B^L\right)^2 + 4\left(v_B^M\right)^2 + \left(v_B^U - n_2\right)^2 + \left(1-\mu_B^U - v_B^U + n_2\right)^2 + 4\left(1-\mu_B^M - v_B^M\right)^2 + \left(1-\mu_B^L - v_B^L\right)^2\Big].$$

Thus, it can be concluded that inequality (14) is valid.

For case 2, inequality (14) is equivalent to

$$\mu_B^U + 2v_B^U - 1 \geq \left(\mu_B^U + 2v_B^U - 1 - 2n_2\right) \times \frac{\sqrt{\left(\mu_B^L\right)^2 + 4\left(\mu_B^M\right)^2 + \left(\mu_B^U\right)^2 + \left(v_B^L\right)^2 + 4\left(v_B^M\right)^2 + \left(v_B^U\right)^2 + \left(1-\mu_B^U - v_B^U\right)^2 + 4\left(1-\mu_B^M - v_B^M\right)^2 + \left(1-\mu_B^L - v_B^L\right)^2}}{\sqrt{\left(\mu_B^L\right)^2 + 4\left(\mu_B^M\right)^2 + \left(\mu_B^U\right)^2 + \left(v_B^L\right)^2 + 4\left(v_B^M\right)^2 + \left(v_B^U - n_2\right)^2 + \left(1-\mu_B^U - v_B^U + n_2\right)^2 + 4\left(1-\mu_B^M - v_B^M\right)^2 + \left(1-\mu_B^L - v_B^L\right)^2}}. \tag{15}$$

If there is the below relationship

$$\sqrt{\left(\mu_B^L\right)^2 + 4\left(\mu_B^M\right)^2 + \left(\mu_B^U\right)^2 + \left(v_B^L\right)^2 + 4\left(v_B^M\right)^2 + \left(v_B^U\right)^2 + \left(1-\mu_B^U - v_B^U\right)^2 + 4\left(1-\mu_B^M - v_B^M\right)^2 + \left(1-\mu_B^L - v_B^L\right)^2} \leq \sqrt{\left(\mu_B^L\right)^2 + 4\left(\mu_B^M\right)^2 + \left(\mu_B^U\right)^2 + \left(v_B^L\right)^2 + 4\left(v_B^M\right)^2 + \left(v_B^U - n_2\right)^2 + \left(1-\mu_B^U - v_B^U + n_2\right)^2 + 4\left(1-\mu_B^M - v_B^M\right)^2 + \left(1-\mu_B^L - v_B^L\right)^2} \tag{16}$$

then we have that inequality (15) is valid.

Since inequality (16) is equivalent to $1 + n_2 - \mu_B^U - 2v_B^U \geq 0$, next, we further divide it into two sub-cases: $\mu_B^U + 2v_B^U \leq 1 + n_2$ and $\mu_B^U + 2v_B^U > 1 + n_2$.

If $\mu_B^U + 2v_B^U > 1 + n_2$, based on inequality (15), we define an auxiliary function:

$$T\left(n_2\right) = \left(\mu_B^U + 2v_B^U - 1\right)\sqrt{\left(\mu_B^L\right)^2 + 4\left(\mu_B^M\right)^2 + \left(\mu_B^U\right)^2 + \left(v_B^L\right)^2 + 4\left(v_B^M\right)^2 + \left(v_B^U - n_2\right)^2 + \left(1-\mu_B^U - v_B^U + n_2\right)^2 + 4\left(1-\mu_B^M - v_B^M\right)^2 + \left(1-\mu_B^L - v_B^L\right)^2} + \left(1 + 2n_2 - \mu_B^U - 2v_B^U\right)\sqrt{\left(\mu_B^L\right)^2 + 4\left(\mu_B^M\right)^2 + \left(\mu_B^U\right)^2 + \left(v_B^L\right)^2 + 4\left(v_B^M\right)^2 + \left(v_B^U\right)^2 + \left(1-\mu_B^U - v_B^U\right)^2 + 4\left(1-\mu_B^M - v_B^M\right)^2 + \left(1-\mu_B^L - v_B^L\right)^2}. \tag{17}$$

To verify that $T(n_2) \geq 0$, we have $T(0) = 0$ and

$$T'(n_2)=\frac{\left(\mu_B^U+2v_B^U-1\right)\left(1-\mu_B^U-2v_B^U+2n_2\right)}{\sqrt{\begin{array}{l}\left(\mu_B^L\right)^2+4\left(\mu_B^M\right)^2+\left(\mu_B^U\right)^2+\left(v_B^L\right)^2+4\left(v_B^M\right)^2+\left(v_B^U-n_2\right)^2\\+\left(1-\mu_B^U-v_B^U+n_2\right)^2+4\left(1-\mu_B^M-v_B^M\right)^2+\left(1-\mu_B^L-v_B^L\right)^2\end{array}}+2\sqrt{\begin{array}{l}\left(\mu_B^L\right)^2+4\left(\mu_B^M\right)^2+\left(\mu_B^U\right)^2+\left(v_B^L\right)^2+4\left(v_B^M\right)^2+\left(v_B^U\right)^2\\+\left(1-\mu_B^U-v_B^U\right)^2+4\left(1-\mu_B^M-v_B^M\right)^2+\left(1-\mu_B^L-v_B^L\right)^2\end{array}}}. \tag{18}$$

If $1-\mu_B^U-2v_B^U+2n_2\geq 0$, it is obvious that $T'(n_2)\geq 0$. Then we have $T(n_2)\geq 0$.

On the other hand, if $1-\mu_B^U-2v_B^U+2n_2<0$, the following inequality can be obtained

$$\begin{aligned}&2\sqrt{\begin{array}{l}\left(\mu_B^L\right)^2+4\left(\mu_B^M\right)^2+\left(\mu_B^U\right)^2+\left(v_B^L\right)^2+4\left(v_B^M\right)^2+\left(v_B^U\right)^2\\+\left(1-\mu_B^U-v_B^U\right)^2+4\left(1-\mu_B^M-v_B^M\right)^2+\left(1-\mu_B^L-v_B^L\right)^2\end{array}}\\&\times\sqrt{\begin{array}{l}\left(\mu_B^L\right)^2+4\left(\mu_B^M\right)^2+\left(\mu_B^U\right)^2+\left(v_B^L\right)^2+4\left(v_B^M\right)^2+\left(v_B^U-n_2\right)^2\\+\left(1-\mu_B^U-v_B^U+n_2\right)^2+4\left(1-\mu_B^M-v_B^M\right)^2+\left(1-\mu_B^L-v_B^L\right)^2\end{array}}\\&\geq\left(\mu_B^U+2v_B^U-1\right)\left(\mu_B^U+2v_B^U-1-2n_2\right),\end{aligned}$$

from which, we can obtain $T'(n_2)\geq 0$. That is to say, $T(n_2)\geq$ is verified for the case $1-\mu_B^U-2v_B^U+2n_2\geq 0$.

Therefore, we can conclude that inequality (14) is valid under case 2.

Up to now, we have shown that $f(0, \ldots, 0, n_2)\geq 0$. In a similar way, we can prove that f is also non-negative with respect to the other variables, and thus reach the conclusion $f(x_1, y_1, z_1, t, m_1, n_1, x_2, y_2, z_2, t_2, m_2, n_2)\geq 0$.

Based on the above analysis, it can be concluded that inequality (13) is valid.

The same as the proof of (13), we can also prove $\frac{K(\tilde{B},\tilde{C})}{\sqrt{E(\tilde{B})E(\tilde{C})}}\geq\frac{K(\tilde{A},\tilde{C})}{\sqrt{E(\tilde{A})E(\tilde{C})}}$. So, we have done the proof of property 4.

Specially, the cosine similarity measure between ITFNs $\tilde{\alpha}_1=((\mu_1^L,\mu_1^M,\mu_1^U),(v_1^L,v_1^M,v_1^U))$ and $\tilde{\alpha}_2=((\mu_2^L,\mu_2^M,\mu_2^U),(v_2^L,v_2^M,v_2^U))$ can be calculated as:

$$SI\left(\tilde{\alpha}_1,\tilde{\alpha}_2\right)=\frac{K\left(\tilde{\alpha}_1,\tilde{\alpha}_2\right)}{\sqrt{E\left(\tilde{\alpha}_1\right)E\left(\tilde{\alpha}_2\right)}}, \tag{19}$$

where

$$\begin{aligned}K\left(\tilde{\alpha}_1,\tilde{\alpha}_2\right)=&\,\mu_1^L\mu_2^L+4\mu_1^M\mu_2^M+\mu_1^U\mu_2^U+v_1^Lv_2^L+4v_1^Mv_2^M+v_1^Uv_2^U\\&+\left(1-\mu_1^U-v_1^U\right)\left(1-\mu_2^U-v_2^U\right)+4\left(1-\mu_1^M-v_1^M\right)\left(1-\mu_2^M-v_2^M\right)\\&+\left(1-\mu_1^L-v_1^L\right)\left(1-\mu_2^L-v_2^L\right),\end{aligned}$$

$$E(\tilde{\alpha}_1) = (\mu_1^L)^2 + 4(\mu_1^M)^2 + (\mu_1^U)^2 + (v_1^L)^2 + 4(v_1^M)^2 + (v_1^U)^2 + (1-\mu_1^U - v_1^U)^2$$
$$+4(1-\mu_1^M - v_1^M)^2 + (1-\mu_1^L - v_1^L)^2,$$

$$E(\tilde{\alpha}_2) = (\mu_2^L)^2 + 4(\mu_2^M)^2 + (\mu_2^U)^2 + (v_2^L)^2 + 4(v_2^M)^2 + (v_2^U)^2 + (1-\mu_2^U - v_2^U)^2$$
$$+4(1-\mu_2^M - v_2^M)^2 + (1-\mu_2^L - v_2^L)^2.$$

5 Discussion on consensus of ITFMCGDM

For GDM problem, a key that should not be neglected is the consensus reaching process in the aggregation phase (Meng et al., 2024; Guo et al., 2024). To ensure that the decision group forms a higher level of consistency on the decision results, here we discuss the consensus of ITFMCGDM.

Let $A = \{a_1, a_2, \ldots, a_m\}$ be the set of alternatives, $C = \{c_1, c_2, \ldots, c_n\}$ be the set of criteria, and $E = \{e_1, e_2, \ldots, e_q\}$ be the set of DMs. Assume that triangular fuzzy variables $(\mu_{ij}^{L,k}, \mu_{ij}^{M,k}, \mu_{ij}^{U,k})$ and $(v_{ij}^{L,k}, v_{ij}^{M,k}, v_{ij}^{U,k})$ are the membership degree and non-membership degree of the alternative a_i that satisfy the criterion c_j given by the expert e_k, respectively, where $(\mu_{ij}^{L,k}, \mu_{ij}^{M,k}, \mu_{ij}^{U,k})$ and $(v_{ij}^{L,k}, v_{ij}^{M,k}, v_{ij}^{U,k})$ are defined on [0, 1] with $\mu_{ij}^{U,k} + v_{ij}^{U,k} \le 1$. In other words, the evaluation of the alternative a_i w.r.t. the criterion c_j given by the expert e_k is an ITFN $\tilde{\alpha}_{ij}^k = ((\mu_{ij}^{L,k}, \mu_{ij}^{M,k}, \mu_{ij}^{U,k}), (v_{ij}^{L,k}, v_{ij}^{M,k}, v_{ij}^{U,k}))$ ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, q$). By $\boldsymbol{D}^k = (\tilde{\alpha}_{ij}^k)_{m\times n}$, we denote the intuitionistic triangular fuzzy matrix given by expert e_k ($k = 1, 2, \ldots q$)

Based on the cosine similarity measure between ITFSs proposed in section 4, we analysis the consensus problem of ITFMCGDM from two aspects: consensus measurements and consensus adjustments.

For each pair of DMs, the similarity measure between DM_k and DM_l in their evaluation for alternative a_i concerning criterion c_j is given as $sm_{ij}^{kl} = SI(\tilde{\alpha}_{ij}^k, \tilde{\alpha}_{ij}^l)$, ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $k, l = 1, 2, \ldots, q$, $k \neq l$). Then, the similarity matrix $\boldsymbol{SM}^{kl} = (sm_{ij}^{kl})_{m\times n}$ between the intuitionistic triangular fuzzy evaluation matrices $\boldsymbol{D}^k$ and $\boldsymbol{D}^l$ is obtained. In addition, the consensus matrix $\boldsymbol{CM} = (cm_{ij})_{m\times n}$ can be calculated by aggregating all similarity matrices:

$$cm_{ij} = \frac{2}{q(q-1)} \sum\nolimits_{k,l=1,k<l}^{q} sm_{ij}^{kl} \tag{20}$$

According to the consensus matrix, we can define the consensus measures on three levels (Li et al., 2019):

1 Criterion level: the consensus measure for alternative a_i over criterion c_j, denoted as cc_{ij}, can be defined by the element of consensus matrix $\boldsymbol{CM}$ as

$$cc_{ij} = cm_{ij}, i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n \tag{21}$$

This measure is used to identify the positions in the evaluation matrix that have a low consensus level.

2 Alternative level: the consensus measure on alternative a_i, denoted as ca_i, is defined to measure the consensus level among all the DMs for that alternative:

$$ca_i = \frac{\sum_{j=1}^{n} cm_{ij}}{n}, i = 1, 2, \dots, m. \tag{22}$$

3 Decision matrix level: the consensus measure on the decision matrix, called cd, is defined to represent the global consensus level amongst the experts' evaluation information

$$cd = \min_i \{ca_i\}. \tag{23}$$

By taking the min operator, the compromise between some alternatives with high consensus levels and those with low consensus levels can be avoided.

Once the consensus measures for three levels are obtained, we can determine whether the consensus is reached by a comparison between the consensus degree cd and the predefined consensus threshold ε. If $cd \geq \varepsilon$, then the consensus reaching process ends and the evaluation matrices can be used for subsequent decision making.

For cases where the consensus threshold is not reached, the consensus adjustment process is implemented.

First, the non-consensus alternative set IA is obtained by comparing the consensus measure ca_i with the consensus threshold ε, i.e., $IA = \{a_i | ca_i < \varepsilon, i = 1, 2, \dots, m\}$. This rule identifies the rows of the decision matrix that should be revised.

Second, the non-consensus criterion set IC_i is obtained as $IC_i = \{c_j | a_i \in IA \wedge cc_{ij} < \varepsilon, j = 1, 2, \dots, n\}$ to identify the columns of the decision matrix that should be modified for the rows distinguished in IA. On this basis, we can locate the elements in the decision matrix that require consensus improvement.

In terms of consensus reaching, it used to be through multiple feedbacks and adjustments. To improve the efficiency of consensus improvements, we adopt the method based on programming model.

Since the consensus level of the group is judged by taking the min operator of consensus measure on alternative level, all the consensus measures on alternative level without achieving the consensus threshold need to be improved. Therefore, we focus on the consensus measure on alternative level.

For the alternative a_i ($i = 1, 2, \dots, m$) without achieving the consensus threshold, it is necessary to further identify the non-consensus criterion set IC_i under that alternative. All the criteria can be divided into two categories according to whether meet the consensus threshold or not. Without loss of generality, it can be assumed that the first n_i criteria meet the consensus threshold, and the last $n - n_i$ criteria fail to meet the consensus threshold. To ensure that alternative a_i meets the consensus threshold with the overall minimum adjustment, the following programming model is constructed to adjust the individual evaluation information on criterion set IC_i.

$$\text{(Model 1)} \quad T = \min \sum_{j=n_i+1}^{n} \sum_{k=1}^{q} \left(x_{ij}^{L,k} + y_{ij}^{L,k} + x_{ij}^{M,k} + y_{ij}^{M,k} + x_{ij}^{U,k} + y_{ij}^{U,k} + z_{ij}^{L,k} + t_{ij}^{L,k} + z_{ij}^{M,k} + t_{ij}^{M,k} + z_{ij}^{U,k} + t_{ij}^{U,k} \right)$$

s.t.

$$\sum_{j=1}^{n_i} \sum_{k,l=1,k<l}^{q} sm_{ij}^{kl} + \sum_{j=n_i+1}^{n} \sum_{k,l=1,k<l}^{q} sm_{ij}^{kl} \geq \frac{q(q-1)}{2} n\varepsilon$$

$$\mu_{ij}^{L,k} + x_{ij}^{L,k} - y_{ij}^{L,k} \leq \mu_{ij}^{M,k} + x_{ij}^{M,k} - y_{ij}^{M,k}$$

$$\mu_{ij}^{M,k} + x_{ij}^{M,k} - y_{ij}^{M,k} \leq \mu_{ij}^{U,k} + x_{ij}^{U,k} - y_{ij}^{U,k}$$

$$v_{ij}^{L,k} + z_{ij}^{L,k} - t_{ij}^{L,k} \leq v_{ij}^{M,k} + z_{ij}^{M,k} - t_{ij}^{M,k}$$

$$v_{ij}^{M,k} + z_{ij}^{M,k} - t_{ij}^{M,k} \leq v_{ij}^{U,k} + z_{ij}^{U,k} - t_{ij}^{U,k}$$

$$\mu_{ij}^{U,k} + x_{ij}^{U,k} - y_{ij}^{U,k} + v_{ij}^{U,k} + z_{ij}^{U,k} - t_{ij}^{U,k} \leq 1$$

$$x_{ij}^{L,k} y_{ij}^{L,k} = x_{ij}^{M,k} y_{ij}^{M,k} = x_{ij}^{U,k} y_{ij}^{U,k} = z_{ij}^{L,k} t_{ij}^{L,k} = z_{ij}^{M,k} t_{ij}^{M,k} = z_{ij}^{U,k} t_{ij}^{U,k} = 0;$$

$$x_{ij}^{L,k}, y_{ij}^{L,k}, x_{ij}^{M,k}, y_{ij}^{M,k}, x_{ij}^{U,k}, y_{ij}^{U,k}, z_{ij}^{L,k}, t_{ij}^{L,k}, z_{ij}^{M,k}, t_{ij}^{M,k}, z_{ij}^{U,k}, t_{ij}^{U,k} \geq 0,$$

$$j = n_i + 1, \ldots, n; \; k = 1, 2, \ldots, q$$

where $sm_{ij}^{kl} = SI(\tilde{\alpha}_{ij}^{k}, \tilde{\alpha}_{ij}^{l})$ is the similarity measure between DM_k and DM_l on their evaluation for alternative a_i under criterion c_j, ε is the predefined consensus threshold. The first constraint ensures that the consensus measure on alternative a_i can reach the given threshold, constraints (2)–(6) can ensure that they are still ITFNs after adding the corresponding non-negative deviation variables: $x_{ij}^{L,k}$, $y_{ij}^{L,k}$, $x_{ij}^{M,k}$, $y_{ij}^{M,k}$, $x_{ij}^{U,k}$, $y_{ij}^{U,k}$, $z_{ij}^{L,k}$, $t_{ij}^{L,k}$, $z_{ij}^{M,k}$, $t_{ij}^{M,k}$, $z_{ij}^{U,k}$, $t_{ij}^{U,k}$.

After obtaining the individual intuitionistic triangular fuzzy matrices that all meet the consensus requirements, we can aggregate them to get the collective intuitionistic triangular fuzzy matrix $\boldsymbol{D} = (\tilde{\alpha}_{ij})_{m \times n}$ and carry out the subsequent decision process.

6 The ITFMCDM method based on extended prospect theory and PROMETHEE

Two subsections are included in this part, the first one discusses the criteria weights determination method considering the interactions among criteria, and the second one introduces the PROMETHEE method based on the extended prospect theory.

6.1 A method to determine the weights of criteria under interaction

The information aggregation in traditional multi-criteria decision-making method assumes that the criteria are independent of each other, while in practical decision making problems, multiple decision criteria are usually interrelated. To deal with the interactions among criteria, Sugeno (1974) proposed the concept of fuzzy measure.

As the fuzzy measure is defined on the power set of the criteria, when the number of criteria is n, 2^n parameters need to be determined. Such complexity limits its practical application. For this reason, Grabisch (1997) further proposed the k-additive measures, among them, the 2-additive measure only involves the relative importance of criteria and the interaction between two criteria, which can better solve the contradiction between complexity and expressive ability.

Theorem 6.1 (See Grabisch, 1997): Let μ be a fuzzy measure on $N = \{1, 2, \ldots, n\}$, then μ is called a 2-additive measure if and only if there exist $\mu(i)$ and $\mu(i, j)$ for all $i, j \in N$ that satisfy the below conditions:

1. $\mu(i) \geq 0\ (\forall i \in N)$
2. $\sum_{i,j \subseteq N} \mu(i, j) - (|N| - 2) \sum_{i \in N} \mu(i) = 1$
3. $\sum_{i \in S \setminus j} (\mu(i, j) - \mu(i)) \geq (|S| - 2)\mu(j) (\forall S \subseteq N),$ subject to $j \in S$ and $|S| \geq 2$.

Shapley function is an important allocation index in cooperative game theory, which determines the optimal income distribution scheme according to the expected value of each player's marginal contribution to the alliance.

Since the interactions exist among the criteria in MCDM, Marichal (2000) introduced Shapley function into fuzzy measure to reflect the weights of criteria. To facilitate the application, Meng and Tang (2013) further proposed the following Shapley value on 2-additive measure:

$$\Phi_i(\mu, N) = \frac{3 - |N|}{2} \mu(i) + \frac{1}{2} \sum_{j \in N \setminus i} (\mu(i, j) - \mu(j)), \forall i \in N. \tag{24}$$

Based on the above analysis and in combination with the AITFWA operator defined in Section 2, we give the following 2-additive Shapley arithmetic intuitionistic triangular fuzzy aggregation (2ASAITFA) operator:

$$\begin{aligned} &2ASAITFA(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \\ &= \sum_{i=1}^{n} \Phi_i(\mu, N)\tilde{\alpha}_i \\ &= \Big(\Big(\sum_{i=1}^{n} \Phi_i(\mu, N)\mu_i^L, \sum_{i=1}^{n} \Phi_i(\mu, N)\mu_i^M, \sum_{i=1}^{n} \Phi_i(\mu, N)\mu_i^U \Big), \\ &\Big(\sum_{i=1}^{n} \Phi_i(\mu, N)v_i^L, \sum_{i=1}^{n} \Phi_i(\mu, N)v_i^M, \sum_{i=1}^{n} \Phi_i(\mu, N)v_i^U \Big) \Big) \end{aligned} \tag{25}$$

where $\Phi_i(\mu, N)$ is the Shapley value shown in formula (24).

It is easy to verify that the operator satisfies the idempotency, boundedness and monotony.

Under the condition that the weights of criteria are incompletely known, the following programming model can be constructed to obtain the optimal 2-additive measure of the criteria, aiming at minimising the similarity degree among the column vectors of criteria.

$$\text{Model 2} \quad \min \sum\nolimits_{j,t=1,t\neq j}^{n} \Phi_j(\mu, C) SI\left(D_j, D_t\right)$$

s.t.

$$\sum_{c_r \in S \backslash c_j} \left(\mu\left(c_j, c_r\right) - \mu\left(c_r\right)\right) \geq (|S|-2)\mu\left(c_j\right), \forall S \subseteq C, \forall c_j \in S, |S| \geq 2,$$

$$\sum_{c_j, c_r \subseteq C} \mu\left(c_j, c_r\right) - (|C|-2) \sum_{c_j \in C} \mu\left(c_j\right) = 1,$$

$$\mu\left(c_j\right) \in W_j, \mu\left(c_j\right) \geq 0, \ j = 1, 2, \ldots, n.$$

where $\phi(\mu, C)$ is the Shapley value of the criterion c_j (j = 1, 2, …, n), μ is the 2-additive measure defined on C, D_j is the j^{th} column of the collective intuitionistic triangular fuzzy matrix $\boldsymbol{D} = (\tilde{\alpha}_{ij})_{m\times n}$, and SI is the similarity measure between ITFSs defined in Section 4.

According to formula (24), we have

$$\text{Model 3} \quad \min \sum\nolimits_{j,t=1,t\neq j}^{n} SI\left(D_j, D_t\right) \left(\frac{3-n}{2} \mu(c_j) + \frac{1}{2} \sum_{c_r \in C \backslash c_j} \left(\mu\left(c_j, c_r\right) - \mu\left(c_r\right)\right) \right)$$

s.t.

$$\sum_{c_r \in S \backslash c_j} \left(\mu\left(c_j, c_r\right) - \mu\left(c_r\right)\right) \geq (|S|-2)\mu\left(c_j\right), \forall S \subseteq C, \forall c_j \in S, |S| \geq 2,$$

$$\sum_{c_j, c_r \subseteq C} \mu\left(c_j, c_r\right) - (|C|-2) \sum_{c_j \in C} \mu\left(c_j\right) = 1,$$

$$\mu\left(c_j\right) \in W_j, \mu\left(c_j\right) \geq 0, \ j = 1, 2, \ldots, n.$$

After the optimal 2-additive measures on criteria set are obtained by solving model 3, we can further use formula (24) to get the Shapley values, namely the weights, for the criteria.

6.2 The PROMETHEE method based on extended prospect theory

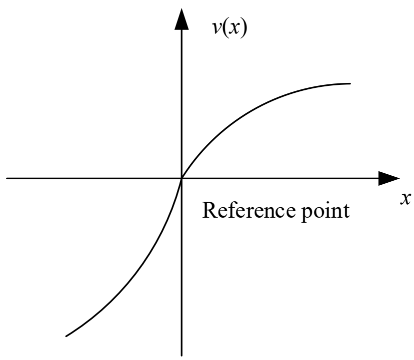
According to PROMETHEE method introduced in Section 2, one of the main steps is to compare the evaluation information of alternatives under the criteria in pairs, and put the comparison results into the preference function. On this basis, the preference net flow of each alternative compared with others is obtained to rank the alternatives.

In the above process of comparing the evaluation information among the alternatives, the subjective psychological characteristics of DMs should be taken into consideration. Prospect theory (PT) proposed by Kahneman and Tversky (1979) is an effective tool to describe human decision-making behaviour under risk. PT assumes that individuals are risk aversion for gains and risk pursuit for losses, which is expressed as the below value function

$$v(x) = \begin{cases} x^{\alpha} & \text{if } x \geq 0 \\ -\lambda(-x)^{\beta} & \text{if } x < 0 \end{cases} \tag{26}$$

where α and β are parameters related to gains and losses, respectively, the parameter λ denotes the degree of loss aversion, which is usually larger than 1. Kahneman and Tversky (1979) experimentally determined that $\alpha = \beta = 0.88$, and $\lambda = 2.25$. The value function of PT can be described by an S-shaped function as shown in Figure 1.

Figure 1 The value function of PT



As mentioned in Section 1, in the case of MCDM, the reference point is usually not an exact value, but corresponds to an interval $\overline{r} = [r^L, r^U]$. The DMs are not sensitive to the change of evaluation information within the reference interval, and the value function can be expressed as

$$v'(x) = \begin{cases} x^{\gamma} & \text{if } 0 \leq x \leq \dfrac{r^U - r^L}{2} \\ -(-x)^{\gamma} & \text{if } \dfrac{r^L - r^U}{2} \leq x < 0 \end{cases} \tag{27}$$

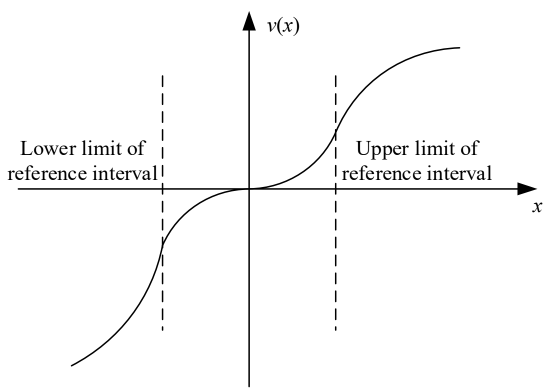
where $\gamma \geq 1$.

The value function defined on the reference interval is symmetrically distributed, and the slope is relatively gentle to reflect the insensitivity of DMs. However, when the evaluation information is lower than or higher than the reference interval, it will cause the DMs' strong satisfaction or dissatisfaction, and show obvious risk aversion. Combing the above two cases, we can get the value function of the EPT as below:

$$v''(x)=\begin{cases}\left(\dfrac{r^U-r^L}{2}\right)+\left(x-\dfrac{r^U-r^L}{2}\right)^{\alpha} & \text{if } x>\dfrac{r^U-r^L}{2}\\ x^{\gamma} & \text{if } 0\le x\le\dfrac{r^U-r^L}{2}\\ -(-x)^{\gamma} & \text{if } \dfrac{r^L-r^U}{2}\le x<0\\ -\left(\dfrac{r^U-r^L}{2}\right)^{\gamma}-\lambda\left(\dfrac{r^L-r^U}{2}-x\right)^{\beta} & \text{if } x<\dfrac{r^L-r^U}{2}\end{cases} \tag{28}$$

where the values of α, β and λ are the same as that in the traditional PT. The parameter γ complies with $\gamma \geq 1$, and is determined by the DM. The value function can be intuitively expressed as the curve shown in Figure 2.

Figure 2 The value function of extended PT



In addition, the DMs are usually accustomed to using linguistic variables to make qualitative judgements in the actual decision-making process (Tian, 2024). To reduce the difficulty of decision making and better reflect the uncertainty of subjective judgement, the corresponding relationship between the linguistic scale and the ITFNs is established as shown in Table 1.

Table 1 Linguistic variables and their ITFN representations

Linguistic variables	*ITFNs*
Extremely high (EH)	((0.80, 0.85, 0.90), (0, 0.05, 0.10))
Very high (VH)	((0.70, 0.75, 0.80), (0.10, 0.15, 0.20))
High (H)	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))
Medium high (MH)	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))
Medium (M)	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))
Medium low (ML)	((0.30, 0.35, 0.40), (0.50, 0.55, 0.60))
Low (L)	((0.20, 0.25, 0.30), (0.60, 0.65, 0.70))
Very low (VL)	((0.10, 0.15, 0.20), (0.70, 0.75, 0.80))
Extremely low (EL)	((0, 0.05, 0.10), (0.80, 0.85, 0.90))

To sum up the above analysis, the steps of ITFMCGDM based on the extended prospect theory are given as follows:

Step 1 The DM e_k evaluates the alternative a_i w.r.t. criterion c_j by linguistic variable l_{ij}^k ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, q$) and forms the linguistic evaluation matrix $L^k = (l_{ij}^k)_{m\times n}$.

Step 2 Based on the linguistic variables and their ITFN representations, the corresponding individual intuitionistic triangular fuzzy matrices $D^k = (\tilde{\alpha}_{ij}^k)_{m\times n}$ can be obtained.

Step 3 If all criteria c_j ($j = 1, 2, \ldots, n$) are benefit (i.e., the larger the value, the greater the preference), then there is no need to normalise the criteria values. Otherwise, we normalise the intuitionistic triangular fuzzy matrix $\boldsymbol{D}^k = (\tilde{a}_{ij}^k)_{m\times n}$ into

$\boldsymbol{D}'^k = (\tilde{\alpha}_{ij}'^k)_{m\times n}$ ($k = 1, 2, \ldots, q$), where $\tilde{\alpha}_{ij}'^k = \begin{cases} \tilde{\alpha}_{ij}^k & \text{for benefit criterion } c_j \\ \overline{\tilde{\alpha}_{ij}^k} & \text{for cost criterion } c_j \end{cases}$

($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$), $\overline{\tilde{\alpha}_{ij}^k}$ is the complement of $\tilde{\alpha}_{ij}^k$ such that $\overline{\tilde{\alpha}_{ij}^k} = ((v_{ij}^{L,k}, v_{ij}^{M,k}, v_{ij}^{U,k}), (\mu_{ij}^{L,k}, \mu_{ij}^{M,k}, \mu_{ij}^{U,k}))$.

Step 4 Calculate the similarity matrix between each pair of individual intuitionistic triangular fuzzy matrices, and get the consensus matrix $CM = (cm_{ij})_{m\times n}$.

Step 5 Calculate the consensus measures on criterion level, alternative level, and decision matrix level respectively. Compare the consensus degree cd and the predefined consensus threshold ε. If $cd \geq \varepsilon$, go to step 6. Otherwise, for each alternative without achieving the consensus threshold, model 1 can be used to adjust the individual evaluation information.

Step 6 Aggregate the individual evaluation matrices into collective intuitionistic triangular fuzzy matrix $\boldsymbol{D} = (\tilde{\alpha}_{ij})_{m\times n}$. Determine the optimal 2-additive measure on criteria set C by model 3, and calculate the Shapley value of each criterion by formula (25).

Step 7 By using the intuitionistic triangular fuzzy score function, the final intuitionistic triangular fuzzy matrix $\boldsymbol{D} = (\tilde{\alpha}_{ij})_{m\times n}$ can be transformed into score matrix $\boldsymbol{S} = (S(\tilde{\alpha}_{ij}))_{m\times n}$. To avoid the arbitrariness of individual preferences, the mean value of each column of the score matrix $\boldsymbol{S}$ is taken as the reference point under each criterion. For reference points r_j, $j = 1, 2, 3, 4$, the corresponding reference intervals can be obtained by incorporating the parameter δ as

$$\overline{r_j} = \left[r_j^L, r_j^U\right] = \left[r_j(1-\delta), r_j(1+\delta)\right]. \tag{29}$$

Step 8 By the comparison between $S(\tilde{\alpha}_{ij})$ and $\overline{r}_j$ ($i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$), the extended prospect value matrix $V = (v''_{ij})_{m\times n}$ can be obtained by formula (28).

Step 9 Based on the value matrix $V = (v''_{ij})_{m\times n}$ and the given preferred function, the weighted preferred value $\Gamma(a_i, a_s)$ of the alternative a_i over a_s for all criteria can be calculated by formula (4).

Step 10 The positive outranking flow $\phi^+(a_i)$ and the negative outranking flow $\phi^-(a_i)$ of alternative a_i can be obtained by formula (5).

Step 11 Calculate the net flow of the alternative a_i as $\phi(a_i) = \phi^+(a_i) - \phi^-(a_i)$ ($i = 1, 2, \ldots, m$), and get the ranking result of the alternatives.

7 Application of the proposed ITFMCGDM method

To illustrate the application of the proposed algorithm and compare the new method with previous ones, the following example on the selection of social parties of pension institutions is introduced.

Aging of population is a global problem. As the most populous country in the world, China is facing a rapidly growing trend of population aging. According to the latest census data, by the end of 2020, the elderly population aged 65 or above was 190.64 million in China, accounting for 13.5% of the total population, which is approaching the level of deep aging. It is expected that by the middle of this century, the proportion will be close to 30%. To improve the utilisation efficiency of pension resources and effectively deal with the increasing social pension pressure, China is accelerating the market-oriented reform in the pension service field and encouraging social capital to enter the pension service industry. One of the important ways is to hand over the public pension institutions to social capital parties to operate and manage. To ensure the operation effect of the new pension institutions, the key lies in the selection of appropriate social capital parties.

There is a public pension service project is open to public bidding. After preliminary screening, eight social capital parties are shortlisted, that is, the alternative set, denoted as $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$ The competent authorities appoint three experts $E = \{e_1, e_2, e_3\}$ to carry out a further review from four aspects, including social credit status, elderly service ability, operation and management level, and sustainable development ability, shown as $C = \{c_1, c_2, c_3, c_4\}$. Next, we use the GDM method proposed in section 6 to solve this problem.

Step 1 The individual linguistic decision matrices offered by three experts are listed in Tables 2–4.

Step 2 By the linguistic variables and their ITFN representations shown in Table 1, we can get the individual intuitionistic triangular fuzzy decision matrices as listed in Tables 5–7.

Step 3 Since all criteria are benefit, there is no need to modify the intuitionistic triangular fuzzy decision matrices.

Table 2 Linguistic decision matrix $\boldsymbol{L}^1$ offered by the expert e_1

e_l	c_1	c_2	c_3	c_4
a_1	MH	H	VH	H
a_2	M	H	M	MH
a_3	EH	VH	MH	M
a_4	M	M	H	EH
a_5	MH	MH	EH	H
a_6	H	EH	H	VH
a_7	VH	M	H	ML
a_8	H	H	VH	VH

Table 3 Linguistic decision matrix $\boldsymbol{L}^2$ offered by the expert e_2

e_l	c_1	c_2	c_3	c_4
a_1	MH	MH	H	MH
a_2	M	MH	H	MH
a_3	H	MH	VH	VH
a_4	MH	M	VH	H
a_5	VH	EH	M	ML
a_6	MH	H	H	MH
a_7	M	VH	EH	H
a_8	MH	M	MH	H

Table 4 Linguistic decision matrix $\boldsymbol{L}^3$ offered by the expert e_3

e_l	c_1	c_2	c_3	c_4
a_1	M	MH	MH	H
a_2	M	H	VH	H
a_3	MH	ML	M	ML
a_4	M	H	M	H
a_5	ML	VH	MH	M
a_6	M	M	ML	ML
a_7	MH	MH	M	VH
a_8	MH	M	H	MH

Table 5 Intuitionistic triangular fuzzy decision matrix $\boldsymbol{D}^1$ offered by the expert e_1

e_l	c_1	c_2	c_3	c_4
a_1	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.70, 0.75, 0.80), (0.10, 0.15, 0.20))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))
a_2	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))
a_3	((0.80, 0.85, 0.90), (0, 0.05, 0.10))	((0.70, 0.75, 0.80), (0.10, 0.15, 0.20))	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))
a_4	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.80, 0.85, 0.90), (0, 0.05, 0.10))
a_5	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.80, 0.85, 0.90), (0, 0.05, 0.10))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))

Table 5 Intuitionistic triangular fuzzy decision matrix $\boldsymbol{D}^1$ offered by the expert e_1 (continued)

e_l	c_1	c_2	c_3	c_4
a_6	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.80, 0.85, 0.90), (0, 0.05, 0.10))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.70, 0.75, 0.80), (0.10, 0.15, 0.20))
a_7	((0.70, 0.75, 0.80), (0.10, 0.15, 0.20))	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.30, 0.35, 0.40), (0.50, 0.55, 0.60))
a_8	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.70, 0.75, 0.80), (0.10, 0.15, 0.20))	((0.70, 0.75, 0.80), (0.10, 0.15, 0.20))

Table 6 Intuitionistic triangular fuzzy decision matrix $\boldsymbol{D}^2$ offered by the expert e_2

e_l	c_1	c_2	c_3	c_4
a_1	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))
a_2	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))
a_3	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.70, 0.75, 0.80), (0.10, 0.15, 0.20))	((0.70, 0.75, 0.80), (0.10, 0.15, 0.20))
a_4	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.70, 0.75, 0.80), (0.10, 0.15, 0.20))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))
a_5	((0.70, 0.75, 0.80), (0.10, 0.15, 0.20))	((0.80, 0.85, 0.90), (0, 0.05, 0.10))	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.30, 0.35, 0.40), (0.50, 0.55, 0.60))
a_6	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))
a_7	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.70, 0.75, 0.80), (0.10, 0.15, 0.20))	((0.80, 0.85, 0.90), (0, 0.05, 0.10))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))
a_8	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))

Table 7 Intuitionistic triangular fuzzy decision matrix $\boldsymbol{D}^3$ offered by the expert e_3

e_l	c_1	c_2	c_3	c_4
a_1	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))
a_2	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.70, 0.75, 0.80), (0.10, 0.15, 0.20))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))
a_3	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.30, 0.35, 0.40), (0.50, 0.55, 0.60))	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.30, 0.35, 0.40), (0.50, 0.55, 0.60))
a_4	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))
a_5	((0.30, 0.35, 0.40), (0.50, 0.55, 0.60))	((0.70, 0.75, 0.80), (0.10, 0.15, 0.20))	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))
a_6	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.30, 0.35, 0.40), (0.50, 0.55, 0.60))	((0.30, 0.35, 0.40), (0.50, 0.55, 0.60))
a_7	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.70, 0.75, 0.80), (0.10, 0.15, 0.20))
a_8	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))

Step 4 Calculate the similarity matrix between each pair of individual intuitionistic triangular fuzzy matrices,

$$SM^{12} = \begin{pmatrix} 1 & 0.837 & 0.986 & 0.981 \\ 1 & 0.981 & 0.917 & 1 \\ 0.954 & 0.934 & 0.934 & 0.810 \\ 0.977 & 1 & 0.986 & 0.954 \\ 0.934 & 0.876 & 0.753 & 0.98 \\ 0.981 & 0.954 & 1 & 0.934 \\ 0.837 & 0.977 & 0.954 & 0.810 \\ 0.981 & 0.917 & 0.934 & 0.986 \end{pmatrix},$$

$$SM^{13} = \begin{pmatrix} 0.977 & 0.977 & 0.934 & 1 \\ 1 & 1 & 0.837 & 0.981 \\ 0.876 & 0.701 & 0.977 & 0.986 \\ 1 & 0.917 & 0.917 & 0.954 \\ 0.909 & 0.934 & 0.876 & 0.810 \\ 0.917 & 0.753 & 0.810 & 0.934 \\ 0.934 & 0.977 & 0.917 & 0.701 \\ 0.981 & 0.917 & 0.986 & 0.943 \end{pmatrix},$$

$$SM^{23} = \begin{pmatrix} 0.977 & 1 & 0.981 & 0.981 \\ 1 & 0.986 & 0.986 & 0.981 \\ 0.981 & 0.909 & 0.837 & 0.701 \\ 0.977 & 0.917 & 0.837 & 1 \\ 0.701 & 0.991 & 0.977 & 0.909 \\ 0.977 & 0.917 & 0.810 & 1 \\ 0.977 & 1 & 0.753 & 0.986 \\ 1 & 1 & 0.981 & 0.981 \end{pmatrix},$$

and get the consensus matrix

$$CM = \begin{pmatrix} 0.985 & 0.938 & 0.967 & 0.987 \\ 1 & 0.989 & 0.913 & 0.987 \\ 0.937 & 0.848 & 0.916 & 0.832 \\ 0.985 & 0.944 & 0.913 & 0.969 \\ 0.848 & 0.934 & 0.869 & 0.900 \\ 0.958 & 0.874 & 0.873 & 0.956 \\ 0.916 & 0.985 & 0.874 & 0.832 \\ 0.987 & 0.944 & 0.967 & 0.970 \end{pmatrix}.$$

Step 5 Based on the consensus matrix, we can get the consensus measure on the alternative level as below:

$$ca_1 = 0.969, ca_2 = 0.972, ca_3 = 0.883, ca_4 = 0.953,$$
$$ca_5 = 0.888, ca_6 = 0.916, ca_7 = 0.902, ca_8 = 0.967.$$

Furthermore, we can get the consensus measure on the decision matrix level $cd = \min_i ca_i = 0.883$. Compared with the given threshold $\varepsilon = 0.9$, the group consensus level is not reached. To ensure that the group consensus level meets the threshold requirements, all the alternatives without achieving the consensus threshold need to be improved by model 1. Take alternative a_3 as an example, the following programming model is constructed to adjust the individual evaluation of this alternative under the criteria without meeting the consensus threshold.

$$T = \min \sum_{k=1}^{3} \Big[\big(x_{32}^{L,k} + y_{32}^{L,k} + x_{32}^{M,k} + y_{32}^{M,k} + x_{32}^{U,k} + y_{32}^{U,k} + z_{32}^{L,k} + t_{32}^{L,k} + z_{32}^{M,k}$$
$$+ t_{32}^{M,k} + z_{32}^{U,k} + t_{32}^{U,k} \big) + \big(x_{34}^{L,k} + y_{34}^{L,k} + x_{34}^{M,k} + y_{34}^{M,k} + x_{34}^{U,k} + y_{34}^{U,k} + z_{34}^{L,k}$$
$$+ t_{34}^{L,k} + z_{34}^{M,k} + t_{34}^{M,k} + z_{34}^{U,k} + t_{34}^{U,k} \big) \Big]$$

s.t.

$$\sum_{k,l=1,k<l}^{3} \left(SI\left(\tilde{\alpha}_{31}^{k}, \tilde{\alpha}_{31}^{l}\right) + SI\left(\tilde{\alpha}_{32}^{k}, \tilde{\alpha}_{32}^{l}\right) + SI\left(\tilde{\alpha}_{33}^{k}, \tilde{\alpha}_{33}^{l}\right) + SI\left(\tilde{\alpha}_{34}^{k}, \tilde{\alpha}_{34}^{l}\right) \right) \ge 12\varepsilon$$

$$\tilde{\alpha}_{3j}^{k} = \Big(\big(\mu_{3j}^{L,k} + x_{3j}^{L,k} - y_{3j}^{L,k}, \mu_{3j}^{M,k} + x_{3j}^{M,k} - y_{3j}^{M,k}, \mu_{3j}^{U,k} + x_{3j}^{U,k} - y_{3j}^{U,k} \big),$$
$$\big(\mu_{3j}^{L,k} + x_{3j}^{L,k} - y_{3j}^{L,k}, \mu_{3j}^{M,k} + x_{3j}^{M,k} - y_{3j}^{M,k}, \mu_{3j}^{U,k} + x_{3j}^{U,k} - y_{3j}^{U,k} \big) \Big)$$

$$\mu_{3j}^{L,k} + x_{3j}^{L,k} - y_{3j}^{L,k} \le \mu_{3j}^{M,k} + x_{3j}^{M,k} - y_{3j}^{M,k}$$

$$\mu_{3j}^{M,k} + x_{3j}^{M,k} - y_{3j}^{M,k} \le \mu_{3j}^{U,k} + x_{3j}^{U,k} - y_{3j}^{U,k}$$

$$v_{3j}^{L,k} + z_{3j}^{L,k} - t_{3j}^{L,k} \le v_{3j}^{M,k} + z_{3j}^{M,k} - t_{3j}^{M,k}$$

$$v_{3j}^{M,k} + z_{3j}^{M,k} - t_{3j}^{M,k} \le v_{3j}^{U,k} + z_{3j}^{U,k} - t_{3j}^{U,k}$$

$$\mu_{3j}^{U,k} + x_{3j}^{U,k} - y_{3j}^{U,k} + v_{3j}^{U,k} + z_{3j}^{U,k} - t_{3j}^{U,k} \le 1$$

$$x_{3j}^{L,k} y_{3j}^{L,k} = x_{3j}^{M,k} y_{3j}^{M,k} = x_{3j}^{U,k} y_{3j}^{U,k} = z_{3j}^{L,k} t_{3j}^{L,k} = z_{3j}^{M,k} t_{3j}^{M,k} = z_{3j}^{U,k} t_{3j}^{U,k} = 0$$

$$x_{3j}^{L,k}, y_{3j}^{L,k}, x_{3j}^{M,k}, y_{3j}^{M,k}, x_{3j}^{U,k}, y_{3j}^{U,k}, z_{3j}^{L,k}, t_{3j}^{L,k}, z_{3j}^{M,k}, t_{3j}^{M,k}, z_{3j}^{U,k}, t_{3j}^{U,k} \ge 0$$

$$j = 2, 4, k = 1, 2, 3$$

By using LINGO software to solve the programming model, the adjusted individual evaluation information about alternative a_3 can be obtained as follows:

$$\tilde{\alpha}_{32}^{1} = ((0.7, 0.75, 0.8), (0.1, 0.2, 0.2)), \tilde{\alpha}_{34}^{2} = ((0.7, 0.75, 0.8), (0.1, 0.2, 0.2)),$$

$$\tilde{\alpha}_{32}^{3} = ((0.3, 0.4, 0.4), (0.5, 0.51, 0.6)), \tilde{\alpha}_{34}^{3} = ((0.3, 0.4, 0.4), (0.5, 0.54, 0.6)).$$

Similarly, we can obtain the adjusted individual evaluation information about alternative a_5 under the criteria without meeting the consensus threshold as

$$\tilde{\alpha}_{53}^{1} = ((0.8, 0.85, 0.9), (0, 0.1, 0.1)), \tilde{\alpha}_{51}^{2} = ((0.7, 0.75, 0.8), (0.1, 0.2, 0.2)),$$

$$\tilde{\alpha}_{53}^{2} = ((0.4, 0.47, 0.5), (0.4, 0.44, 0.5)), \tilde{\alpha}_{51}^{3} = ((0.3, 0.4, 0.4), (0.5, 0.52, 0.6)).$$

Step 6 Without loss of generality, it is assumed that the weights of the experts are equal. The collective intuitionistic triangular fuzzy decision matrix $\boldsymbol{D} = (\tilde{\alpha}_{ij})_{m\times n}$ is obtained by aggregating the individual evaluation matrices that meet the consensus requirement.

Considering the interactions among the criteria, the following programming model is constructed to determine the optimal 2-additive measure on criteria with partial weight information.

$$\min -5.873\left(\mu(c_1)+\mu(c_2)+\mu(c_3)+\mu(c_4)\right)+2.93\mu(c_1,c_2)+2.941\mu(c_1,c_3)$$
$$+2.929\mu(c_1,c_4)+2.944\mu(c_2,c_3)+2.933\mu(c_2,c_4)+2.943\mu(c_3,c_4)$$

s.t.

$$\sum_{c_r\in S\setminus c_j}\left(\mu(c_j,c_r)-\mu(c_r)\right)\geq(|S|-2)\mu(c_j),\ \forall S\subseteq C,\ \forall c_j\in S,\ |S|\geq 2,$$

$$\sum_{c_j,c_r\subseteq C}\mu(c_j,c_r)-(|C|-2)\sum_{c_j\in C}\mu(c_j)=1,$$

$$\mu(c_1)+\mu(c_2)\leq\mu(c_1,c_2),\ \mu(c_1)+\mu(c_3)\leq\mu(c_1,c_3),\ \mu(c_1)+\mu(c_4)\leq\mu(c_1,c_4),$$

$$\mu(c_2)+\mu(c_3)\leq\mu(c_2,c_3),\ \mu(c_2)+\mu(c_4)\leq\mu(c_2,c_4),\ \mu(c_3)+\mu(c_4)\geq\mu(c_3,c_4),$$

Solving this model, we can get the optimal 2-additive measures on the criteria:

$$\mu(c_1)=0.25,\ \mu(c_2)=0.3,\ \mu(c_3)=0.15,\ \mu(c_4=0.25)$$

$$\mu(c_1,c_2)=0.55,\ \mu(c_1,c_3)=0.4,\ \mu(c_1,c_3)=0.7,$$

$$\mu(c_2,c_3)=0.45,\ \mu(c_2,c_4)=0.55,\ \mu(c_3,c_4)=0.25.$$

Following this, we can further calculate the corresponding Shapley value, namely the criteria weights as below:

$$\Phi_1(\mu,C)=0.35,\ \Phi_2(\mu,C)=0.3,\ \Phi_3(\mu,C)=0.075,\ \Phi_4(\mu,C)=0.275.$$

Table 8 The collective intuitionistic triangular fuzzy decision matrix ***D***

e_l	c_1	c_2	c_3	c_4
a_1	((0.47, 0.52, 0.57), (0.33, 0.38, 0.43))	((0.53, 0.58, 0.63), (0.27, 0.32, 0.37))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.57, 0.62, 0.67), (0.23, 0.28, 0.33))
a_2	((0.40, 0.45, 0.50), (0.40, 0.45, 0.50))	((0.57, 0.62, 0.67), (0.23, 0.28, 0.33))	((0.57, 0.62, 0.67), (0.23, 0.28, 0.33))	((0.53, 0.58, 0.63), (0.27, 0.32, 0.37))
a_3	((0.63, 0.68, 0.73), (0.17, 0.22, 0.27))	((0.50, 0.57, 0.60), (0.30, 0.35, 0.40))	((0.53, 0.58, 0.63), (0.27, 0.32, 0.37))	((0.47, 0.53, 0.57), (0.33, 0.40, 0.43))
a_4	((0.43, 0.48, 0.53), (0.37, 0.42, 0.47))	((0.47, 0.52, 0.57), (0.33, 0.38, 0.43))	((0.57, 0.62, 0.67), (0.23, 0.28, 0.33))	((0.67, 0.72, 0.77), (0.13, 0.18, 0.23))
a_5	((0.50, 0.57, 0.60), (0.30, 0.36, 0.40))	((0.67, 0.72, 0.77), (0.13, 0.18, 0.23))	((0.57, 0.62, 0.67), (0.23, 0.30, 0.33))	((0.43, 0.48, 0.53), (0.37, 0.42, 0.47))
a_6	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))	((0.50, 0.55, 0.60), (0.30, 0.35, 0.40))
a_7	((0.53, 0.58, 0.63), (0.27, 0.32, 0.37))	((0.53, 0.58, 0.63), (0.27, 0.32, 0.37))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.53, 0.58, 0.63), (0.27, 0.32, 0.37))
a_8	((0.53, 0.58, 0.63), (0.27, 0.32, 0.37))	((0.47, 0.52, 0.57), (0.33, 0.38, 0.43))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))	((0.60, 0.65, 0.70), (0.20, 0.25, 0.30))

Step 7 By using the intuitionistic triangular fuzzy score function, the final intuitionistic triangular fuzzy matrix can be transformed into the score matrix

$$S = \begin{pmatrix} 0.55 & 0.619 & 0.684 & 0.652 \\ 0.491 & 0.652 & 0.652 & 0.619 \\ 0.716 & 0.591 & 0.619 & 0.557 \\ 0.523 & 0.555 & 0.652 & 0.748 \\ 0.591 & 0.748 & 0.651 & 0.523 \\ 0.587 & 0.684 & 0.587 & 0.587 \\ 0.619 & 0.619 & 0.684 & 0.619 \\ 0.619 & 0.555 & 0.684 & 0.684 \end{pmatrix}.$$

Let $\delta = 0.05$, then the reference interval under each criterion can be obtained by formula (29) as follows:

$\overline{r}_1 = [0.558, 0.617], \overline{r}_2 = [0.596, 0.659], \overline{r}_3 = [0.619, 0.684], \overline{r}_4 = [0.592, 0.655].$

Step 8 Except for $\alpha = \beta = 0.88$, and $\lambda = 2.25$, which are experimentally determined in (Kahneman and Tversky, 1979), we set $\gamma = 1.5$. By formula (28), the extended prospect value matrix can be obtained

$$V = \begin{pmatrix} -0.0196 & -0.0008 & 0.0058 & 0.0048 \\ -0.2142 & 0.0037 & 0 & -0.0003 \\ 0.1357 & -0.0286 & -0.0059 & -0.1246 \\ -0.1235 & -0.1423 & 0 & 0.1294 \\ 0.0002 & 0.1242 & 0 & -0.2207 \\ 0 & 0.0441 & -0.1148 & -0.0284 \\ 0.0092 & -0.0008 & 0.0058 & -0.0003 \\ 0.0092 & -0.1423 & 0.0058 & 0.0501 \end{pmatrix}.$$

Step 9 Based on the value matrix V, and the following linear preference function proposed in Brans et al. (1986)

$$P_j(a_i, a_s) = \begin{cases} \dfrac{d_j(a_i, a_s)}{p}, & \text{if } 0 \le d_j(a_i, a_s) \le p \\ 1, & \text{if } d_j(a_i, a_s) > p \end{cases}.$$

Let the preference threshold $p = 0.3$, we can get the weighted preferred relations among the alternatives by formula (4)

$$\Pi = \begin{pmatrix} 0 & 0.2332 & 0.1493 & 0.2642 & 0.2081 & 0.0606 & 0.0047 & 0.1415 \\ 0.0046 & 0 & 0.1477 & 0.1460 & 0.2020 & 0.0545 & 0.0046 & 0.1460 \\ 0.1811 & 0.3500 & 0 & 0.4161 & 0.2462 & 0.1856 & 0.1475 & 0.2612 \\ 0.1142 & 0.2247 & 0.2343 & 0 & 0.2762 & 0.1856 & 0.1475 & 0.2612 \\ 0.1481 & 0.3706 & 0.1543 & 0.4108 & 0 & 0.1091 & 0.1251 & 0.2665 \\ 0.0677 & 0.2903 & 0.1609 & 0.3305 & 0.1763 & 0 & 0.0450 & 0.1864 \\ 0.0366 & 0.2622 & 0.1446 & 0.2978 & 0.2140 & 0.0667 & 0 & 0.1415 \\ 0.0752 & 0.3084 & 0.1631 & 0.1563 & 0.2603 & 0.1129 & 0.0462 & 0 \end{pmatrix}$$

Step 10 According to the formula (5), the positive outranking flow $\phi^+(a_i)$ and the negative outranking flow $\phi^-(a_i)$ of each alternative can be calculated.

$$\phi^+(a_1) = 0.1517, \phi^+(a_2) = 0.1008, \phi^+(a_3) = 0.2554, \phi^+(a_4) = 0.1733,$$

$$\phi^+(a_5) = 0.2264, \phi^+(a_6) = 0.17967, \phi^+(a_7) = 0.1658, \phi^+(a_8) = 0.1603.$$

$$\phi^-(a_1) = 0.0892, \phi^-(a_2) = 0.2913, \phi^-(a_3) = 0.1649, \phi^-(a_4) = 0.2888,$$

$$\phi^-(a_5) = 0.2260, \phi^-(a_6) = 0.1090, \phi^-(a_7) = 0.0703, \phi^-(a_8) = 0.1737.$$

Step 11 Then, we can get the net flow of each alternative as

$$\phi(a_1) = 0.0624, \phi(a_2) = -0.1906, \phi(a_3) = 0.0905, \phi(a_4) = -0.1155,$$

$$\phi(a_5) = 0.0004, \phi(a_6) = 0.0706, \phi(a_7) = 0.0955, \phi(a_8) = -0.0134.$$

Following this, the ranking result of the alternatives can be obtained as:

$a_7 \succ a_3 \succ a_6 \succ a_1 \succ a_5 \succ a_8 \succ a_4 \succ a_2$.

To illustrate the rationality of the proposed method, we further use some other methods to deal with the above example.

1 Based on the score matrix S obtained above, a simple method is to directly carry out weighted average of the score values of the alternatives under each criterion. By comparing the comprehensive score values of the alternatives $S(a_1) = 0.6106$, $S(a_2) = 0.5866$, $S(a_3) = 0.6275$, $S(a_4) = 0.6042$, $S(a_5) = 0.6239$, $S(a_6) = 0.6161$, $S(a_7) = 0.6238$, $S(a_8) = 0.6226$, we can get the alternatives ranked as:

   $a_3 \succ a_5 \succ a_7 \succ a_8 \succ a_6 \succ a_1 \succ a_4 \succ a_2$.

   Compared with the ranking results obtained by the method in this paper, the difference between them is obvious. In particular, the optimal alternative obtained from the two methods are different, one is alternative a_3, while the other is a_7. The main reason is that the method proposed in this paper considers the influence of the decision makers' subjective psychological characteristics.

2 If we carry out comprehensive evaluation information aggregation of the alternatives on the basis of the extended prospect value matrix V, the comprehensive extended prospect values of the alternatives can be obtained $V(\mathrm{a}_1) = -0.0053$, $V(a_2) = -0.0739$, $V(a_3) = 0.0042$, $V(a_4) = -0.0503$, $V(a_5) = -0.0234$, $V(a_6) = -0.0032$, $V(a_7) = 0.0033$, $V(a_8) = -0.0252$. Thus, the ranking result of the alternatives is

   $a_3 \succ a_7 \succ a_6 \succ a_1 \succ a_5 \succ a_8 \succ a_4 \succ a_2$.

   The difference between this ranking result and that obtained by the proposed method lies only in the order relationship between alternatives a_3 and a_7. The main reason for this difference is that the proposed method not only considers the influence of decision makers' subjective psychology, but also avoids the complete compensability among criteria by introducing the PROMETHEE method.

3 On the basis of the score matrix S, the reference points of classic PT can be used to get the prospect value matrix

$$V' = \begin{pmatrix} -0.1107 & -0.0352 & 0.0489 & 0.0435 \\ -0.2878 & 0.0377 & 0.0010 & -0.0198 \\ 0.1642 & -0.1233 & -0.1107 & -0.2075 \\ -0.2020 & -0.2245 & 0.0010 & 0.1597 \\ 0.0067 & 0.1549 & -0.0034 & -0.2982 \\ -0.0034 & 0.0793 & -0.2020 & -0.1225 \\ 0.0475 & -0.0352 & 0.0489 & -0.0198 \\ 0.0475 & -0.2245 & 0.0489 & 0.0846 \end{pmatrix}.$$

   Similar to the proposed method, the net flow of each alternative can be obtained by PROMETHEE method $\phi(a_1) = 0.0313$, $\phi(a_2) = -0.1589$, $\phi(a_3) = -0.0334$, $\phi(a_4) = -0.1998$, $\phi(a_5) = 0.0519$, $\phi(a_6) = 0.0639$, $\phi(a_7) = 0.1755$, $\phi(a_8) = 0.0695$. Thus, the ranking result of the alternatives can be obtained as:

$a_7 \succ a_8 \succ a_6 \succ a_5 \succ a_1 \succ a_3 \succ a_2 \succ a_4$.

Although the optimal alternative is the same, the ranking results obtained by this method and the one proposed in this paper are obviously different. The main reason is that the reference intervals-based EPT is adopted in this paper, while the reference points-based PT is adopted in above method.

4 Based on the score matrix *S*, similar to some existing methods (Kahneman and Tversky, 1979), we further take the value function of PT as the preference function of PROMETHEE, and the net flow of each alternative can be obtained as $\phi(a_1) = -0.0685$, $\phi(a_2) = -0.1284$, $\phi(a_3) = -0.0514$, $\phi(a_4) = -0.1049$,\ $\phi(a_5) = -0.0546$, $\phi(a_6) = -0.0537$, $\phi(a_7) = -0.0235$, $\phi(a_8) = -0.0398$. From this, all the alternatives can be ranked as:

$a_7 \succ a_8 \succ a_3 \succ a_6 \succ a_5 \succ a_1 \succ a_4 \succ a_2$.

The ranking results obtained from the two methods also have obvious differences. In contrast to the method proposed in this paper, this method introduces PT in the pairwise comparison of the evaluation information, that is, the reference points of each comparison are different. However, in line with the principle of uniformity, all the alternatives should have a common reference point under one criterion.

Through the above analysis, the advantages of the proposed method are not only the combination of PT and PROMETHEE method, but also the extension of the reference point of traditional PT to the reference interval, which can more objectively reflect the actual decision situation, so as to get more reasonable decision results.

8 Conclusions

For the ranking of ITFNs, a novel score function of ITFNs is defined in this paper. Compared with some existing ranking method, it has better distinguishing effect and robustness. In order to depict the similarity relationship between two ITFSs, the cosine similarity measure of ITFSs is defined, and the properties are proved. Aiming at the lack of consensus analysis in the existing ITFMCGDM methods, this paper defines the consensus measure on three levels based on the defined similarity measure. For the group consensus level unachieved the threshold, a programming model is constructed to assure that the amount of adjustment is minimal. Considering the interactions among the criteria, the optimal 2-additive measures of the criteria are obtained by the programming model, and the corresponding Shapley values are further obtained as the weight of the criteria. In addition, to reflect the influence of decision makers' subjective psychological characteristics, the EPT is introduced, which is characterised by changing the reference points of the classical PT into the reference intervals, so as to be more consistent with the actual decision situation. On the basis of the extended prospect value matrix, the ranking results of the alternatives are obtained by PROMETHEE method. Finally, the application of the proposed method is illustrated by an example of social capital parties' selection in the reform of public pension institutions, and the comparison with some other methods is introduced. The limitation of the proposed method lies in that some parameters involved in the determinations of the reference intervals and the value function within the reference intervals are given by DMs. In order to further popularise the application of the method, it is necessary to combine the theory of psychology and behavioural science to determine the parameters experimentally.

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