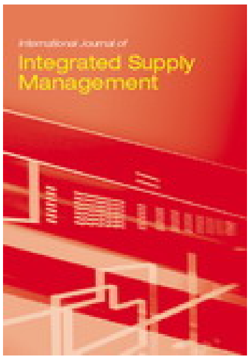

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Discrete-time EOQ with lost sales, binomial demand, and geometric lead time: inventory level distribution and performance analysis

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Discrete-time EOQ with lost sales, binomial demand, and geometric lead time: inventory level distribution and performance analysis

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Abstract: This paper presents a stochastic analytical model for the discrete-time EOQ (DT-EOQ) with lost sales, where both consumption and supply processes are modelled as Bernoulli trials. This approach allows for the representation of binomial daily demand and geometrically distributed lead time, capturing various demand patterns, including smooth, intermittent, or rare occurrences. The study provides a comprehensive mathematical derivation of the closed-form expression for the steady-state probabilities of on-hand inventory levels. This leads to concise and exact formulations for key performance measures such as the average inventory level, inventory cycle length, stock-outs per cycle, and fill rate. Notably, the proposed estimator for the average inventory level shows improved accuracy over existing formulas. Finally, some numerical examples are provided to demonstrate the effectiveness of the proposed modelling approach.

Keywords: stochastic modelling; discrete-time EOQ; DT-EOQ; lost sales; Markov chain; partitioning technique.

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1 Introduction

Inventory control is a fundamental problem with extensive real-world implications, forming the basis of decades of analytical modelling and decision-making research in operations management (Axsäter, 2015).

In the domain of inventory control, two main review mechanisms are recognised: continuous and periodic review policies (see, e.g., Ivanov et al., 2021). Periodic review policies fix the length of the inventory cycle to a specific value, known as the ‘inventory period’, which dictates when decisions about the quantity of items to be ordered are made. These quantities generally vary from period to period. Recent reviews by Perera and Sethi (2023a, 2023b) provide a comprehensive classification of decades of research in this field. Conversely, the continuous review policy, that sees its canonical implementation in the economic order quantity (EOQ) policy (Harris, 1913), involves placing a fixed-size order Q each time the inventory position, which is continuously monitored, falls to or below a reorder point r. As a result, the inventory period is not fixed and typically varies from cycle to cycle when demand and/or lead time are affected by stochastic phenomena.

The EOQ model has evolved over the years to address more complex, real-world problems, such as stochastic demand and lead time. Examples are works assuming compound Poisson demand patterns (see, e.g., Matheus and Gelders, 2000) or addressing safety stock management under controllable lead time and batch shipments in multi-stage supply chains (Castellano et al., 2024). However, these approaches often require iterative solutions, which can be computationally intensive, and lack exact estimators for key performance measures. Recent research has expanded the EOQ framework to incorporate features such as perishable products (see, e.g., Boxma et al., 2022), quality issues (see, e.g., Khan et al., 2011), time discounting (see, e.g., Roy and Chaudhuri, 2011), environmental aspects (see, e.g., Taleizadeh et al., 2020), supply disruptions (see, e.g., Babai et al., 2023) and integrated inventory models for multi-product, multi-level supply chains with joint replenishment and loss sharing (Sultanov and Hasanov, 2024). Despite these advances, the authors of this paper emphasise the need for further exploration of the classical EOQ model under stochastic demand and stochastic lead time assumptions, particularly to develop more accurate estimators for key performance metrics, such as the average inventory level. This need remains highly relevant even in the context of Industry 4.0/5.0, where supply chain management increasingly integrates AI-enabled planning systems and digital twin technologies. In these environments, analytical stochastic models continue to play a critical role: they provide interpretable, computationally efficient estimations that support decision-making and serve as foundational components for model validation and hybrid simulation frameworks, as also emphasised in recent resilience-focused supply chain studies (Ivanov, 2025). For instance, the model proposed in this work can be incorporated into digital twin architectures to describe discrete inventory dynamics under uncertainty, or used to benchmark data-driven inventory policies, especially in the presence of intermittent demand.

This challenge is especially pronounced in the lost sales scenario, which introduces additional complexity compared to the backorder case in a stochastic environment. In lost sales scenarios, demand that goes unmet during an inventory cycle creates additional inventory availability in the next cycle, influencing holding costs and extending the inventory cycle length. As a result, inventory control under the lost sales assumption is widely regarded as one of the most challenging cases to model, especially when considering continuous review policies, as highlighted by Bijvank and Vis (2011). To the best of the authors' knowledge, few studies in the literature offer exact metrics for the EOQ policy with lost sales under stochastic conditions. Moreover, these studies typically focus on continuous-time, continuous-state models (see, e.g., Mohebbi and Posner, 1998, which become less applicable when addressing intermittent demand, as they assume a continuous inventory level, reducing accuracy when inventory levels are low.

Although the classical EOQ model assumes continuous time, recent advancements by Ang et al. (2013) and Lagodimos et al. (2018) have introduced a discrete-time EOQ (DT-EOQ) model, which aligns more closely with real-world applications where time and inventory changes are discrete. In the DT-EOQ model, time is discretised such that information about the inventory position is available at the end of any time unit and then the review can take place. As a result, the concept of continuous review still applies, but reviews occur regularly at the end of each time unit (which is a fraction of the inventory cycle length). The authors demonstrated an equivalence between the DT-EOQ and the (S, T) periodic review model and, assuming backordering, showed that the optimal policy in the continuous-time framework does not hold in the discrete-time framework.

In this work, the DT-EOQ model is applied under the assumptions of lost sales, binomial daily demand, and geometric lead time. By modelling the inventory system as a Markov chain, the closed-form solution for the steady-state probability distribution of on-hand inventory levels is derived. This result yields a set of exact, compact formulas for key performance measures, including the average inventory level, which is particularly challenging to estimate in lost sales scenarios. The new estimator proposed for the average inventory level has demonstrated greater accuracy and precision compared to the classical formulas used in the literature. Additionally, this metric is expressed in terms of average lead time demand, fill rate, and expected stock-out per cycle, introducing a generalised form that is potentially adaptable to various demand and lead time distributions.

The remainder of this paper is structured as follows: Section 2 outlines the problem statement and assumptions. Section 3 details the modelling approach, while Section 4 derives the closed-form solution for the system state probabilities. Sections 5 and 6 present the performance measure calculations and application insights, respectively. Finally, Section 7 offers concluding remarks and directions for future research.

2 Problem statement

This paper presents a DT-EOQ model with lost sales which is solved in an exact analytical way by providing a closed-form solution for the probability distribution of the on-hand inventory level and compact formulas for the main performance measures.

Under the DT-EOQ model (see Lagodimos et al., 2018), time forms a sequence of indivisible time units which are used to measure the passing of time in a discrete manner. The inventory cycle length T is then split in a number of time units, each of them accounting for a fraction of the cumulative demand in the inventory cycle.

This discrete-time framework is here applied by properly defining the length of the time unit so that the fraction of the cumulative demand per time unit cannot exceed 1 item. Specifically, that demand is here assumed to take discrete values, 0 or 1, according to a Bernoulli trial with a fixed consumption probability denoted as p_2 in the following. Although the consumption probability p_2 is assumed to be constant within each scenario, the resulting demand per day is not fixed. Since demand is modelled as a sequence of Bernoulli trials over the time units of a day, the daily demand follows a binomial distribution, and thus retains inherent stochastic variability. This allows the model to realistically capture intermittent or variable demand patterns (see Section 6.1 for deeper details). The proposed model also considers a positive stochastic lead time. Specifically, at the very time unit in which the inventory level reaches the reorder point r, a replenishment order for the fixed quantity Q is triggered. The order quantity Q is assumed to arrive after a number of time units (the lead time) determined by a geometrical distribution of parameter p_1, called supply probability in the following. As an example, a setting of $p_1 = 0.1$ generates a geometric lead time with mean value of $\frac{1}{p_1} = \frac{1}{0.1} = 10$ time units.

The aforementioned assumptions, i.e., binomial daily demand and geometrically distributed lead time, although they may seem restrictive, are motivated by the following considerations. First, these assumptions are justified in real-world applications where items undergo intermittent/rare demand and the supply process is unreliable. These situations are common in modern supply chains where the responsive part results in a single inventory location that has the role of absorbing the most of the demand variability (e.g., e-commerce sector with a large variety of products, automotive spare parts aftermarket sector, etc.) and sees an unreliable supply process. This latter is because low volume items typically wait in the order queue at the manufacturer's level for an economic batch size to be reached until being released into production, thus resulting in less predictable and unreliable lead times.

Moreover, those assumptions allows us to model the inventory system as a discrete-time discrete-state Markov chain and to provide an exact closed-form solution for the probability distribution of the on-hand inventory level for the EOQ policy with lost sales where the inventory level is a discrete variable. At the best of the Authors' knowledge, the proposed model is the first one available in the literature achieving this result, thus representing a significant leap forward in the extent of stochastic EOQ modelling. Along this direction, further research may be carried out by generalising the distribution form of demand and lead time, as well as by adding new model features (e.g., inventory decay, multiple items, etc.).

Figure 1 A sample of the on hand inventory level generated by the simulator with $p_1 = 0.1$, $p_2 = 0.4$, $r = 5$, and $Q = 8$ (see online version for colours)

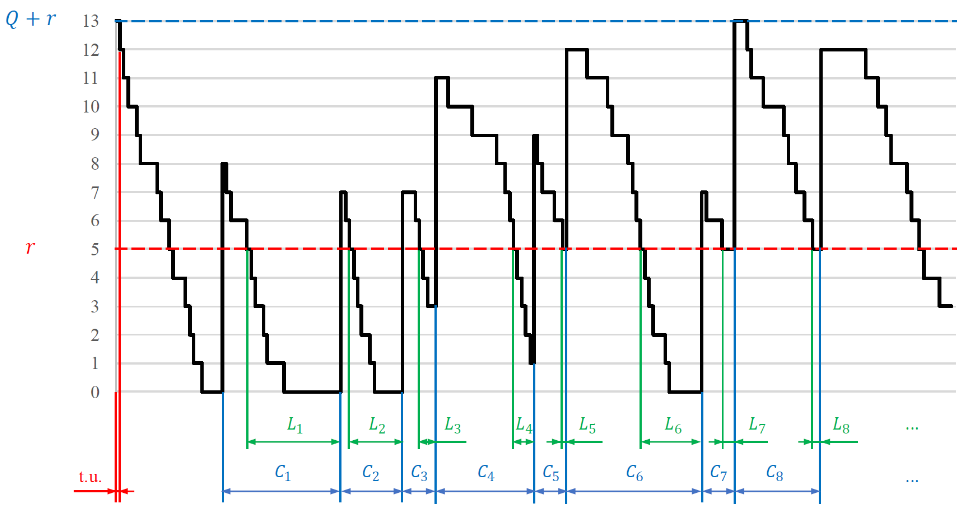


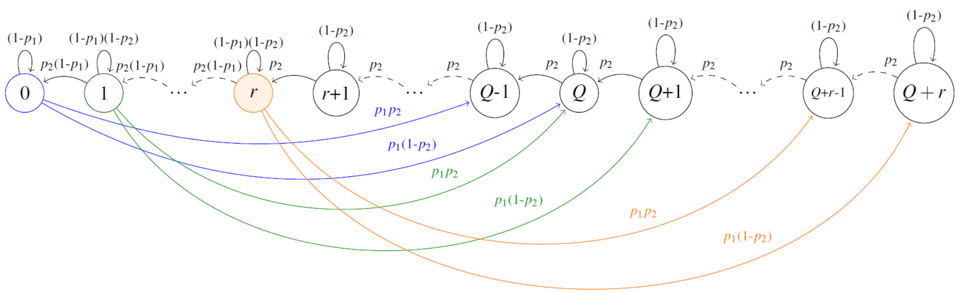
Figure 1 shows a sample of the on-hand inventory over time generated by a simulation model which works according to the aforementioned assumptions. The simulator moves over time in discrete steps, i.e., the time units, keeping trace of the current state of the system in terms of on-hand inventory; transitions are evaluated in each time unit as a function of the current system state and by applying the consumption probability p_2 and the supply probability p_1. As can be seen by looking at Figure 1 (related to the setting $p_1 = 0.1$, $p_2 = 0.4$, $r = 5$, and $Q = 8$), the dynamics of a continuous-review system is cleary generated by the proposed DT-EOQ model, as different inventory cycles have different time lengths (C_i) and different lead times (L_i) while maintaining the same supply lot of fixed size Q. It can also be noticed that, when a supply lot of $Q = 8$ arrives in a time unit, the on-hand inventory can increase by either 8 or 7 items depending on the non-occurrence or the occurrence of a consumption event in the same time unit, respectively.

Similarly as in Lagodimos et al. (2018), also here a dualism exists between the continuous-review DT-EOQ model and the periodic-review (s, S) model. Specifically, as regards the latter, let us consider the particular case in which the review period is one time unit and the demand in the period can be either 0 or 1. In this condition, the inventory level at the beginning of the review period cannot be less than s and so, the replenishment lot is always the same and equal to $Q = S - s$.

3 Modelling approach

The DT-EOQ model with lost sales is here modelled as a discrete-state discrete-time Markov process whose transition graph is depicted in Figure 2. The system states correspond to the on-hand inventory levels, denoted as $n = 0, 1, ..., Q + r$, representing the number of items available in stock. The maximum inventory level is $(Q + r)$, where r is the reorder point, i.e., the highest level at which a replenishment can take place, and Q is the fixed order size.

Figure 2 Transition graph for the single-echelon inventory system with (r, Q) inventory policy and lost sales (see online version for colours)



The model is formulated according to the assumption that $Q > r + 1$. At the end of Subsection 4.3 it will be shown that this constraint can be relaxed to $Q > r$. This assumption, together with the lost sale behaviour, guarantees that there can not be more than one order in queue, as it is commonly accepted in the literature (Hill and Johansen, 2006). The generalisation of the model to cases in which $Q \leq r$ is not considered a priority at this development stage and is left for future research.

As shown in Figure 2, when the system is in state $(Q + r)$ it can only transit in $(Q + r - 1)$ if a consumption occurs with probability p_2 in the current time unit, or remain in state $(Q + r)$ with probability $(1 - p_2)$. This behaviour holds for every state between $(r + 1)$ and $(Q + r)$ determining a reduction over time of the inventory level towards the reorder point r driven by the stochastic demand. At a certain time unit, the system will enter state (r) indicating that the inventory position has reached the reorder point. According to the EOQ policy, an order of size Q is placed and the corresponding replenishment lot will arrive after a positive lead time. The proposed Markov chain models the lead time as a geometric process with parameter p_1. Consequently, in any state (n) with $n = 0, ..., r$, the supply event related to the order placed at state (r) occurs with a constant probability p_1 and fails to do so with probability $(1 - p_1)$.

At each time unit, the system state gradually decreases toward $n = 0$ based on the consumption probability p_2 until a supply event occurs. In the very time unit in which the supply event occurs, the lot of size Q is received and the on-hand inventory increases by a quantity of Q, if in the same time unit no consumption takes place, or a quantity of $Q-1$ if a consumption takes place. So, for any state (n) with $n=0,1,...,r$, the system transits to either state $(n+Q)$ or state $(n+Q-1)$ with probability $p_1(1-p_2)$ and p_1, p_2, respectively. Note that the system can remain in states (n) with $n=0,1,...,r$ with a probability that is scaled by the factor $(1-p_1)$, being this the probability of not receiving the supply lot in the generic time unit. When the system reaches state $n=0$, it is unable to fulfill any further demand, resulting in lost sales since no backlog is maintained. The system stays in this state with probability $(1-p_1)$ and exits only when a replenishment event takes place, which occurs with probability p_1.

The discrete Markov chain of Figure 2 can be solved numerically by calculating the state probabilities for given parameter values. However, as the state space grows, numerical methods become increasingly impractical. A closed-form solution is therefore crucial for ensuring efficient and rapid computations, particularly when comparing different scenarios. Moreover, such an approach allows for the derivation of compact and simple formulas for key performance indicators commonly used to evaluate supply systems, such as average inventory, cycle time, and service level. Therefore, this study aims to provide a detailed mathematical derivation of the closed-form expressions for state probabilities (detailed in Section 4) and the corresponding performance measures, which are ultimately provided as concise formulas in Section 5.

Table 1 Notation correspondence

Symbol	*Description*
r	Reorder point
Q	Order size
p_1	Supply probability
p_2	Consumption probability
n	System state (inventory position), $n=0,1,...,Q+r$
π_i	Partition probability of partition $[i]$ 'in isolation'
$\mathbf{P}(n)^{[i]}$	Steady-state probability 'in isolation' of state n belonging to the sub-chain related to partition $[i]$ 'in isolation'
$\mathbf{P}(n)$	Steady-state probability of state n
α	$=1+\dfrac{p_1}{(1-p_1)p_2}$, supplementary parameter
β	$=1+\dfrac{p_2}{(1-p_2)p_1}$, supplementary parameter
γ	$=\dfrac{1}{\alpha-1}=\dfrac{p_2(1-p_1)}{p_1}$, supplementary parameter

Table 1 summarises the notation used throughout the paper, while the proposed approach is based on the *partitioning technique* described in the sequel.

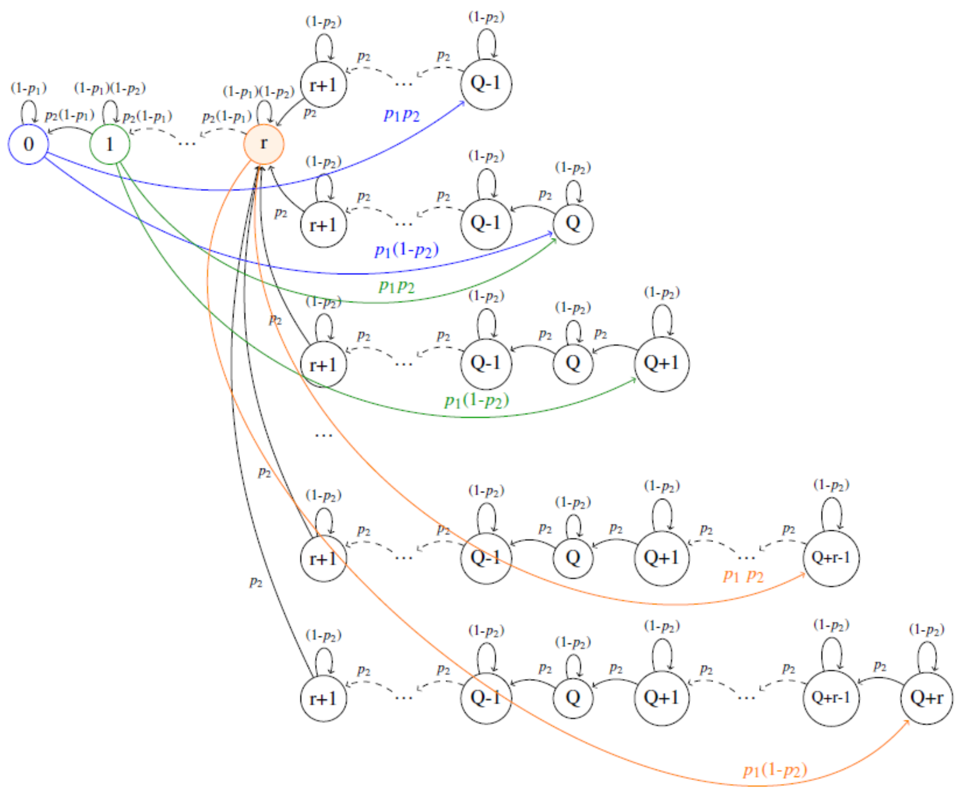
3.1 Partitioning technique

The *partitioning technique* was first introduced in Gebennini et al. (2013, 2017) where it was applied to a Markov model of a two-machine one-buffer production line. Here, the technique is implemented to derive the closed-form solution of the inventory system under analysis described by the transition graph of Figure 2.

The transition graph of Figure 2 can be exploded into the equivalent transition graph depicted in Figure 3. This approach is particularly useful as it enables the recognition of homogeneous behavioural patterns. Specifically, it allows for the identification of distinct state partitions, each corresponding to smaller (and simpler) sub-chains, thereby simplifying the overall mathematical manipulation.

In the equivalent transition graph the inventory level reached after the replenishment [see states (n), with $n = Q - 1, Q, \ldots, Q + r$] is emphasised. Note that the inventory level reached after the replenishment depends on both the inventory position at the beginning of the time unit when a replenishment occurs and the consumption process. For example, state (Q) can be reached either from state (0) if a replenishment takes place and no consumption occurs or from state (1) if both a replenishment and a consumption occur.

Figure 3 Exploded transition graph with $r + 2$ state-partitions (see online version for colours)



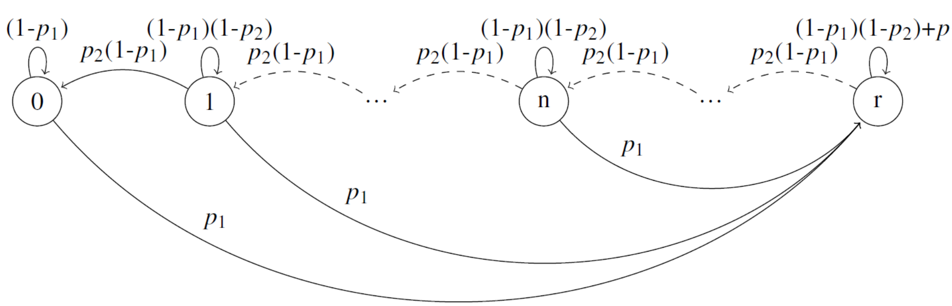
As a consequence, in Figure 3 any state (n), with $n > r$, is exploded in a number of sub-states. As an example, state $(r + 1)$ in the original transition graph (Figure 2) is split into $r + 1$ sub-states. This is because, independently on the inventory level reached after the replenishment, the inventory level must drop down to state $(r + 1)$ before another order can be placed. The same happens for states (n) with $n = r + 1, \ldots, Q - 1$. Conversely, states (n) with $n > Q - 1$ are split into a lower number of sub-states. This is because the system can reach and stay at those states only if the replenishment occurs at specific inventory positions. For example, state (Q) in the original transition graph is split into r (instead of $r+1$) sub-states because it is not possible for the system to enter state (Q) if the replenishment occurs at state (0) and a consumption occurs in the same time unit (leading to a maximum inventory after the replenishment equal to $Q-1$). By applying the same reasoning, state $(Q+1)$ in the original transition graph is split into $r-1$ sub-states, state $(Q+2)$ in the original transition graph is split into $r-2$ sub-states, and so on. The last state $(Q+r)$ is a single state also in the equivalent transition graph since the system enters that state only if the replenishment takes place at the reorder point and no consumption occurs. As mentioned above, the equivalent transition graph of Figure 3 shows some homogeneous kinds of behaviour which correspond to the state partitions.

The first state partition consists of states (n) with $n = 0, 1, ..., r$, which represent the states where a replenishment event can occur. This sub-chain is called 'partition $[r]$' in the sequel. Other $r+1$ state partitions (and the corresponding sub-chains) can be identified on the right side of the graph of Figure 3. These partitions, denoted as 'partitions $[i]$', are indexed by i, where $i = Q-1, ..., Q+r$, indicating the inventory level reached after a replenishment occurs.

Hence, a total of $r+2$ partitions can be identified and it is possible to apply the so-called *partitioning technique* with the aim of facilitating the mathematical treatment of the problem. The fundamental concept is that, in a steady-state system, the probability of entering a partition must be equal to the probability of leaving it. Since these partitions are mutually exclusive, each one can be treated separately, allowing it to be 'isolated' and analysed independently. As a result, the corresponding (simpler) sub-chain can be solved as if it were independent of the others. This approach enables the computation of the state probabilities 'in isolation'. We denote as $\mathbf{P}(n)^{[i]}$ the probability 'in isolation' of state (n) belonging to the sub-chain related to partitions $[i]$ 'in isolation'.

Specifically, partition $[r]$ 'in isolation' can be depicted by the transition graph in Figure 4. In this representation, the probability of leaving the partition, given by the probability of being in any state (n) (where $n = 0, 1, ..., r$) and a replenishment occurs, is equal to the probability of entering the partition, which happens exclusively at state (r).

Figure 4 Partition-$[r]$ 'in isolation'



As shown in Figure 4, the sub-chain corresponding to partition $[r]$ 'in isolation' is simple and can be described by the following transition equations:

$$\mathbf{P}(0)^{[r]} = (1-p_1)\,\mathbf{P}(0)^{[r]} + (1-p_1)\,p_2\mathbf{P}(1)^{[r]} \tag{1}$$

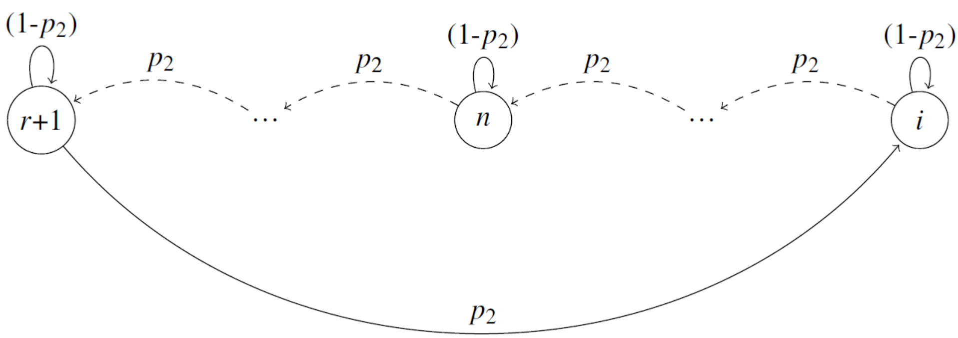
$$\mathbf{P}(n)^{[r]} = (1-p_1)(1-p_2)\mathbf{P}(n)^{[r]} + (1-p_1)p_2\mathbf{P}(n+1)^{[r]} \quad \forall n = 1, ..., r-1 \tag{2}$$

$$\mathbf{P}(r)^{[r]} = (1-p_2+p_1p_2)\mathbf{P}(r)^{[r]} + \sum_{n=0}^{r-1} p_1\mathbf{P}(n)^{[r]} \tag{3}$$

$$\sum_{n=0}^{r} \mathbf{P}(n)^{[r]} = 1 \tag{4}$$

Partitions $[i]$ with $i = Q-1, ..., Q+r$, depicted on the right side of the transition graph of Figure 3, show the same (simple) schema in terms of transition probabilities when taken 'in isolation'. The generic partition $[i]$ 'in isolation' is represented in the transition graph of Figure 5, where the index i indicates the inventory level reached immediately after replenishment, corresponding to the highest inventory level within that partition.

Figure 5 Partition-$[i]$ 'in isolation', with $i = Q-1, ..., Q+r$



The transition equations which describe the generic partitions $[i]$ 'in isolation' are as follows:

$$\mathbf{P}(n)^{[i]} = (1-p_2)\mathbf{P}(n)^{[i]} + p_2\mathbf{P}(n+1)^{[i]} \quad \forall n = r+1, ..., i-1 \quad \forall i = Q-1, ..., Q+r \tag{5}$$

$$\sum_{n=r+1}^{i} \mathbf{P}(n)^{[i]} = 1 \quad \forall i = Q-1, ..., Q+r \tag{6}$$

Given that the $r+2$ partitions of the original model can be analysed 'in isolation', the next step is to determine the probability for the system to be in each partition in a time unit. This requires calculating the so-called *partition probabilities*: $\Pi = \{\pi_r, \pi_{Q-1}, \pi_Q ..., \pi_{Q+r}\}$.

First, let us consider partition $[r]$. By introducing the partition probability $\pi_{[r]}$, the steady-state probability of any state (n) (where $n = 0, ..., r$) in the original system, denoted as $\mathbf{P}(n)$, can be expressed as the product of $\pi_{[r]}$ and the probability 'in isolation' of that state $\left(\mathbf{P}(n)^{[r]}\right)$. More formally,

$$\mathbf{P}(n) = \pi_{[r]}\mathbf{P}(n)^{[r]}, \quad n = 0, ..., r. \tag{7}$$

Similarly, by introducing the partition probability $\pi_{[i]}$, with $i = Q-1, ..., Q+r$, which represents the probability for the original system to be in partition $[i]$, the steady-state probability of any state (n) with $n = r+1, ..., Q+r$ can be expressed as:

$$\mathbf{P}(n) = \begin{cases} \sum_{i=Q-1}^{Q+r} \pi_{[i]} \mathbf{P}(n)^{[i]}, & n = r+1, ..., Q-1 \\ \sum_{i=n}^{Q+r} \pi_{[i]} \mathbf{P}(n)^{[i]}, & n = Q, ..., Q+r \end{cases} \tag{8}$$

The interpretation of equation (8) is as follows:

- States (n) with $n = r+1, ..., Q-1$ are included in all partitions $[i]$ with $i = Q-1, ..., Q+r$.
- States (n) with $n = Q, ..., Q+r$ are not present in all partitions $[i]$ but only in a subset of them. Specifically, they belong only to partitions $[i]$ with $i = n, ..., Q+r$. For instance, state (Q) belongs to partitions $[i]$ with $i = Q, ..., Q+r$ but it does not belong to partition $[Q-1]$, which represents the partition entered when the inventory level is at zero and both a consumption event and a replenishment take place in the same time unit.

To sum up, the application of the *partitioning technique* to the transition graph of Figure 3 is based on the following steps:

1. Solution of partition $[r]$ 'in isolation'.
2. Solution of partition $[i]$, with $i = Q-1, ..., Q+r$, 'in isolation'.
3. Determination of the partition probabilities $\Pi = \{\pi_{[r]}, \pi_{[Q-1]}, \pi_{[Q]} ..., \pi_{[Q+r]}\}$.
4. Determination of the state probabilities for the original system at the steady-state.

As described in the subsequent section, the application of the aforementioned steps leads to simple and compact closed-form expressions of the state probabilities for the Markov process representing the DT-EOQ model with lost sales.

4 Closed-form solution for state probabilities

The objective of this section is to provide the reader with simple closed-form expressions of the state probabilities for the model described in Section 3. These relationships are formalised in equation (20), where the probability $\mathbf{P}(n)$ of any state (n) (for $n = 0, 1, ..., Q+r$) can be directly expressed as a function of the system parameters p_1, p_2, r, and Q. The approach that has led to the closed-form solution is based on the steps of the *partitioning technique* described in Section 3.1 and applied in the sequel.

4.1 Solution of partition [r] in isolation

Partition $[r]$ comprises states (n) with $n = 0, 1, ..., r$. When taken 'in isolation' it is represented according to the transition graph depicted in Figure 4 and the transition equations (1)–(4) where the notation $\mathbf{P}(n)^{[r]}$ denotes the probability 'in isolation' for state (n) belonging to partition $[r]$.

From equation (1) we have:

$$\mathbf{P}(1)^{[r]} = \frac{p_1}{(1-p_1)p_2}\mathbf{P}(0)^{[r]}.$$

From equation (2) we have:

$$\mathbf{P}(n+1)^{[r]} = \frac{p_1 + (1-p_1)p_2}{(1-p_1)p_2}\mathbf{P}(n)^{[r]} \quad \forall n = 1, ..., r-1,$$

or, in terms of $\mathbf{P}(0)^{[r]}$ only,

$$\mathbf{P}(n)^{[r]} = \frac{\big(p_1 + (1-p_1)p_2\big)^{(n-1)} p_1}{(1-p_1)^n p_2^n}\mathbf{P}(0)^{[r]} \qquad \forall n = 1, ..., r. \tag{9}$$

It may be convenient at this stage to introduce the parameter α, depending on p_1 and p_2 according to the following relationship:

$$\alpha = 1 + \frac{p_1}{(1-p_1)p_2}. \tag{10}$$

Following the mathematical steps outlined in Appendix A1, the solution of partition $[r]$ 'in isolation' is as follows:

$$\mathbf{P}(n)^{[r]} = \begin{cases} \dfrac{1}{\alpha^r} & n = 0 \\ \dfrac{\alpha - 1}{\alpha}\alpha^{n-r} & n = 1, ..., r \end{cases} \tag{11}$$

4.2 Solution of partitions [i] with i = Q – 1, ..., Q + r in isolation

Partitions $[i]$ with $i = Q-1, ..., Q+r$, depicted on the right side of the transition graph of Figure 3, show the same (simple) schema in terms of transition probabilities when taken 'in isolation'. The reader may refer to the transition graph in Figure 5, which show the generic partitions $[i]$ in isolation, and to the corresponding transition equations (5)–(6). We recall here that the index i represents the inventory level attained immediately after replenishment, which corresponds to the highest state within each partition.

For the generic partitions $[i]$ 'in isolation', from equation (5) we have:

$$\mathbf{P}\,(n)^{[i]} = \mathbf{P}\,(n+1)^{[i]} \quad \forall n = r+1, ..., i-1.$$

In words, all the states belonging to the same partition $[i]$ 'in isolation' have the same probability. By considering the normalisation equation (6), the state probabilities 'in isolation' basically depend on the number of states constituting the partition $[i]$ under analysis. Specifically, the solution for the generic partition $[i]$ 'in isolation' is as follows:

$$\mathbf{P}\,(n)^{[i]} = \frac{1}{i-r} \quad \forall n = r+1, ..., i \qquad \forall i = Q-1, ..., Q+r \tag{12}$$

where the term $(i-r)$ represents the number of states constituting partition $[i]$ (i.e., states (n) with $n = r+1, r+2, ..., i$).

4.3 Determination of the partition probabilities

In order to find the expressions for the partition probabilities it is necessary to analyse the transitions which occur between the different partitions of the original system as represented in the transition graph of Figure 3. It can be noted that partition $[Q-1]$ can be entered only from state (0) if a consumption occurs during the same time unit when the replenishment quantity arrives. In addition, no other partition can be entered from state (0) if a consumption occurs during the replenishment. This means that, at the steady state, the probability of leaving partition $[r]$ from state (0) given that a consumption has occurred during the replenishment must equal the probability of entering partition $[Q-1]$. Recall also that the probability of entering partition $[Q-1]$ must equal the probability of leaving that partition. This is because the partition can be entered from one state only, i.e., state $(Q-1)$, and left from one state only, i.e., state $(r+1)$. Hence, the probability of leaving partition $[r]$ from state (0) given that a consumption has occurred during the replenishment (given by $\pi_{[r]}\mathbf{P}(0)^{[r]}p_1p_2$) must equal the probability of leaving partition $[Q-1]$ which occurs if the system is in state $(r+1)$ of that partition and a consumption occurs (i.e., $\pi_{[Q-1]}\mathbf{P}\,(r+1)^{[Q-1]}p_2$). Formally,

$$\pi_{[r]}\mathbf{P}(0)^{[r]}p_1p_2 = \pi_{[Q-1]}\mathbf{P}\,(r+1)^{[Q-1]}p_2. \tag{13}$$

On the other extreme, partition $[Q+r]$ can be entered only from state (r) if no consumption occurs during the replenishment, and that partition is the only one that can be entered from that state under those conditions. Hence,

$$\pi_{[r]}\mathbf{P}(r)^{[r]}p_1(1-p_2) = \pi_{[Q+r]}\mathbf{P}\,(r+1)^{[Q+r]}p_2. \tag{14}$$

Any other partition $[i]$ with $i = Q, ..., Q+r-1$ can be reached:

- from state $(i-Q)$, belonging to partition $[r]$ by definition, if no consumption occurs during the replenishment
- from state $(i-Q+1)$, still belonging to partition $[r]$, if the consumption occurs during the replenishment.

So,

$$\begin{aligned}&\pi_{[r]}\mathbf{P}(i-Q)^{[r]}p_1(1-p_2) + \pi_{[r]}\mathbf{P}(i-Q+1)^{[r]}p_1p_2\\&= \pi_{[i]}\mathbf{P}\,(r+1)^{[i]}p_2, \quad i = Q, ..., Q+r-1.\end{aligned} \tag{15}$$

Finally, since the system can be in one and only one partition at any time unit, the sum of the partition probabilities must equal 1. So,

$$\pi_{[r]} + \sum_{i=Q-1}^{Q+r} \pi_{[i]} = 1. \tag{16}$$

The problem is then reduced to a set of $r+3$ linear equations, of which $r+2$ are independent linear equations, in $r+2$ unknowns.

The proposed solution, after some mathematical manipulations which are reported in Appendix A2, is as follows:

$$\pi_{[i]} = \begin{cases} \dfrac{\alpha^r}{1-p_1+Q\frac{p_1}{p_2}\alpha^r}, & i=r \\ \dfrac{p_1(Q-1-r)}{1-p_1+Q\frac{p_1}{p_2}\alpha^r}, & i=Q-1 \\ \dfrac{(Q-r)(\alpha-p_1-1)}{1-p_1+Q\frac{p_1}{p_2}\alpha^r}, & i=Q \\ \dfrac{(\alpha-1)^2}{\alpha^{Q+1}\left(1-p_1+Q\frac{p_1}{p_2}\alpha^r\right)}(i-r)\alpha^i, & i=Q+1,...,Q+r-1 \\ \dfrac{Q}{\beta}\dfrac{p_1}{p_2}\dfrac{\alpha^r}{1-p_1+Q\frac{p_1}{p_2}\alpha^r}, & i=Q+r, \end{cases} \tag{17}$$

where α is from equation (10), and

$$\beta = 1 + \frac{p_2}{(1-p_2)p_1}. \tag{18}$$

It can be noticed that when $Q = r+1$ the probability of partition $[Q-1]$ becomes zero, meaning that the sufficient condition to assure that there is not more than one order in queue is

$$Q > r. \tag{19}$$

Moreover, looking at equations (11) and (12), it is clear that the model can also work in the case $r = 0$. Even the extreme situation $(r, Q) = (0, 1)$ can be represented, since in this case $Q = r+1$ and then the probability of partition $[Q-1]$ is zero as previously stated [this is required for eliminating the case $i = Q-1 = 0$ in equation (12)].

4.4 Determination of the state probabilities

The closed-form expression for the steady-state probability of each state n, where $n = 0, 1, ..., Q+r$, in the original system is given by $\mathbf{P}(n)$ and is formulated as follows:

$$\mathbf{P}(n) = \begin{cases} \dfrac{p_2}{p_1(\gamma+Q\alpha^r)}, & n=0 \\ \dfrac{p_2}{p_1(\gamma+1)(\gamma+Q\alpha^r)}\alpha^n, & n=1,...,r \\ \dfrac{\alpha^r}{\gamma+Q\alpha^r}, & n=r+1,...,Q-1 \\ \dfrac{\alpha^r-p_2}{\gamma+Q\alpha^r}, & n=Q \\ \dfrac{\alpha^r}{\gamma+Q\alpha^r} - \dfrac{p_2}{p_1(\gamma+1)(\gamma+Q\alpha^r)}\alpha^{n-Q}, & n=Q+1,...,Q+r \end{cases} \tag{20}$$

where α is from equation (10), and

$$\gamma = \frac{1}{\alpha-1} = \frac{p_2(1-p_1)}{p_1}. \tag{21}$$

The mathematical derivation of equation (20) is reported in Appendix A3.

Here, it can also be further verified that the model can handle the case $(r, Q) = (0, 1)$. Considering, in equation (20), the terms for the only admissible states (n) with $n = 0$ and $n = Q$ (this latter representing the state $n = 1$ in this specific case), it can be observed that

$$\mathbf{P}(0) = \frac{p_2}{p_1(\gamma + 1)}, \ \mathbf{P}(1) = \frac{1 - p_2}{\gamma + 1} = 1 - \mathbf{P}(0). \tag{22}$$

Equation (20) represents a relevant result since it makes it possible to compute the state probabilities for any values of the system parameters p_1, p_2, r, and Q without the need of explicit state-space enumeration. This is a useful result by itself, allowing to quickly find the probability distribution without computation efforts and delays. In addition, the closed-form solution of equation (20) allows us to obtain exact formulas for a number of fundamental performance measures, as described in Section 5.

5 Performance measures

This section reports the closed-form expressions of the most relevant performance measures, whose computation is based on the state probabilities at the steady-state [see equation (20)]. The derivation of the compacted formulas is left to the reader since a simplification approach similar to the one seen in Appendix A2 is adopted.

5.1 Length of the inventory cycle

The length of the inventory cycle is the average time between two consecutive order arrivals. This is equal to the reciprocal of the probability that an order Q arrives, i.e., the reciprocal of the probability that the inventory position is equal to or less than the reorder point r and a supply occurs. Formally,

$$C = \frac{1}{p_1 \sum_{i=0}^{r} \mathbf{P}(i)} = \frac{Q}{p_2} + \frac{\gamma}{p_2 \alpha^r} \text{ [time units].} \tag{23}$$

It can be noted that the length of the inventory cycle is composed of two terms: the first one $\left(\frac{Q}{p_2}\right)$ represents the average time interval related to the working stock, while the second term is the additional contribution generated by the average stock-out which, in the lost sale case, translates in an unsold quantity that determines an additional stock availability. At this stage, it can already be guessed that the average amount of stock-out per cycle is actually $\frac{\gamma}{\alpha^r}$.

5.2 Stock-out probability per time unit

The stock-out probability per time unit is the probability of not satisfying a demand during a generic time unit. This occurs when the inventory level is zero, a consumption occurs but no replenishment takes place. Hence,

$$P_{SO} = p_2(1 - p_1)\mathbf{P}(0) = \frac{p_2 \gamma}{\gamma + Q\alpha^r}. \tag{24}$$

5.3 Average amount of stock-out per cycle

This performance measure represents the expected demand that cannot be satisfied during a generic inventory cycle. Formally, this is equal to the product of the stockout probability per time unit and the lenght of the inventory cycle in time units. So,

$$SO_C = CP_{SO} = \frac{\gamma}{\alpha^r} \text{ [items]}. \tag{25}$$

As expected, this measure does not depend on the order size Q, but only on the reorder point r and the probabilistic parameters p_1 and p_2 according to the expressions of α and γ. The order size Q, affecting the length of the inventory cycle and then the average number of inventory cycles per years, impacts the expected stock-out per year.

5.4 Service level per demanded unit (fill rate)

The fill rate represents the fraction of demand that is satisfied in an inventory cycle with the available stock that, in the lost-sale case, must also account for the average stock-out per cycle.

$$SL_U = 1 - \frac{SO_C}{Q + SO_C} = 1 - \frac{\gamma}{\gamma + Q\alpha^r} = \frac{Q\alpha^r}{\gamma + Q\alpha^r}. \tag{26}$$

5.5 Average inventory level

By definition, the average inventory level is the mean of the random variable representing the inventory level. So,

$$\bar{n} = \sum_{i=0}^{Q+r} i\mathbf{P}(i) = Q - \left(\frac{Q-1}{2} - r + \frac{p_2}{p_1}\right)\frac{Q\alpha^r}{\gamma + Q\alpha^r} \text{ [items]}. \tag{27}$$

Let us now introduce M as the random variable representing the lead time demand. It is clear that, from the model assumptions, the mean value of M is

$$\bar{M} = \frac{p_2}{p_1}. \tag{28}$$

With some mathematical steps, equation (27) can be rewritten as

$$\bar{n} = \frac{Q}{2} + \left[r - \bar{M} + \frac{1 + SO_C}{2}\right] SL_U, \tag{29}$$

where SO_C and SL_U are from equations (25) and (26), respectively.

Equation (29), being expressed in terms of steady state average performance measures, can be accounted as a generalised estimator of the average inventory level for the lost sales case. Notice that it is different from the one commonly adopted by the scientific community, that is

$$\bar{n}^c = \frac{Q}{2} + r - \bar{M} + SO_C. \tag{30}$$

Hence, equation (30) cannot be considered an exact estimator for the average inventory level in the case of lost sales, and further studies are needed to prove if equation (29) can effectively be assumed as the exact generalised one. Remaining within the scope of this paper, the reader may refer to Subsection 6.2 for a numerical comparison between equations (29) and (30).

5.6 Average inventory level at the beginning of the cycle

In order to compute this measure, it is necessary to observe the average inventory level at the occurrence of a generic supply event (i.e., the first time unit at the beginning of a cycle) and re-scale it to the length C of the cycle. Hence,

$$\begin{aligned}\bar{n}_0 &= C\left(\sum_{i=0}^{r}(i+Q-1)\,p_1p_2\mathbf{P}(i) + \sum_{i=0}^{r}(i+Q)\,p_1(1-p_2)\mathbf{P}(i)\right)\\ &= \frac{\gamma}{\alpha^r} + Q + r - (p_2+\gamma) \qquad (31)\end{aligned}$$

Since

$$p_2 + \gamma = \frac{p_2}{p_1} = \bar{M},$$

and considering equation (25), equation (31) can be rewritten as

$$\bar{n}_0 = Q + r - \bar{M} + SO_C, \qquad (32)$$

resulting in the classical formulae commonly adopted by the scientific community to compute the average inventory level at the beginning of the cycle in the lost sales case.

5.7 Total cost

Let us consider the following parameters:

- u: the unit purchase cost [$/item]
- a: the cost per order [$/order]
- h: the cost of holding a unit in stock for an entire year [$/unit/year]
- s: the unit stock-out cost [$/item], which corresponds to the loss of money when a lost sale occurs
- N: number of time units per day
- w: working days per year.

The derivation of the purchase cost (PC), holding cost (HC), ordering cost (OC) and stock-out cost (SC) is:

$$PC = uQ\frac{Nw}{C}\ \text{[\$/year]}, \qquad (33)$$

$$OC = a\frac{Nw}{C} \text{ [\$/year]}, \tag{34}$$

$$HC = h\bar{n} \text{ [\$/year]}, \tag{35}$$

$$SC = sSO_C\frac{Nw}{C} \text{ [\$/year]}, \tag{36}$$

where C is the lenght of the inventory cycle expressed in number of time units according to equation (23), $\bar{n}$ is the average inventory according to equation (27), and $\bar{SO}_C$ is the average stock-out per cycle according to equation (25). Hence, the total annual cost TC results as follows:

$$TC(r, Q) = PC + OC + HC + SC \text{ [\$/year]}. \tag{37}$$

The optimal inventory control policy is defined by the values $(r^{\text{opt}}, Q^{\text{opt}})$ which minimise the total annual cost TC, as given by equation (37). It can be noted that the equation of TC is transcendent in r, thus it is not possible to invert it to directly find the values $(r^{\text{opt}}, Q^{\text{opt}})$. However, a numerical search procedure can be carried out in order to find the optimal values $(r^{\text{opt}}, Q^{\text{opt}})$ similarly as in Mohebbi and Posner (1998).

It is worth noting that the cost structure adopted in this study follows a classical EOQ-based modelling approach, particularly suited to single-echelon inventory systems under stochastic demand and lost sales conditions. While broader cost formulations, such as those proposed in Ivanov and Rozhkov (2020), include additional components like transportation, manufacturing, and write-off costs, these are typically associated with multi-echelon or production-integrated models. In our setting, such factors are either constant, external to the inventory decision process, or subsumed within the unit purchase cost. Nonetheless, the proposed model remains flexible and could be extended to accommodate a more comprehensive cost structure in future research.

6 Application insights

In this section, some insights about how to derive the probabilistic parameters as a function of the daily demand signal and the average lead time are presented and clarified with some numerical examples.

6.1 Probabilistic parameters definition

The proposed model, as it is conceived, makes it possible to represent a daily demand of binomial form, with certain limitations, by opportunely tuning the time unit size and the probability p_2. The lead time has a geometric distribution with parameter p_1.

Hence, the time unit size (or, better, the number of time units per day), as well as p_1 and p_2, can be computed with respect to the actual problem to be represented. To do this, let us consider the scheme reported in Figure 6, where the number of time units per day is represented by the variable N.

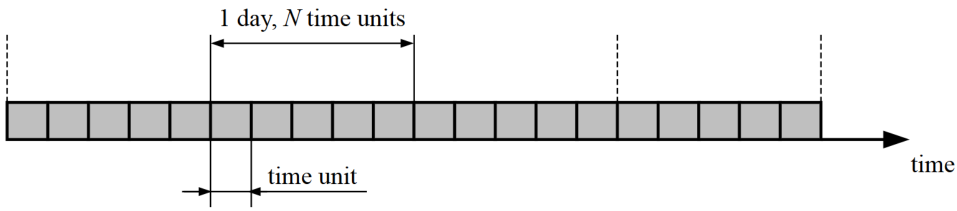
Suppose now that we want to model a daily demand with mean $\bar{D}$ and variance σ_D^2. Since the proposed model draws a single item (i.e. unit of demand) with probability p_2

in each time unit, and there are N time units per day, the daily demand is modelled as a binomial distribution with

$$\bar{D} = Np_2, \tag{38}$$

$$\sigma_D^2 = Np_2(1 - p_2). \tag{39}$$

Figure 6 Example of time line discretisation with 1 day composed of $N = 5$ time units



By means of equation (39), the number of time units per day N and the probability p_2 to see a unit of demand in a time unit can be uniquely computed given the wanted daily demand signal with mean $\bar{D}$ and variance σ_D^2, settling for the assumption that the daily demand can be conveniently assumed to be of binomial form. Hence,

$$p_2 = 1 - \frac{\sigma_D^2}{\bar{D}}, \tag{40}$$

$$N = \frac{\bar{D}^2}{\bar{D} - \sigma_D^2}. \tag{41}$$

Note that $N \in \mathbb{R}^+$ and can be less than one in the case a very rare daily demand needs to be represented. In this case, there will be more days covered by the single time unit.

Being $0 < p_2 < 1$, the following conditions must be verified:

$$\begin{cases} 1 - \frac{\sigma_D^2}{\bar{D}} > 0 \\ 1 - \frac{\sigma_D^2}{\bar{D}} < 1 \end{cases} \Rightarrow \begin{cases} \sigma_D^2 < \bar{D} \\ \frac{\sigma_D^2}{\bar{D}} > 0 \end{cases} \tag{42}$$

While the second condition in equation (42) is implicitly satisfied by the fact that the demand is positive, the first condition actually imposes a boundary on the variability of the daily demand that the model can handle. Specifically:

$$\bar{D}CV_D^2 < 1, \tag{43}$$

where $CV_D^2 = \frac{\sigma_D^2}{\bar{D}^2}$ is the squared coefficient of variation of the daily demand.

As can be deduced from Table 2 reporting the maximum mean value of the demand that can be modelled with respect to the squared coefficient of variation as expressed by equation (43), the proposed model well suits cases in which the demand is intermittent and even rare (here one time unit can cover more days). Cases of greater average daily demand can be modelled provided that the degree of variability becomes progressively low, meaning that the proposed model cannot handle daily demand signals characterised by both high average value and high variance. This does not appear to be excessively limiting in modern market scenarios where cases of high variation in daily demand mainly refer to lumpy phenomena, that are mathematically addressed in different way (see, e.g., Janssen et al., 1998) and are out of the scope of the proposed model.

Table 2 Upper bound of the daily demand mean value as a function of CV_D^2

	Gaussian like	*Smooth*	*Intermittent*	*Rare*
CV_D^2	... $\longleftrightarrow$ 0.1	0.2 $\longleftrightarrow$ 0.9	1 $\longleftrightarrow$ 40	50 $\longleftrightarrow$ ...
$\bar{D}_{\max}$ [items/day]	... $\longleftrightarrow$ 10	5.0 $\longleftrightarrow$ 1.1	1 $\longleftrightarrow$ 0.025	0.020 $\longleftrightarrow$ ...

Cases in which a vendor serves a number z of buyers with a high responsive service can be addressed as well. This situation is common, e.g., in the Ho.Re.Ca. sector when shipments are carried out on a daily basis or in some other highly responsive supply chains. Let us consider the case in which all the buyers have the same daily demand, represented by a random variable d so that:

$$\begin{cases} \bar{d}_i = \bar{d}_j = \bar{d} \\ \sigma_{d_i}^2 = \sigma_{d_j}^2 = \sigma_d^2 \\ CV_{d_i}^2 = CV_{d_j}^2 = CV_d^2 = \frac{\sigma_d^2}{\bar{d}^2} \end{cases} \qquad \forall i, j = 1, ..., z, i \neq j. \tag{44}$$

and that the demands are independent of each other. The vendor will then see a demand represented by a random variable D so that, from the central limit theorem:

$$\bar{D} = \sum_{i=1}^{z} \bar{d} = z\bar{d}, \tag{45}$$

$$\sigma_D^2 = \sum_{i=1}^{z} \sigma_d^2 = z\sigma_d^2. \tag{46}$$

and then

$$CV_D^2 = \frac{z\sigma_d^2}{z^2\bar{d}^2} = \frac{1}{z} CV_d^2. \tag{47}$$

By imposing the condition of equation (43) we have

$$\bar{D} CV_D^2 = z\bar{d}\frac{1}{z} CV_d^2 = \bar{d} CV_d^2 < 1, \tag{48}$$

meaning that such condition is satisfied as long as the demand of the individual buyer satisfies it. Thus, the applicability of the model is independent of the number of buyers. So, the presented model can be used for any aggregation of identical (similar) and independent buyers whose individual daily demand satisfies equation (43) provided that the resulting aggregated daily demand can be reasonably assumed binomial, or at least that the binomial form is not discarded. When buyers daily demands are different, i.e., there are some dominant buyers, equation (43) must be specifically verified.

Defining $\bar{L}$ as the average order lead time expressed in days, the parameter p_1 can be derived as follows:

$$p_1 = \frac{1}{N\bar{L}} \tag{49}$$

Table 3 Application examples with different daily demand form

Rare daily demand ($\bar{D}$ = 0.05 [items/day], $\bar{L}$ = 10 [days], N = 1, p_1 = 0.1, p_2 = 0.05)

$r \setminus Q$	$\bar{n}$ [items]										$\bar{C}$ [days]									
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
0	0.655	1.184	1.696	2.202	2.706	3.209	3.711	4.213	4.714	5.215	29.00	49.00	69.00	89.00	109.0	129.0	149.0	169.0	189.0	209.0
1		2.000	2.522	3.034	3.541	4.045	4.549	5.051	5.553	6.055		42.79	62.79	82.79	102.8	122.8	142.8	162.8	182.8	202.8
2			3.493	4.000	4.504	5.007	5.509	6.011	6.512	7.013			60.87	80.87	100.9	120.9	140.9	160.9	180.9	200.9
3				4.997	5.499	6.000	6.501	7.002	7.502	8.003				80.27	100.3	120.3	140.3	160.3	180.3	200.3
4					6.499	6.999	7.500	8.000	8.500	9.000					100.1	120.1	140.1	160.1	180.1	200.1
5						8.000	8.500	9.000	9.500	10.00						120.0	140.0	160.0	180.0	200.0
6							9.500	10.00	10.50	11.00							140.0	160.0	180.0	200.0
7								11.00	11.50	12.00								160.0	180.0	200.0
8									12.50	13.00									180.0	200.0
9										14.00										200.0
	SO_C [items]										SL_U									
0	0.450000000										0.690	0.816	0.870	0.899	0.917	0.930	0.940	0.947	0.952	0.957
1	0.139655172											0.935	0.956	0.966	0.973	0.977	0.980	0.983	0.985	0.986
2	0.043341260												0.986	0.989	0.9^{214}	0.9^{228}	0.9^{239}	0.9^{246}	0.9^{252}	0.9^{257}
3	0.013450736													0.9^{267}	0.9^{273}	0.9^{278}	0.9^{282}	0.9^{283}	0.9^{285}	0.9^{287}
4	0.004174366														0.9^{316}	0.9^{331}	0.9^{340}	0.9^{349}	0.9^{354}	0.9^{358}
5	0.001295493															0.9^{378}	0.9^{382}	0.9^{384}	0.9^{386}	0.9^{387}
6	0.000402050																0.9^{443}	0.9^{450}	0.9^{455}	0.9^{460}
7	0.000124774																	0.9^{484}	0.9^{486}	0.9^{488}
8	0.000038723																		0.9^{557}	0.9^{561}
9	0.000012017																			0.9^{588}

Table 3 Application examples with different daily demand form (continued)

Intermittent daily demand ($\bar{D}$ = 0.4 [items/day], $\bar{L}$ = 10 [days], N = 1, p_1 = 0.1, p_2 = 0.4)																				
$r \setminus Q$	$\bar{n}$ [items]										$\bar{C}$ [days]									
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
0	0.130	0.393	0.727	1.105	1.512	1.938	2.377	2.828	3.286	3.750	11.50	14.00	16.50	19.00	21.50	24.00	26.50	29.00	31.50	34.00
1		0.547	0.937	1.360	1.802	2.257	2.722	3.193	3.669	4.149		12.04	14.54	17.04	19.54	22.04	24.54	27.04	29.54	32.04
2			1.271	1.744	2.224	2.709	3.198	3.688	4.181	4.674			13.01	15.51	18.01	20.51	23.01	25.51	28.01	30.51
3				2.253	2.770	3.282	3.791	4.298	4.804	5.309				14.31	16.81	19.31	21.81	24.31	26.81	29.31
4					3.425	3.959	4.485	5.005	5.522	6.035					15.88	18.38	20.88	23.38	25.88	28.38
5						4.725	5.262	5.792	6.315	6.835						17.64	20.14	22.64	25.14	27.64
6							6.106	6.641	7.168	7.691							19.57	22.07	24.57	27.07
7								7.537	8.067	8.591								21.62	24.12	26.62
8									9.000	9.524									23.77	26.27
9										10.48										25.99
	SO_C [items]										SL_U									
0						3.6000					0.217	0.357	0.455	0.526	0.581	0.625	0.660	0.690	0.714	0.735
1						2.8174						0.415	0.516	0.587	0.640	0.680	0.713	0.740	0.762	0.780
2						2.2049							0.576	0.645	0.694	0.731	0.760	0.784	0.803	0.819
3						1.7256								0.699	0.743	0.777	0.802	0.823	0.839	0.853
4						1.3505									0.787	0.816	0.838	0.856	0.870	0.881
5						1.0569										0.850	0.869	0.883	0.895	0.904
6						0.8271											0.894	0.906	0.916	0.924
7						0.6473												0.925	0.933	0.939
8						0.5066													0.947	0.952
9						0.3965														0.962

Table 3 Application examples with different daily demand form (continued)

Smooth daily demand ($\bar{D}$ = 6 *[items/day]*, $\bar{L}$ = 10 *[days]*, N = 10, p_1 = 0.01, p_2 = 0.6)																				
$r \setminus Q$	$\bar{n}$ *[items]*										$\bar{C}$ *[days]*									
	60	*70*	*80*	*90*	*100*	*110*	*120*	*130*	*140*	*150*	*60*	*70*	*80*	*90*	*100*	*110*	*120*	*130*	*140*	*150*
50	32.37	37.48	42.56	47.63	52.69	57.75	62.79	67.83	72.86	77.89	14.30	15.96	17.63	19.30	20.96	22.63	24.30	25.96	27.63	29.30
60		43.70	48.96	54.18	59.36	64.52	69.65	74.77	79.87	84.96		15.30	16.97	18.64	20.30	21.97	23.64	25.30	26.97	28.64
70			56.03	61.37	66.66	71.90	77.10	82.28	87.43	92.57			16.41	18.08	19.74	21.41	23.08	24.74	26.41	28.08
80				69.12	74.49	79.79	85.05	90.27	95.47	100.6				17.60	19.27	20.94	22.60	24.27	25.94	27.60
90					82.78	88.13	93.43	98.68	103.9	109.1					18.87	20.54	22.20	23.87	25.54	27.20
100						96.84	102.2	107.4	112.7	117.9						20.20	21.86	23.53	25.20	26.86
110							111.2	116.5	121.7	127.0							21.58	23.24	24.91	26.58
120								125.8	131.0	136.2								23.00	24.67	26.34
130									140.5	145.7									24.46	26.13
140										155.3										25.96
	SO_C *[items]*										SL_U									
50	25.78										0.699	0.731	0.756	0.777	0.795	0.810	0.823	0.835	0.844	0.853
60	21.82											0.762	0.786	0.805	0.821	0.835	0.846	0.856	0.865	0.873
70	18.46												0.813	0.830	0.844	0.856	0.867	0.876	0.883	0.890
80	15.62													0.852	0.865	0.876	0.885	0.893	0.900	0.906
90	13.22														0.883	0.893	0.901	0.908	0.914	0.919
100	11.19															0.908	0.915	0.921	0.926	0.931
110	9.467																0.927	0.932	0.937	0.941
120	8.012																	0.942	0.946	0.949
130	6.780																		0.954	0.957
140	5.737																			0.963

Table 4 Comparison between analytical and simulated scenarios

p_1	p_2	r	Q	*Analytical*				*Simulated*				Δ%			
				$\bar{n}$ *[items]*	C *[time units]*	SO_C *[items]*	$\bar{n}_0$ *[items]*	$\bar{n}$ *[items]*	C *[time units]*	SO_C *[items]*	$\bar{n}_0$ *[items]*	$\bar{n}$	C	SO_C	$\bar{n}_0$
0.05	0.2	0	1	0.1667	24.0000	3.8000	0.8000	0.1667	23.9996	3.7998	0.8000	–0.0082%	–0.0017%	–0.0044%	0.0016%
			6	2.0204	49.0000	3.8000	5.8000	2.0204	48.9973	3.7990	5.7999	0.0009%	–0.0055%	–0.0256%	–0.0021%
			16	6.7071	99.0000	3.8000	15.8000	6.7071	98.9830	3.7985	15.7997	0.0001%	–0.0172%	–0.0403%	–0.0018%
		5	6	4.7468	35.9083	1.1817	8.1817	4.7480	35.9088	1.1805	8.1823	0.0250%	0.0012%	–0.0992%	0.0073%
			16	9.9470	85.9083	1.1817	18.1817	9.9464	85.9089	1.1813	18.1811	–0.0062%	0.0006%	–0.0286%	–0.0030%
		15	16	19.4752	80.5713	0.1143	27.1143	19.4757	80.5677	0.1142	27.1147	0.0029%	–0.0045%	–0.0959%	0.0017%
	0.4	0	1	0.0698	21.5000	7.6000	0.6000	0.0698	21.4988	7.5992	0.6001	0.0067%	–0.0054%	–0.0110%	0.0127%
			6	1.3676	34.0000	7.6000	5.6000	1.3679	33.9958	7.5975	5.6002	0.0195%	–0.0123%	–0.0333%	0.0028%
			16	5.4915	59.0000	7.6000	15.6000	5.4912	59.0017	7.6010	15.6000	–0.0066%	0.0029%	0.0136%	0.0001%
		5	6	2.7315	25.2407	4.0963	7.0963	2.7308	25.2430	4.0976	7.0958	–0.0233%	0.0093%	0.0312%	–0.0060%
			16	7.6402	50.2407	4.0963	17.0963	7.6393	50.2482	4.0987	17.0963	–0.0129%	0.0150%	0.0583%	0.0003%
		15	16	15.5346	42.9750	1.1900	24.1900	15.5359	42.9791	1.1905	24.1916	0.0080%	0.0097%	0.0475%	0.0066%
	0.6	0	1	0.0323	20.6667	11.4000	0.4000	0.0323	20.6634	11.3985	0.3999	0.0021%	–0.0157%	–0.0131%	–0.0136%
			6	1.0000	29.0000	11.4000	5.4000	1.0000	29.0017	11.4004	5.4002	0.0021%	0.0057%	0.0037%	0.0034%
			16	4.6131	45.6667	11.4000	15.4000	4.6137	45.6599	11.3953	15.3999	0.0120%	–0.0148%	–0.0410%	–0.0006%
		5	6	1.7738	22.4787	7.4872	6.4872	1.7735	22.4811	7.4892	6.4872	–0.0149%	0.0108%	0.0259%	–0.0004%
			16	6.1223	39.1454	7.4872	16.4872	6.1215	39.1484	7.4885	16.4868	–0.0126%	0.0079%	0.0174%	–0.0028%
		15	16	12.2558	32.0494	3.2296	22.2296	12.2560	32.0471	3.2286	22.2295	0.0021%	–0.0070%	–0.0308%	–0.0006%

Table 4 Comparison between analytical and simulated scenarios (continued)

				Analytical				*Simulated*				$\Delta\%$			
p_1	p_2	r	Q	$\bar{n}$ *[items]*	C *[time units]*	SO_C *[items]*	$\bar{n}_0$ *[items]*	$\bar{n}$ *[items]*	C *[time units]*	SO_C *[items]*	$\bar{n}_0$ *[items]*	$\bar{n}$	C	SO_C	$\bar{n}_0$
0.1	0.2	0	1	0.2857	14.0000	1.8000	0.8000	0.2857	13.9995	1.8002	0.8000	–0.0040%	–0.0033%	0.0101%	0.0034%
			6	2.5385	39.0000	1.8000	5.8000	2.5386	39.0008	1.8001	5.8001	0.0073%	0.0021%	0.0046%	0.0018%
			16	7.4607	89.0000	1.8000	15.8000	7.4607	89.0141	1.8006	15.8001	–0.0003%	0.0159%	0.0354%	0.0009%
		5	6	6.4841	30.9881	0.1976	9.1976	6.4838	30.9902	0.1978	9.1975	–0.0037%	0.0068%	0.0998%	–0.0018%
			16	11.5549	80.9881	0.1976	19.1976	11.5551	80.9923	0.1977	19.1974	0.0016%	0.0052%	0.0203%	–0.0010%
		15	16	21.4992	80.0119	0.0024	29.0024	21.4998	80.0187	0.0024	29.0031	0.0031%	0.0085%	–0.7115%	0.0023%
	0.4	0	1	0.1304	11.5000	3.6000	0.6000	0.1304	11.5003	3.6002	0.6001	0.0018%	0.0028%	0.0065%	0.0084%
			6	1.9375	24.0000	3.6000	5.6000	1.9373	23.9998	3.6006	5.5999	–0.0103%	–0.0009%	0.0163%	–0.0018%
			16	6.6122	49.0000	3.6000	15.6000	6.6125	48.9976	3.5993	15.6000	0.0038%	–0.0049%	–0.0199%	–0.0002%
		5	6	4.7246	17.6422	1.0569	8.0569	4.7246	17.6424	1.0570	8.0569	–0.0011%	0.0013%	0.0130%	0.0003%
			16	9.9028	42.6422	1.0569	18.0569	9.9036	42.6453	1.0567	18.0574	0.0086%	0.0072%	–0.0203%	0.0028%
		15	16	19.4802	40.2277	0.0911	27.0911	19.4797	40.2297	0.0910	27.0912	–0.0024%	0.0049%	–0.1144%	0.0003%
	0.6	0	1	0.0625	10.6667	5.4000	0.4000	0.0625	10.6675	5.4005	0.4000	–0.0150%	0.0077%	0.0097%	–0.0043%
			6	1.5263	19.0000	5.4000	5.4000	1.5263	18.9995	5.3999	5.4000	–0.0008%	–0.0027%	–0.0021%	0.0007%
			16	5.9065	35.6667	5.4000	15.4000	5.9064	35.6684	5.4003	15.4001	–0.0029%	0.0050%	0.0054%	0.0003%
		5	6	3.4727	13.8487	2.3092	7.3092	3.4734	13.8475	2.3087	7.3097	0.0208%	–0.0086%	–0.0221%	0.0068%
			16	8.5720	30.5153	2.3092	17.3092	8.5711	30.5145	2.3095	17.3085	–0.0109%	–0.0028%	0.0117%	–0.0039%
		15	16	17.4614	27.3705	0.4223	25.4223	17.4617	27.3705	0.4221	25.4225	0.0014%	0.0001%	–0.0525%	0.0009%

As the model is conceived, the lead time assumes a geometric distribution and p_1 determines its scale factor. Looking at Figure 3, it points out that partition $[r]$, that models the lead time effect, behaves as a phase-type distribution; this opens opportunities to model more complex lead time forms by extending its structure. Here the challenge is to find the right structures permitting the mathematical simplifications to obtain closed-form solutions.

In addition to equation (43), the other constraint to be accomplished with is $Q > r$, as shown in equation (19).

6.2 Numerical examples

Table 3 presents three illustrative application scenarios constructed to demonstrate the flexibility and applicability of the proposed inventory model under different demand conditions. While these examples are not derived from specific company data, they are simulation-based case studies reflecting realistic inventory situations commonly found in industry.

- The rare demand case ($\bar{D}$ = 0.05 [items/day]) emulates low-rotation items such as spare parts in industrial maintenance or aerospace logistics.
- The intermittent demand case ($\bar{D}$ = 0.4 [items/day]) reflects irregular but recurring consumption patterns typical in service operations or niche product categories.
- The smooth demand case ($\bar{D}$ = 6 [items/day]) represents high-volume, stable-demand products frequently found in fast-moving consumer goods supply chains.

In all three scenarios, a geometrically distributed lead time with an expected value of ten days is assumed. This highlights the impact of supply-side uncertainty, which remains a key challenge in modern supply chains even when demand is predictable.

For each scenario, different combinations of reorder point r and order quantity Q are evaluated, and the corresponding values of key performance measures are computed: average inventory level ($\bar{n}$), average cycle length ($\bar{C}$), average stock-out per cycle (SO_C), and unit service level (SL_U).

From a practical perspective, the table provides several managerial insights:

- In rare and intermittent demand environments, increasing r significantly improves the service level, though the marginal gain decreases as r increases.
- In the smooth demand scenario, although demand variability is low, the uncertainty in lead time has a major effect on service level performance. Achieving high service levels still requires substantial reorder points due to this supply-side variability.
- The ability to compute these indicators quickly and accurately allows practitioners to perform 'what-if' analyses, evaluate different service level targets, and identify suitable (r, Q) configurations without relying on extensive simulation or historical data.

Table 5 Precision analysis of the average inventory level computed by the new equation [$\bar{n}$, equation (29)] and the classical one [$\bar{n}^c$, equation (30)]

p_1	p_2	r	Q	*Analytical* $\bar{n}$ *[items]*	*Analytical* $\bar{n}^c$ *[items]*	*Simulated* $\bar{n}$ *[items]*	$\Delta\%$ $\bar{n}$	$\Delta\%$ $\bar{n}^c$
0.05	0.2	0	1	0.1667	0.3000	0.1667	0.0082%	80.01%
			6	2.0204	2.8000	2.0204	–0.0009%	38.58%
			16	6.7071	7.8000	6.7071	–0.0001%	16.30%
		5	6	4.7468	5.1817	4.7480	–0.0250%	9.13%
			16	9.9470	10.1817	9.9464	0.0062%	2.37%
		15	16	19.4752	19.1143	19.4757	–0.0029%	–1.86%
	0.4	0	1	0.0698	0.1000	0.0698	–0.0067%	43.32%
			6	1.3676	2.6000	1.3679	–0.0195%	90.07%
			16	5.4915	7.6000	5.4912	0.0066%	38.40%
		5	6	2.7315	4.0963	2.7308	0.0233%	50.00%
			16	7.6402	9.0963	7.6393	0.0129%	19.07%
		15	16	15.5346	16.1900	15.5359	–0.0080%	4.21%
	0.6	0	1	0.0323	–0.1000	0.0323	–0.0021%	–409.99%
			6	1.0000	2.4000	1.0000	–0.0021%	139.99%
			16	4.6131	7.4000	4.6137	–0.0120%	60.39%
		5	6	1.7738	3.4872	1.7735	0.0149%	96.63%
			16	6.1223	8.4872	6.1215	0.0126%	38.65%
	0.6	15	16	12.2558	14.2296	12.2560	–0.0021%	16.10%
0.1	0.2	0	1	0.2857	0.3000	0.2857	0.0040%	5.00%
			6	2.5385	2.8000	2.5386	–0.0073%	10.30%
			16	7.4607	7.8000	7.4607	0.0003%	4.55%
		5	6	6.4841	6.1976	6.4838	0.0037%	–4.41%
			16	11.5549	11.1976	11.5551	–0.0016%	–3.09%
		15	16	21.4992	21.0024	21.4998	–0.0031%	–2.31%
	0.4	0	1	0.1304	0.1000	0.1304	–0.0018%	–23.33%
			6	1.9375	2.6000	1.9373	0.0103%	34.21%
			16	6.6122	7.6000	6.6125	–0.0038%	14.93%
		5	6	4.7246	5.0569	4.7246	0.0011%	7.03%
			16	9.9028	10.0569	9.9036	–0.0086%	1.55%
		15	16	19.4802	19.0911	19.4797	0.0024%	–2.00%
	0.6	0	1	0.0625	–0.1000	0.0625	0.0150%	–260.02%
			6	1.5263	2.4000	1.5263	0.0008%	57.24%
			16	5.9065	7.4000	5.9064	0.0029%	25.29%
		5	6	3.4727	4.3092	3.4734	–0.0208%	24.06%
			16	8.5720	9.3092	8.5711	0.0109%	8.61%
		15	16	17.4614	17.4223	17.4617	–0.0014%	–0.23%

Note: $\Delta\%$ is against the simulated results.

These examples illustrate how the model can serve as a decision-support tool for inventory control in diverse operational settings, offering both speed and accuracy in evaluating policy performance and in identifying the best combination of (r, Q). Moreover, by incorporating cost parameters, as discussed in Section 5, the model also enables the determination of the combination of reorder point r and order quantity Q that minimises total inventory cost. It is worth noting that the optimal Q obtained through this process corresponds to the classical EOQ only when derived from cost minimisation; otherwise, Q can be treated as a configurable parameter to support scenario analysis or service-level targeting.

Finally, for the sake of completeness, a simulation tool was developed in Python programming language to reproduce the same behaviour and assumptions of the modelled inventory policy. Simulations were carried out for some settings of p_1, p_2, r, and Q executing, for each scenario, a single long run of 10^9 time units. The results, reported in Table 4, show a very good alignment between the analytical and the simulated values, definitely excluding errors in the mathematical formulation.

In addition, Table 5 focuses on the estimation of the average inventory level. It can be noted that the exact formulae here developed always produces precise results, while the classical formulae so far adopted by the scientific community presents large estimation errors especially in the cases in which the stock-out per cycle is not negligible, thus affecting the actual possibility to correctly estimate the holding cost.

7 Conclusions and further research

This paper presents an exact closed-form analytical solution for the steady-state probability distribution of the on-hand inventory level under a DT-EOQ model with lost sales. The inventory system is modelled as a discrete-time, discrete-state Markov chain, where both the consumption and supply processes are described as Bernoulli trials.

The proposed analytical approach enables the derivation of simple and compact expressions for key performance measures. In particular, the paper introduces a new, exact, and generalised formula for computing the average inventory level in the lost sales scenario. This result also demonstrates that the classical approximations widely adopted in the literature are not exact under stochastic lead times and discrete demand.

The model is well suited for practical applications involving intermittent or rare demand, as well as more stable demand environments where the supply process remains unreliable. These conditions are commonly observed in modern supply chains, where a central inventory location is often responsible for absorbing most of the demand variability (e.g., in e-commerce fulfillment centres or spare parts logistics).

Beyond its immediate analytical contributions, the model opens up several promising avenues for future research. First, the framework can be extended by generalising the assumptions on demand and lead time distributions. In particular, the use of phase-type distributions could allow for a more flexible representation of lead time variability, while compound demand processes would offer a better fit for erratic demand patterns, characterised by burstiness, low predictability, and high variance. These extensions are especially relevant in the context of maintenance and spare parts management, as emphasised by Cavalieri et al. (2008), where demand behaviour often deviates significantly from standard stochastic assumptions.

In addition, the model could be enriched by incorporating other practical features such as inventory decay, stock limits, budget constraints, and multi-item systems. From a methodological perspective, the exact analytical approach proposed here may serve as a foundation for developing a broader class of discrete-time inventory models with tractable and interpretable solutions.

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Declarations

All authors declare that they have no conflicts of interest.

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Appendix

A1 Determination of solution of partition $[r]$ in isolation

By recalling equation (9) and the normalisation equation (4), the following relationship can be obtained where the only unknown is $\mathbf{P}(0)^{[r]}$:

$$\mathbf{P}(0)^{[r]} + \sum_{n=1}^{r} \frac{\big(p_1 + (1-p_1)p_2\big)^{(n-1)} p_1}{(1-p_1)^n p_2^n} \mathbf{P}(0)^{[r]} = 1. \tag{50}$$

Since the parameter α is expressed according to Equation (10), it can be noted that:

$$\frac{\big(p_1 + (1-p_1)p_2\big)^{(n-1)} p_1}{\big((1-p_1)p_2\big)^n} = \frac{\alpha - 1}{\alpha}\alpha^n$$

By substituting in equation (50), we have:

$$\mathbf{P}(0)^{[r]}\Big(1+\frac{\alpha-1}{\alpha}\sum_{n=1}^{r}\alpha^{n}\Big)=1,$$

or, by recalling that $\sum_{n=1}^{r}\alpha^{n}$ is a well known summation,

$$\mathbf{P}(0)^{[r]}=\frac{1}{\alpha^{r}}$$

Based on equation (9), the solution of partition $[r]$ 'in isolation' can be expressed as follows:

$$\mathbf{P}(n)^{[r]}=\begin{cases}\dfrac{1}{\alpha^{r}} & n=0\\ \dfrac{\alpha-1}{\alpha}\alpha^{n-r} & n=1,\ldots,r\end{cases}$$

A2 Determination of partition probabilities

Consider equations (13), (14), (15) and (16).

By recalling the expressions of the probabilities in isolation according to equations (11) and (12), it is possible to express all partition probabilities $\pi_{[i]}$, with $i=1Q-1,\ldots,Q+r$ as functions of the partition probability $\pi_{[r]}$.

Specifically, from equation (13) the partition probability $\pi_{[Q-1]}$ can be simply expressed as a function of $\pi_{[r]}$ as follows:

$$\pi_{[Q-1]}=\frac{p_1(Q-1-r)}{\alpha^{r}}\pi_{[r]}. \tag{51}$$

Equation (14) leads to the following relationship between $\pi_{[Q+r]}$ and $\pi_{[r]}$:

$$\pi_{[Q+r]}=Q\frac{\alpha-1}{\alpha}\frac{p_1(1-p_2)}{p_2}\pi_{[r]}=Q\frac{\alpha-1}{\alpha(\beta-1)}\pi_{[r]}, \tag{52}$$

where similarly as the definition of α, the parameter β has been introduced as follows:

$$\beta=1+\frac{p_2}{(1-p_2)p_1}. \tag{53}$$

As regards equation (15), it can be noted that if $i=Q$ then $\mathbf{P}(i-Q)^{[r]}=\mathbf{P}(0)^{[r]}=\frac{1}{\alpha^{r}}$ and $\mathbf{P}(i-Q+1)^{[r]}=\mathbf{P}(1)^{[r]}=\frac{\alpha-1}{\alpha^{r}}$, so that the relationship between $\pi_{[Q]}$ and $\pi_{[r]}$ simplifies as follows:

$$\pi_{[Q]}=\frac{\alpha-p_1-1}{\alpha^{r}}(Q-r)\pi_{[r]}. \tag{54}$$

For $i=Q+1,\ldots,Q+r-1$, equation (15) can be generalised as follows:

$$\pi_{[r]}\frac{\alpha-1}{\alpha}\left(\frac{p_1(1-p_2)}{p_2}\alpha^{i-Q-r}+p_1\alpha^{i-Q+1-r}\right)=\pi_{[i]}\frac{1}{i-r},$$

that, after some algebraic manipulations, results in:

$$\pi_{[i]} = \frac{(\alpha-1)^2}{\alpha^{Q+1}}(i-r)\alpha^{i-r}\pi_{[r]}, \; i = Q+1, ..., Q+r-1. \tag{55}$$

Finally, by substituing into the normalisation equation given by equation (16), we obtain:

$$\pi_{[r]}\Bigg(1 + \frac{p_1(Q-1-r)}{\alpha^r} + \frac{\alpha - p_1 - 1}{\alpha^r}(Q-r) + \sum_{i=Q+1}^{Q+r-1} \frac{(\alpha-1)^2}{\alpha^{Q+1}}(i-r)\alpha^{i-r} + Q\frac{\alpha-1}{\alpha(\beta-1)}\Bigg) = 1 \tag{56}$$

where the only unknown is $\pi_{[r]}$.

After some basic mathematical manipulations the equation can be rewritten as:

$$\pi_{[r]}\Bigg(1 + \frac{(\alpha-1)(Q-r) - p_1}{\alpha^r} + \sum_{i=Q+1}^{Q+r-1} \frac{(\alpha-1)^2}{\alpha^{Q+1}}(i-r)\alpha^{i-r} + Q\frac{\alpha-1}{\alpha(\beta-1)}\Bigg) = 1 \tag{57}$$

Let us consider the term:

$$\sum_{i=Q+1}^{Q+r-1} \frac{(\alpha-1)^2}{\alpha^{Q+1}}(i-r)\alpha^{i-r}$$

By applying the index change $j = i - r$, we can rewrite it to:

$$\sum_{i=Q+1}^{Q+r-1} \frac{(\alpha-1)^2}{\alpha^{Q+1}}(i-r)\alpha^{i-r} = \frac{(\alpha-1)^2}{\alpha^{Q+1}} \sum_{j=Q-r+1}^{Q-1} j\alpha^j$$

From the summation theory we know that:

$$\sum_{i=1}^{n} ix^i = x\frac{1-x^n}{(1-x)^2} - \frac{nx^{n+1}}{1-x} \quad \text{with} \quad x \neq 1$$

So,

$$\sum_{i=k}^{n} ix^i = \sum_{i=1}^{n} ix^i - \sum_{i=1}^{k-1} ix^i = x^k\left(\frac{(1-x^{n-k+1})}{(1-x)^2} - \frac{(nx^{n-k+1} - k + 1)}{1-x}\right)$$

In our case, $k = Q - r + 1$ and $n = Q - 1$, so $n - k + 1 = r - 1$ and

$$\frac{(\alpha-1)^2}{\alpha^{Q+1}} \sum_{j=Q-r+1}^{Q-1} j\alpha^j = \frac{(\alpha-1)^2}{\alpha^{Q+1}}\alpha^{Q-r+1}\left(\frac{1-\alpha^{r-1}}{(1-\alpha)^2} - \frac{(Q-1)\alpha^{r-1} - Q + r}{1-\alpha}\right) = \frac{1-\alpha^{r-1}}{\alpha^r} + \frac{\alpha-1}{\alpha}(Q-1) - \frac{(\alpha-1)(Q-r)}{\alpha^r}$$

Hence, equation (57) can be simplified to:

$$\pi_{[r]}\left(\frac{1-p_1}{\alpha^r}+Q\frac{\alpha-1}{\alpha}\frac{\beta}{\beta-1}\right)=1$$

Recalling α and β from equations (10) and (53), respectively, we note that:

$$\frac{\beta(\alpha-1)}{\beta-1}=\alpha(\alpha-1)(1-p_1)\text{ and }(1-p_1)=\frac{p_1}{p_2}\frac{1}{\alpha-1}$$

that bring us to the final form:

$$\pi_{[r]}=\frac{\alpha^r}{1-p_1+Q\frac{p_1}{p_2}\alpha^r} \tag{58}$$

The remaining partition probabilities can be obtained by substituting equation (58) in equations (51), (52), (54) and (55).

A3 Determination of state probabilities

It may be convenient to introduce the parameter γ which is related to α [see equation (10)] as follows:

$$\gamma=\frac{1}{\alpha-1}=\frac{p_2(1-p_1)}{p_1}.$$

By merging equations (7) and (8) and by substituting according to equations (11) and (12), we have:

$$\mathbf{P}(n)=\begin{cases}\pi_{[r]}\mathbf{P}(0)^{[r]}=\pi_{[r]}\dfrac{1}{\alpha^r}, & n=0\\ \pi_{[r]}\mathbf{P}(n)^{[r]}=\pi_{[r]}\dfrac{\alpha-1}{\alpha}\alpha^{n-r}, & n=1,...,r\\ \displaystyle\sum_{i=Q-1}^{Q+r}\pi_{[i]}\mathbf{P}\left(n\right)^{[i]}=\sum_{i=Q-1}^{Q+r}\pi_{[i]}\frac{1}{i-r}, & n=r+1,...,Q-1\\ \displaystyle\sum_{i=n}^{Q+r}\pi_{[i]}\mathbf{P}\left(n\right)^{[i]}=\sum_{i=n}^{Q+r}\pi_{[i]}\frac{1}{i-r}, & n=Q,...,Q+r\end{cases}$$

where $\mathbf{P}(n)$ denotes the steady-state probability of any state (n) with $n=0,1,...,Q+r$ in the original system. By substituting according to the expressions of the *partition probabilities* [see equation (17)], the following can be derived.

For the state (0):

$$\mathbf{P}(0)=\pi_{[r]}\frac{1}{\alpha^r}=\frac{p_2}{p_1(\gamma+Q\alpha^r)}$$

For the states (n), with $n=1,...,r$:

$$\mathbf{P}(n)=\pi_{[r]}\frac{\alpha-1}{\alpha}\alpha^{n-r}=\frac{\alpha-1}{\alpha\left(1-p_1+Q\frac{p_1}{p_2}\alpha^r\right)}\alpha^n=\frac{p_2}{p_1(\gamma+1)(\gamma+Q\alpha^r)}\alpha^n$$

For the states (n), with $n = r+1, ..., Q-1$:

$$\begin{aligned}
\mathbf{P}(n) &= \sum_{i=Q-1}^{Q+r} \pi_{[i]} \frac{1}{i-r} = \pi_{[Q-1]} \frac{1}{Q-1-r} + \pi_{[Q]} \frac{1}{Q-r} \\
&+ \sum_{i=Q+1}^{Q+r-1} \pi_{[i]} \frac{1}{i-r} + \pi_{[Q+r]} \frac{1}{Q} \\
&= \frac{p_1(Q-1-r)}{1-p_1+Q\frac{p_1}{p_2}\alpha^r} \frac{1}{Q-1-r} + \frac{(Q-r)(\alpha-p_1-1)}{1-p_1+Q\frac{p_1}{p_2}\alpha^r} \frac{1}{Q-r} \\
&+ \sum_{i=Q+1}^{Q+r-1} \frac{(\alpha-1)^2}{\alpha^{Q+1}\left(1-p_1+Q\frac{p_1}{p_2}\alpha^r\right)} (i-r)\alpha^i \frac{1}{i-r} \\
&+ \frac{Q}{\beta} \frac{p_1}{p_2} \frac{\alpha^r}{1-p_1+Q\frac{p_1}{p_2}\alpha^r} \frac{1}{Q} \\
&= \frac{1}{1-p_1+Q\frac{p_1}{p_2}\alpha^r} \left(\alpha - 1 + \frac{(\alpha-1)^2}{\alpha^{Q+1}} \sum_{i=Q+1}^{Q+r-1} \alpha^i + \frac{p_1\alpha^r}{p_2\beta} \right) \\
&= \frac{\alpha^r}{1-p_1+Q\frac{p_1}{p_2}\alpha^r} \left(1 + \frac{p_1}{p_2\beta} - \frac{1}{\alpha} \right) = \frac{\alpha^r}{\gamma + Q\alpha^r}
\end{aligned}$$

having considered that, from the summation theory,

$$\sum_{k=m}^{n} x^k = \frac{x^m - x^{n+1}}{1-x} \text{ with } x \neq 1, \text{ and then } \sum_{i=Q+1}^{Q+r-1} \alpha^i = \frac{\alpha^{Q+1} - \alpha^{Q+r}}{1-\alpha},$$

and that,

$$1 + \frac{p_1}{p_2\beta} - \frac{1}{\alpha} = \frac{p_1}{p_2}.$$

For the state (Q):

$$\begin{aligned}
\mathbf{P}(Q) &= \sum_{i=Q}^{Q+r} \pi_{[i]} \frac{1}{i-r} = \pi_{[Q]} \frac{1}{Q-r} + \sum_{i=Q+1}^{Q+r-1} \pi_{[i]} \frac{1}{i-r} + \pi_{[Q+r]} \frac{1}{Q} \\
&= \frac{1}{1-p_1+Q\frac{p_1}{p_2}\alpha^r} \left(\alpha - p_1 - 1 + \frac{(\alpha-1)^2}{\alpha^{Q+1}} \sum_{i=Q+1}^{Q+r-1} \alpha^i + \frac{p_1\alpha^r}{p_2\beta} \right) \\
&= \frac{1}{1-p_1+Q\frac{p_1}{p_2}\alpha^r} \left(\alpha - p_1 - 1 + \frac{(\alpha-1)^2}{\alpha^{Q+1}} \frac{\alpha^{Q+1} - \alpha^{Q+r}}{1-\alpha} + \frac{p_1\alpha^r}{p_2\beta} \right) \\
&= \frac{p_1}{p_2} \frac{\alpha^r - p_2}{1-p_1+Q\frac{p_1}{p_2}\alpha^r} = \frac{\alpha^r - p_2}{\gamma + Q\alpha^r}
\end{aligned}$$

For the states (n), with $n = Q+1, ..., Q+r$:

$$\mathbf{P}(n) = \sum_{i=n}^{Q+r} \pi_{[i]} \frac{1}{i-r} = \sum_{i=n}^{Q+r-1} \pi_{[i]} \frac{1}{i-r} + \pi_{[Q+r]} \frac{1}{Q}$$

$$= \frac{1}{1 - p_1 + Q\frac{p_1}{p_2}\alpha^r}\left(\frac{(\alpha-1)^2}{\alpha^{Q+1}}\sum_{i=n}^{Q+r-1}\alpha^i + \frac{p_1\alpha^r}{p_2\beta}\right)$$

$$= \frac{1}{1 - p_1 + Q\frac{p_1}{p_2}\alpha^r}\left(\frac{(\alpha-1)^2}{\alpha^{Q+1}}\frac{\alpha^n - \alpha^{Q+r}}{1-\alpha} + \frac{p_1\alpha^r}{p_2\beta}\right)$$

$$= \frac{\alpha^r}{(\gamma + Q\alpha^r)} - \frac{p_2}{p_1(1+\gamma)(\gamma+Q\alpha^r)}\alpha^{n-Q}$$

By summarising:

$$\mathbf{P}(n) = \begin{cases} \dfrac{p_2}{p_1(\gamma+Q\alpha^r)}, & n = 0 \\ \dfrac{p_2}{p_1(\gamma+1)(\gamma+Q\alpha^r)}\alpha^n, & n = 1, ..., r \\ \dfrac{\alpha^r}{\gamma+Q\alpha^r}, & n = r+1, ..., Q-1 \\ \dfrac{\alpha^r - p_2}{\gamma+Q\alpha^r}, & n = Q \\ \dfrac{\alpha^r}{\gamma+Q\alpha^r} - \dfrac{p_2}{p_1(\gamma+1)(\gamma+Q\alpha^r)}\alpha^{n-Q}, & n = Q+1, ..., Q+r \end{cases}$$

where

$$\alpha = 1 + \frac{p_1}{(1-p_1)p_2} \text{ and } \gamma = \frac{1}{\alpha - 1} = \frac{p_2(1-p_1)}{p_1}.$$