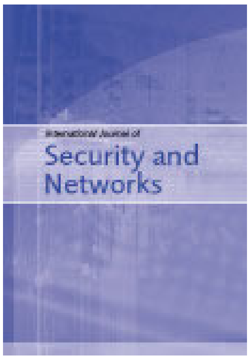

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Pursuing multi-agent Nash equilibria amidst DoS attacks with stochastic perturbations

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Abstract: In this study, we have developed a comprehensively distributed framework aimed at examining the collaborative dynamics of multi-agent systems (MAS) amidst denial-of-service (DoS) attacks, with a particular focus on the impact of stochastic perturbations within an insecure network milieu. We have engineered a highly robust, fully distributed control protocol that guarantees the attainment of Nash equilibrium in the presence of DoS assaults. The investigation encompasses first-and second-order systems, with a detailed exploration of general non-quadratic and quadratic payoff games. The theoretical findings are substantiated through simulation experiments, which validate our analysis and reveal that the observer and control protocol proposed herein exhibit enhanced compatibility with second-order systems under identical stochastic perturbation conditions.

Keywords: DoS attacks; random perturbations; multi-agent systems; MAS; Nash equilibrium; decentralised control protocols.

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1 Introduction

Multi-agent systems (MAS) have emerged as a focal point of interest in today's highly interconnected global landscape due to their proven efficacy and adaptability in managing intricate tasks and decision-making processes. These systems, comprising a constellation of autonomous entities, leverage local communication and collaborative efforts to surmount challenges that exceed the scope of individual capabilities. The burgeoning application of MAS extends across diverse domains, including automated control (Grimes and Breen, 2023), economic market analysis (Zhang and Zhang, 2024), and social network management (Nazari et al., 2024), where they enhance system scalability, flexibility, and robustness through a decentralised architecture of intelligent agents. Despite these advantages, the widespread adoption of MAS has also rendered them susceptible to cyber threats, with denial-of-service (DoS) attacks being particularly pernicious (Yang et al., 2021). Such attacks undermine the stability and performance of MAS by depleting system resources or severing communication pathways, impeding their operational integrity.

The repercussions of DoS attacks on MAS are far-reaching. In the event of an attack, agents may be hindered from receiving essential information or engaging in effective communication with their peers, potentially obstructing the decision-making processes of the entire system and even leading to its complete failure. For example, within the context of automated control systems, DoS attacks can precipitate the interruption of production workflows, incurring substantial economic losses; within social networks, such attacks can impede the flow of information, adversely affecting the communication between users (Xiong et al., 2024). Consequently, the pursuit of strategies to sustain the stability of MAS amidst DoS attacks while concurrently optimising overall system benefits has emerged as a critical area of research (Raffaele et al., 2024; Muhammad et al., 2024).

The Nash equilibrium, a cornerstone in game-theoretic analysis, is instrumental in elucidating the behaviours of agents engaged in mutual interactions and competitions. It delineates a scenario wherein no individual can enhance their utility by altering their strategy unilaterally, contingent upon the strategies adopted by others (Nash, 1950; Nash, 1951). This equilibrium is pivotal for designing and operationalising enduring MAS, as it mitigates the systemic oscillations that arise from the frequent alterations in individual strategies (Feng et al., 2023; Chen et al., 2023). The utility of the Nash equilibrium concept is evident across a spectrum of practical applications within MAS, including traffic flow management in autonomous vehicles (Nan et al., 2022), competitive dynamics among firms in economic markets (Mansour et al., 2022), and the propagation of information within social networks (Guo et al., 2022). In these contexts, the Nash equilibrium serves as a valuable tool for designers, enabling them to comprehend and anticipate the behaviours of agents and, consequently, to engineer more efficacious systems and strategies (Ye et al., 2023; Yi et al., 2022).

Without external disruptions, the quest for Nash equilibrium is a well-established problem within theoretical game theory. However, this conventional approach to Nash equilibrium typically presupposes that all decision-makers possess complete visibility into the decisions of their counterparts, which can engender substantial communication overheads and incur significant costs in real-world engineering systems, presenting certain challenges (Nabetani et al., 2011; Frihauf et al., 2011). Especially when the system is under the duress of DoS attacks, the traditional methodologies for Nash equilibrium resolution may no longer suffice, necessitating the emergence of distributed Nash equilibrium search problems (Hu et al., 2022). Such attacks introduce randomness and uncertainty, making system dynamics intricate and unpredictable. Furthermore, random perturbations are an inescapable facet of DoS attacks, potentially arising from the attacker's strategic maneuvers, fluctuations in network conditions, or intrinsic system noise. The perturbations profoundly influence the decision-making processes within MAS, escalating the complexity of seeking Nash equilibrium under adversarial conditions. The inherent unpredictability introduced by random disturbances outstrips the capabilities of traditional deterministic approaches to address such issues.

Consequently, the prevailing research trajectory in the distributed Nash equilibrium optimisation is oriented toward developing robust search strategies to counteract these disruptions (Romano et al., 2020; Ai and Wang, 2021). Among the robust control techniques that have been integrated into the Nash equilibrium search domain are extended state observers (Ye, 2020), high-gain observers (Huang et al., 2021), the internal model principle (Zhang et al., 2020; Romano et al., 2020), and finite-time convergent disturbance observers (Ai, 2020). The evolution of these robust control methodologies furnishes novel strategies and instrumentalities for optimising distributed Nash equilibrium. This enables a more productive pursuit and sustenance of Nash equilibrium in the presence of uncertainties and disturbances encountered in practical engineering scenarios, thereby surmounting the intricacies of more sophisticated engineering challenges.

While some studies have addressed the implications of DoS attacks on MAS, most of these efforts have been directed toward specific attacks or have relied on simplified models. A comprehensive understanding of DoS attacks characterised by random disturbances and their consequential effects on Nash equilibrium attainment remains underexplored in theoretical and empirical research. Consequently, this manuscript endeavours to bridge this gap by introducing a novel framework to analyse and address the multi-agent Nash equilibrium problem in the context of DoS attacks with random disturbances.

The principal contributions of this manuscript are delineated as follows:

- Developing an observer model incorporating stochastic disturbances offers a more faithful representation of perturbation scenarios encountered in real-world applications. It is demonstrated that this approach is particularly adept at handling second-order systems under identical stochastic disturbance parameters.
- We introduce a Nash equilibrium-seeking algorithm designed to operate under conditions of DoS attacks,

enabling the effective pursuit of Nash equilibrium within MAS amidst random disturbances induced by such attacks.

- We show the substantiation of the proposed methods' efficacy and robustness through rigorous theoretical analysis and extensive simulation experiments, culminating in the derivation of sufficient conditions for system stability.

The organisation of this manuscript is outlined as follows: Section 2 offers a comprehensive review of pertinent literature encompassing MAS, Nash equilibrium, and DoS attacks. Section 3 delineates our attack model and the methodology for Nash equilibrium resolution in meticulous detail. Section 4 showcases the outcomes of simulation experiments, while Section 5 concludes the paper and posits potential avenues for future research endeavours.

2 The model of signal mixing and separation

The following defines the references (Ye and Hu, 2017).

Definition 1: A game consisting of three elements, denoted as $\Gamma \triangleq \{\mathcal{I}, X, f\}$ where $\mathcal{I}$ is the set of N players, $X = X_1 \times X_2 \ldots \times X_N$, $X_i \subseteq R$ represents the set of strategic actions available to the agents. $f = (f_1, f_2, \ldots, f_N)$, f_i: $X \rightarrow R$ represents the payment function of agent i.

Definition 2: The Nash equilibrium can be denoted as $(x_1^*, x_2^*, \cdots, x_n^*) \in X_1 \times X_2 \times \cdots \times X_n$, where x_i^* represents the strategy chosen by player i in the equilibrium state. In this combination of strategies, for each player i, the strategy x_i^* is the best response to the strategies of all other players. That is, given that the strategies of all other players, $\mathbf{x}_{-i}^*$ are fixed, no other strategy x_i can yield a higher utility for player i. Mathematically, this is expressed as for all $x_i \in X_i$, $i \in \mathcal{I}$, the inequality: $f_i(x_i^*, \mathbf{x}_{-i}^*) \le f_i(x_i, \mathbf{x}_{-i}^*)$ holds.

To address the problem of searching for Nash equilibrium points in distributed scenarios within games, a graph $g = (V, E)$ is utilised to describe the information interaction relationships among N participants, where V represents the set of all nodes in a graph $\mathcal{G}$. The adjacency matrix of the graph, denoted as $A = [a_{ij}] \in R^{N \times N}$, represents the relationships between agents and their neighbouring nodes. If participant i can obtain information from participant j, i.e., if $(j, i) \in E$, then $a_{ij} > 0$, otherwise, $a_{ij} = 0$. The weighted degree of node i is defined as $d_i = \sum_{j=1}^{N} a_{ij}$. The degree matrix $D = \text{diag}\{d_i\}$, and the Laplacian matrix of the graph is $L = D - A$.

DoS attacks impede information transfer between agents and their neighbours by blocking communication channels, posing significant challenges for agents seeking a Nash equilibrium. To address this issue, we characterise the behaviour of DoS attacks by setting $a_{ij} = 0$ when an attack is detected and $a_{ij} = 1$, when no attack is present as follows:

$$a_{ij} = \begin{cases} 1, & i \ne j \,\&\, (j, i) \in E \\ 0, & otherwise \end{cases}$$

Assuming that no attack is present at the initial moment, i.e., $a_{ij}(t_0) = 1$. Let $T = T_m \cup T_p$, where $T_m = \{1, \ldots, s\}$ is the set of all connected graphs, and $T_p = \{s + 1, \ldots, q\}$ is the set of all graphs with disconnected connections due to the impact of DoS attacks. $T_p(T_1, T_2)$ denotes the communication graphs that are disconnected during the interval $[T_1, T_2)$.

3 Problem description

The article considers a non-cooperative game involving a set of N agents, where the set of agents is denoted as $\mathcal{I} = \{1, 2, \cdots, N\}$ with $N \ge 2$, Let $x_i \in X_i = R$ represent the action of agent i, and $f_i(\boldsymbol{x})$: $R^N \rightarrow R$ denote the objective function of agent i, where $\boldsymbol{x} = [x_1, x_2, \ldots, x_N]^T \in R^N$. Reference (Ye et al., 2017) designed a Nash equilibrium seeking strategy, as shown in equation (1), and demonstrated that $(\boldsymbol{x}^*, 1_N \otimes \boldsymbol{x}^*)$ is exponentially stable. However, multi-agent networks are susceptible to cyber-attacks, which can lead to communication interruptions. Attackers are constrained by their attack capabilities and costs, ensuring that the network communication topology is not perpetually disrupted. Moreover, there are established algorithms to detect whether the system is under attack (Persis and Tesi, 2015; Bhatia et al., 2024), and network recovery schemes can be employed to ensure that the system's communication remains unobstructed for extended periods (An et al., 2019).

The subsequent issue we address is the convergence performance of the Nash equilibrium in the presence of disturbances, specifically when the multi-agent system is subjected to DoS attacks. We have validated this for both first-order and second-order systems. We have also discussed cases where the objective functions are non-quadratic and quadratic.

The Nash equilibrium-seeking strategy is:

$$u_i = k_i \frac{\partial f_i}{\partial x_i}(\mathbf{y}_i), i \in \mathcal{I} \tag{1}$$

In the context of equation (1), the term $(\partial f_i / \partial x_i)(\boldsymbol{y}_i)$ represents $(\partial f_i / \partial x_i)|_{x=y_i}$, $k_i = \delta \overline{k}_i$, where $\delta > 0$, and $\overline{k}_i$ is a predetermined constant. Furthermore, $\mathbf{y}_i = [y_{i1}, y_{i2}, \ldots y_{iN}]$ represents the vector of player i estimates of the actions of other players, where y_{ij} denotes player i estimate of player j action, the update rule for y_{ij} is given by:

$$\dot{y}_{ij} = -\left(\sum_{k=1}^{N} a_{ik} \left(y_{ij} - y_{kj} \right) + a_{ij} \left(y_{ij} - x_j \right) \right) \tag{2}$$

After obtaining local information, each agent adjusts its behaviour to minimise cost. Thus, the mathematical model is established as follows:

$$\begin{cases} \min f_i\left(x_i, \mathbf{x}_{-i}\right), & i \in \mathcal{I} \\ s.t.\ \dot{x}_i(t) = u_i(t), & t \in [0, T) \end{cases} \tag{3}$$

In this context, $f_i(\cdot)$ and u_i represent the cost function and control input for each intelligent agent, respectively.

Random disturbances are an inevitable phenomenon in practical engineering hence, we also need to introduce random disturbances $d_i(t)$ into the constraint conditions of equation (3), that is

$$\dot{x}_i(t) = u_i + d_i(t), i \in \mathcal{I} \tag{4}$$

Assumption 1: The graph $\mathcal{G}$ is strongly connected, ensuring that each player can access information from all other players.

Remark: Assumption 1 has been introduced to establish a framework where players can effectively communicate their local information to their peers through established channels. This assumption is critical for our model as it reflects the necessity of information exchange in decentralised systems, which is a common requirement in many real-world scenarios, including economic markets and distributed computing networks.

Assumption 2: A function f: $R^N \to R$ is classified as a C^2 function if it possesses first and second-order continuous derivatives.

Assumption 3: (Frihauf et al., 2012): It is posited that a minimum of one, and possibly several, isolated Nash equilibria $\mathbf{x}^* = \left[x_1^*, x_2^*, \cdots, x_N^*\right]^T$, with the condition that for every player $i \in \mathcal{I}$, the first partial derivative $\partial f_i(\boldsymbol{x}^*)/\partial x_i = 0$ equals zero and the second partial derivative $\partial^2 f_i\left(\mathbf{x}^*\right)/\partial x_i^2 < 0$ is less than zero.

Assumption 4: (Frihauf et al., 2012): The matrix is strictly diagonally dominant.

$$B = \begin{bmatrix} \dfrac{\partial^2 f_1\left(\mathbf{x}^*\right)}{\partial x_1^2} & \dfrac{\partial^2 f_1\left(\mathbf{x}^*\right)}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f_1\left(\mathbf{x}^*\right)}{\partial x_1 \partial x_N} \\ \dfrac{\partial^2 f_2\left(\mathbf{x}^*\right)}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f_2\left(\mathbf{x}^*\right)}{\partial x_2^2} & & \dfrac{\partial^2 f_2\left(\mathbf{x}^*\right)}{\partial x_2 \partial x_N} \\ \vdots & & \ddots & \\ \dfrac{\partial^2 f_N\left(\mathbf{x}^*\right)}{\partial x_N \partial x_1} & \dfrac{\partial^2 f_N\left(\mathbf{x}^*\right)}{\partial x_N \partial x_2} & & \dfrac{\partial^2 f_N\left(\mathbf{x}^*\right)}{\partial x_N^2} \end{bmatrix}$$

Remark: Assumptions 2–4 are derived from Frihauf et al. (2012) and are integral to our analysis. These assumptions are designed to ensure that any Nash equilibrium that meets the specified criteria is stable and exponentially stable under the dynamics of gradient play, as outlined in Lemma 2. This is a significant contribution to the stability analysis of Nash equilibria in game-theoretic models, which has practical implications for understanding the long-term behaviour of strategic interactions in various applications.

Lemma 1: (Berman et al., 1994): Suppose the communication graph is strongly connected. Then, there exists a positive definite diagonal matrix P = diag$\{p_i\}$ for $i \in \{1, 2, \cdots, N^2\}$ and a symmetric positive definite matrix $Q\hat{I}\ R^{N_2}, R^{N_2}$ such that the following equation is satisfied: $\left(L\ddot{A}I_{N\times N} + B_0\right)^T P + P\left(L\ddot{A}I_{N\times N} + B_0\right) = Q$, where B_0 = $diag\{a_{ij}\}$, for $i, j \in \mathcal{I}$.

4 Principal findings

This section delineates applying the optimisation strategies presented in equations (1) and (2) to resolve the multi-agent Nash equilibrium seeking amidst DoS attacks, incorporating random perturbations. The analysis commences with exploring Nash equilibrium seeking in first-order systems, succeeded by examining second-order systems. In the context of these systems, general non-quadratic and quadratic strategies are considered.

4.1 Nash equilibrium of first-order systems under dos attacks with stochastic disturbances

For first-order agents, the state update adheres to equation (4), with the control input formulated as follows:

$$u_i = -\hat{d}_i + \nabla_i f_i\left(\mathbf{y}_i\right) \tag{5}$$

Here, $\hat{d}_i$ denotes agent i's estimation of the time-varying disturbance $d_i(t)$, and the gradient term is given by $\nabla_i f_i\left(\mathbf{y}_i\right) = \dfrac{\partial f_i(\mathbf{x})}{\partial x_i}|\mathbf{x} = \mathbf{y}_i$, where $\mathbf{y}_i = [y_{i1}, y_{i2}, \ldots, y_{iN}]$ and y_{ij} is the estimated state of agent j by agent i. Moreover, $\hat{d}_i$ is estimated using equation (6) and y_{ij} is updated via the differential equation (7):

$$\hat{d}_i = \tau h_i\left(x_i + z_i\right) \tag{6}$$

$$\dot{y}_{ij} = -\gamma\overline{\gamma}\left(\sum_{k=1}^{N} a_{ik}\left(y_{ij} - y_{kj}\right) + a_{ij}\left(y_{ij} - x_j\right)\right) \tag{7}$$

In these equations, τ and γ are tunable positive parameters, while $\overline{\gamma}_{ij}$ and k_i are fixed positive constants. The variable z_i is utilised in a supportive capacity. Define $\Psi(\mathbf{y}) = \left[\dfrac{\partial f_1\left(\mathbf{y}_1\right)}{\partial x_1}, \dfrac{\partial f_2\left(\mathbf{y}_2\right)}{\partial x_2}, \cdots, \dfrac{\partial f_N\left(\mathbf{y}_N\right)}{\partial x_N}\right]^T$, the update for z_i is given by:

$$\dot{z}_i = \nabla_i f_i\left(\mathbf{y}_i\right) \tag{8}$$

The closed-loop system, derived from equations (4), (5), (6), and (7), is described as:

$$\begin{aligned}&\dot{\mathbf{x}} = -\hat{\mathbf{d}} + \psi(\mathbf{y}) + \mathbf{d}(t)\\ &\dot{\mathbf{z}} = \psi(\mathbf{y})\\ &\hat{\mathbf{d}} = \tau\mathbf{h}(\mathbf{x}+\mathbf{z})\\ &\dot{\mathbf{y}} = -\gamma\bar{\boldsymbol{\gamma}}\left(\left(L\otimes I_{N\times N} + B_0\right)\mathbf{y} - B_0\left(\mathbf{1}_N\otimes\mathbf{x}\right)\right)\end{aligned} \tag{9}$$

4.1.1 Non-quadratic game

We will now establish that the pursuit strategy, as dictated by the control input (5), disturbance estimation (6), and state estimation update (7), leads to the convergence of participants' actions to the Nash equilibrium in a non-quadratic utility game.

Theorem 1: Under Assumptions 1–3, participants in a non-quadratic utility game update their strategies under equations (5) and (7) and perform state estimation using equation (6). For every $\mathbf{x}^*$ fulfilling Assumptions 3–4, there exists a positive constant δ^* such that for $\forall\delta\in(0,\delta^*)$, under the presence of DoS attacks and considering disturbances, the equilibrium point ($\mathbf{x}^*$, $1_N\otimes\mathbf{x}^*$) is exponentially stable if the following conditions are satisfied:

$$\frac{T_p\left(t_0,t\right)}{t-t_0} \le \frac{\alpha-\xi}{\alpha+\beta} \tag{10}$$

$\alpha = \dfrac{\phi_1}{2\omega_1}$, $\beta = \dfrac{2\lambda_{\max}(A)\omega_2+\omega_1}{2\omega_1^2}$, where $\xi\in(0,\alpha)$ and the constant ϕ_i is detailed in equation (18).

Proof: To streamline subsequent analysis, introduce a supportive system.

$$\dot{x}_i = \bar{h}_i\frac{\partial f_i(\mathbf{x})}{\partial x_i} - \hat{d}_i + d_i(t), i\in\mathcal{I} \tag{11}$$

By linearising equation (9) at $\mathbf{x}^*$, under the assumptions of Assumption 2 and Assumption 3, and for each $\mathbf{x}^*$ satisfying Assumption 4, there exists a function M_1: $G_0\to R$, where $G_0 = \{\mathbf{e}\in R^N \| \|\mathbf{e}\|\le\rho_0\}$, $\mathbf{e} = \mathbf{x}-\mathbf{x}^*$ for $\rho_0 > 0$ that satisfies.

$$\begin{aligned}&c_1\|\mathbf{e}\|^2 \le M_1(\mathbf{e}) \le c_2\|\mathbf{e}\|^2\\ &\left(\frac{\partial M_1(\mathbf{e})}{\partial\mathbf{e}}\right)^T\left(\mathrm{diag}\{\bar{h}_i\}\psi(\mathbf{x})\right) \le -c_3\|\mathbf{e}\|^2\\ &\left\|\frac{\partial M_1(\mathbf{e})}{\partial\mathbf{e}}\right\| \le c_4\|\mathbf{e}\|\end{aligned} \tag{12}$$

where c_1, c_2, c_3, and c_4 are positive constants.

Define a Lyapunov function:

$$V = V_1 + V_2 + V_3 \tag{13}$$

where $V_1 = \frac{1}{2}\mathbf{e}_d^T\mathbf{e}_d$, $V_1 = \frac{1}{2}\mathbf{e}_d^T\mathbf{e}_d$, $V_3 = \bar{\mathbf{y}}^TP\bar{\mathbf{y}}$, $c\in(0,\ 1)$ is a constant. $\bar{\mathbf{y}} = \mathbf{y} - \mathbf{1}_N\otimes\mathbf{x}$, $\mathbf{e} = \mathbf{x}-\mathbf{x}^*$, $\mathbf{e}_d = \hat{\mathbf{d}} - \mathbf{d}(t)$.

Let $\mathbf{E} = \left[\mathbf{e}_d^T, \mathbf{e}^T, \bar{\mathbf{y}}^T\right]^T$, then, it can be derived that $\min\left\{\frac{1}{2}, c_1, p_i^{\min}\right\}\|\mathbf{E}\|^2 \le V \le \min\left\{\frac{1}{2}, c_2, p_i^{\max}\right\}\|\mathbf{E}\|^2$ where $p_i^{\min} = \min\{p_i\}$, $p_i^{\max} = \max\{p_i\}$.

Furthermore, we consider a domain G1 = $\{\mathbf{E}\in R^{N+N} \| \|\mathbf{E}\| \le \rho_1\}$, where ρ_1 is a positive constant. For $t\in[t_k, t_{k+1})$ and for all $k\in Z_+$, it is derived that:

$$\begin{aligned}\dot{V}_1 &= \mathbf{e}_d^T\dot{\mathbf{e}}_d = \mathbf{e}_d^T\left(\dot{\hat{\mathbf{d}}} - \dot{\mathbf{d}}(t)\right) = \mathbf{e}_d^T(\tau\mathbf{h}\left(\dot{\mathbf{x}}+\dot{\mathbf{z}}\right) - \dot{\mathbf{d}}(t))\\ &= \mathbf{e}_d^T\left(-\tau\mathbf{h}\mathbf{e}_d - \dot{\mathbf{d}}(\mathrm{t})\right) \le -\tau\lambda_{\min}(\mathbf{h})\|\mathbf{e}_d\|^2 + \|\mathbf{e}_d\|\|\dot{\mathbf{d}}(t)\|\end{aligned} \tag{14}$$

$$\begin{aligned}\dot{V}_2 &= \left(\frac{\partial W_1(\mathbf{e})}{\partial\mathbf{e}}\right)^T\dot{\mathbf{e}}\\ &= \delta\left(\frac{\partial W_1(\mathbf{e})}{\partial\mathbf{e}}\right)^T\left[diag\{\bar{h}_i\}\psi\left(\bar{\mathbf{y}}+\mathbf{1}_N\otimes\mathbf{e}+\mathbf{1}_N\otimes\mathbf{x}^*\right) - \mathbf{e}_d\right]\\ &= \delta\left(\frac{\partial W_1(\mathbf{e})}{\partial\mathbf{e}}\right)^T\left(diag\{\bar{h}_i\}\psi(\mathbf{x})\right) + \delta\left(\frac{\partial W_1(\mathbf{e})}{\partial\mathbf{e}}\right)^T\\ &\bullet diag\{\bar{h}_i\}\left(\psi(\bar{\mathbf{y}}+\mathbf{1}_N\otimes\mathbf{x}) - \psi(\mathbf{x})\right) + \left(\frac{\partial W_1(\mathbf{e})}{\partial\mathbf{e}}\right)^T(-\mathbf{e}_d)\\ &\le -\delta c_3\|\mathbf{e}\|^2 + \delta l_1\|\mathbf{e}\|\|\bar{\mathbf{y}}\| + c_4\|\mathbf{e}\|\|\mathbf{e}_d\|\end{aligned} \tag{15}$$

where $i_1 > 0$ and satisfies

$$\left\|\left(\partial W_1(\mathbf{e})/\partial\mathbf{e}\right)^T \mathrm{diag}\{\bar{h}_i\}\psi(\bar{\mathbf{y}}+\mathbf{1}_N\otimes\mathbf{x}) - \psi(\mathbf{x})\right\| \le l_1\|\mathbf{e}\|\|\bar{\mathbf{y}}\|$$

$$\begin{aligned}\dot{V}_3 &= \dot{\bar{\mathbf{y}}}^TP\bar{\mathbf{y}} + \bar{\mathbf{y}}^TP\dot{\bar{\mathbf{y}}}\\ &= \left[-\gamma\bar{\gamma}\left(L\otimes I_{N\times N}+B_0\right)\bar{\mathbf{y}} + \mathbf{1}_N\otimes\left(-\mathbf{e}_d+\psi(\mathbf{y})\right)\right]^TP\bar{\mathbf{y}}\\ &+\bar{\mathbf{y}}^TP\left[-\gamma\bar{\gamma}\left(L\otimes I_{N\times N}+B_0\right)\bar{\mathbf{y}} + \mathbf{1}_N\otimes\left(-\mathbf{e}_d+\psi(\mathbf{y})\right)\right]\\ &= -\gamma\bar{\mathbf{y}}^TQ\bar{\mathbf{y}} + \mathbf{2}\bar{\mathbf{y}}^TP(\mathbf{1}_N\otimes\psi(\mathbf{y})) - \mathbf{2}\bar{\mathbf{y}}^TP\left(\mathbf{1}_N\otimes\mathbf{e}_d\right)\\ &= -\gamma\bar{\mathbf{y}}^TQ\bar{\mathbf{y}} + 2\bar{\mathbf{y}}^TP\left(\mathbf{1}_N\otimes\left(\psi(\mathbf{y})-\psi(\mathbf{x})\right)\right)\\ &+2\bar{\mathbf{y}}^TP\left(\mathbf{1}_N\otimes\left(\psi(\mathbf{x})-\psi(\mathbf{x}^*)\right)\right) - 2\bar{\mathbf{y}}^TP\left(\mathbf{1}_N\otimes\mathbf{e}_d\right)\\ &= -\left(\gamma\lambda_{\min}(Q) - 2\sqrt{N}\|P\|\max_{i\in v}\{l_i\}\right)\|\bar{\mathbf{y}}\|^2\\ &+2N\|P\|\max_{i\in v}\{l_i\}\|\bar{\mathbf{y}}\|\|\mathbf{e}\| - 2\sqrt{N}\|P\|\|\bar{\mathbf{y}}\|\|\mathbf{e}_d\|\end{aligned} \tag{16}$$

Case 1: $r\in C_m$

$$\begin{aligned}\dot{V} &= \dot{V}_1+\dot{V}_2+\dot{V}_3\\ &\le -\tau\lambda_{\min}(\mathbf{h})\|\mathbf{e}_d\|^2 + \|\mathbf{e}_d\|\|\dot{d}(t)\| - \delta c_3\|\mathbf{e}\|^2 + \delta l_1\|\mathbf{e}\|\|\bar{\mathbf{y}}\|\\ &+c_4\|\mathbf{e}\|\|\mathbf{e}_d\| - \left(\gamma\lambda_{\min}(Q) - 2\sqrt{N}\|P\|\max_{i\in\mathcal{I}}\{l_i\}\right)\|\bar{\mathbf{y}}\|^2\\ &+2N\|P\|\max_{i\in\mathcal{I}}\{l_i\}\|\bar{\mathbf{y}}\|\|\mathbf{e}\| + 2\sqrt{N}\|P\|\|\bar{\mathbf{y}}\|\|\mathbf{e}_d\|\\ &\le -\left(\gamma\lambda_{\min}(Q) - 2\sqrt{N}\|P\|\max_{i\in\mathcal{I}}\{l_i\} - \frac{\left(\delta l_1 + 2N\|P\|\max_{i\in\mathcal{I}}\{l_i\}\right)\varepsilon_1}{2}\right)\|\bar{\mathbf{y}}\|^2\\ &+\sqrt{N}\|P\|\|\bar{\mathbf{y}}\|^2 - \left(\delta c_3 - \frac{\delta l_1 + 2N\|P\|\max_{i\in v}\{l_i\}}{2\varepsilon_1} - \frac{c_4}{2\varepsilon_2}\right)\|\mathbf{e}\|^2\\ &-\left(\tau\lambda_{\min}(\mathbf{h}) + \sqrt{N}\|P\| - \frac{c_4\varepsilon_2}{2} - \frac{\varepsilon_3}{2}\right)\|\mathbf{e}_d\|^2 + \frac{1}{2\varepsilon_3}\|\dot{d}(t)\|^2\end{aligned} \tag{17}$$

The constants ε_1, ε_2, ε_3 are positive and remain to be ascertained.

It is observed that for t^3t_0, the inequality $\|\dot{\mathbf{d}}(t)\| \le \sqrt{N}\max\{\bar{d}_i\}$ always holds.

By choosing ε_3 to be sufficiently large, it is possible to ensure that $\frac{1}{2\varepsilon_3}\|\dot{\mathbf{d}}(t)\|^2$ is sufficiently small.

The selection of ε_1 and ε_2 should satisfy the following condition:

$$\phi_1 = \delta c_3 - \frac{\delta l1 + 2N\|P\|\max_{i\in n}\{l_i\}}{2\varepsilon_1} - \frac{c_4}{2\varepsilon_2} > 0 \tag{18}$$

For a given ε_1, there always exists a γ such that:

$$\gamma\lambda_{\min}(Q) - 2\sqrt{N}\|P\|\max_{i\in\mathcal{I}}\{l_i\} - \frac{\left(\delta l_1 + 2N\|P\|\max_{i\in\mathcal{I}}\{l_i\}\right)\varepsilon_1}{2} + \sqrt{N}\|P\| > \phi_1 \tag{19}$$

For given ε_2 and ε_3, there always exists a such that:

$$\phi_3 = \tau\lambda_{min}(\boldsymbol{h}) + \sqrt{N}\|P\| - \frac{c_4\varepsilon_2}{2} - \frac{\varepsilon_3}{2} > \phi_1 \tag{20}$$

upon rearranging $\dot{V}$, we ultimately obtain:

$$\dot{V} \le -\phi_1\|\mathbf{E}\|^2 + \frac{1}{2\varepsilon_3}\|\dot{\mathbf{d}}(t)\|^2 \tag{21}$$

when $\|\mathbf{E}\|^2 \ge \frac{1}{\phi_1\varepsilon_3}\left(\sqrt{N}\max\{\bar{d}_i\}\right)^2$, it follows that:

$$\dot{V} \le -\frac{1}{2}\phi_1\|\mathbf{E}\|^2 \tag{22}$$

we have $\omega_1\|\mathbf{E}\|^2 \le V \le \omega_2\|\mathbf{E}\|^2$, where $\omega_1 = \min\left\{\frac{1}{2}, c_2, \frac{1}{2}p_i^{\max}\right\}$ and $\omega_2 = \max\left\{\frac{1}{2}, c_2, \frac{1}{2}p_i^{\max}\right\}$

$$\dot{V} \le \frac{-\phi_1}{2\omega_1}V = -\alpha V \tag{23}$$

$$V(t) \le e^{-\alpha(t-t_k)}V(t_k), t\in[t_k, t_{k+1}) \tag{24}$$

Case 2: $r\in C_p$

$$\begin{aligned}
\dot{V} &= \dot{V}_1 + \dot{V}_2 + \dot{V}_3 \\
&\le -\tau\lambda_{\min}(\mathbf{h})\|\mathbf{e}_d\|^2 + \|\mathbf{e}_d\|\|\dot{d}(t)\| - \delta c_3\|\mathbf{e}\|^2 + \delta l_1\|\mathbf{e}\|\|\mathbf{y}\| \\
&\quad + c_4\|\mathbf{e}\|\|\mathbf{e}_d\| + 2\sqrt{N}\|P\|\max_{i\in\mathcal{I}}\{l_i\}\|\mathbf{y}\|^2 + 2N\|P\|\max_{i\in\mathcal{I}}\{l_i\}\|\mathbf{y}\|\|\mathbf{e}\| \\
&\quad + 2\sqrt{N}\|P\|\|\mathbf{y}\|\|\mathbf{e}_d\| \\
&\le -\left(\tau\lambda_{\min}(\mathbf{h}) - \frac{\varepsilon_3}{2}\right)\|\mathbf{e}_d\|^2 - \delta c_3\|\mathbf{e}\|^2 + \delta l_1\|\mathbf{e}\|\|\mathbf{y}\| + c_4\|\mathbf{e}\|\|\mathbf{e}_d\| \\
&\quad + 2\sqrt{N}\|P\|\max_{i\in v}\{l_i\}\|\mathbf{y}\|^2 + 2N\|P\|\max_{i\in v}\{l_i\}\|\mathbf{y}\|\|\mathbf{e}\| \\
&\quad + 2\sqrt{N}\|P\|\|\mathbf{y}\|\|\mathbf{e}_d\| + \frac{1}{2\varepsilon_3}\|\dot{d}(t)\|^2 \\
&= \mathbf{E}^T A\mathbf{E} + \frac{1}{2\varepsilon_3}\|\dot{d}(t)\|^2
\end{aligned} \tag{25}$$

where

$$A = \begin{bmatrix} -\delta c_3 & N\|P\|\max_{i\in v}\{l_i\} + \frac{\delta l_i}{2} & \frac{c_4}{2} \\ N\|P\|\max_{i\in v}\{l_i\} + \frac{\delta l_i}{2} & 2\sqrt{N}\|P\|\max_{i\in v}\{l_i\} & -\sqrt{N}\|P\| \\ \frac{c_4}{2} & -\sqrt{N}\|P\| & -\left(\tau\lambda_{\min}(\mathbf{h}) - \frac{\varepsilon_3}{2}\right) \end{bmatrix}$$

then, if

$$0 < \delta < \frac{\frac{c_4}{2}N\|P\|\max_{i\in v}\{l_i\}\left(\tau\lambda_{\min}(k) - \frac{\varepsilon_3}{2}\right) - c_3\left(\sqrt{N}\|P\|\right)^3 + c_4 N\|P\|\max_{i\in v}\{l_i\}\left(\sqrt{N}\|P\|\right)}{2c_3\sqrt{N}\|P\|\max_{i\in v}\{l_i\}\left(\tau\lambda_{\min}(\mathbf{h}) - \frac{\varepsilon_3}{2}\right)\frac{l_i c_4}{4}}$$

matrix A is symmetric positive definite.

$$\dot{V} \le \frac{\lambda_{\max}(A)}{\omega_1}V + \frac{1}{2\varepsilon_3}\|\mathbf{d}(t)\|^2 \tag{26}$$

when $\|\mathbf{E}\|^2\ 3\ \frac{1}{\phi_1\varepsilon_3}\left(\sqrt{N}\max\{\bar{d}_i\}\right)^2$,

$$\begin{aligned}
\dot{V} &\le \frac{\lambda_{\max}(A)}{\omega_1}V + \frac{1}{2\varepsilon_3}\left(\sqrt{N}\max\{\bar{d}_i\}\right)^2 \\
&\le \frac{\lambda_{\max}(A)\omega_2}{\omega_1}\|\mathbf{E}\|^2 + \frac{1}{2}\|\mathbf{E}\|^2 \\
&\le \frac{2\lambda_{\max}(A)\omega_2 + \omega_1}{2\omega_1^2}V = \beta V
\end{aligned} \tag{27}$$

$$V(t) \le e^{\beta(t-t_k)}V(t_k), t\in[t_k, t_{k+1}), r\in C_p \tag{28}$$

$$T_m(t_k, t) = t - t_k - T_p(t_k, t),\ \forall r\in C_m \cup C_p \tag{29}$$

$$V(t) \le e^{\beta T_p(t_k, t) - \alpha T_m(t_k, t)}V(t_k), t\in[t_k, t_{k+1}) \tag{30}$$

At each attack instant $t = t_k$, it is straightforward to deduce that $x_i(t_k) = x_i(t_k^-)$, since player actions cannot jump instantaneously. This implies that $V(t_k) \le V(t-k)$ then

$$V(t) \le e^{\beta T_p(t_k, t)} - \alpha^{T_m(t_k, t)}V(t_k^-) \tag{31}$$

if equation (13) in Theorem 1 holds, then we have

Thus, $V(t) \le e^{-x(t-t_0)}V(t_0)$, which implies that $\mathbf{x}(t)\to\mathbf{x}^*$ and $\mathbf{y}(t)\to 1_N\otimes\mathbf{x}^*$ as $\delta\in(0, \delta^*)$.

4.1.2 Quadratic game

Subsequently, we focus on a scenario characterised by quadratic objective functions for the players. We assume the cost functions for the participants are given by:

$$f_i(\mathbf{x}) = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}k_{jk}^i x_j x_k + \sum_{j=1}^{N}v_j^i x_j + g_i \tag{32}$$

Here, k_{jk}^i, v_j^i and g_i are the coefficients. Additionally, $k_{jk}^i < 0$ and $h_{jk}^i = h_{kj}^i$, , $\forall i, j, k \in \mathcal{I}$.

Assumption 5: The matrix

$$K=\begin{bmatrix} k_{11}^1 & k_{12}^1 & \cdots & k_{1N}^1 \\ k_{21}^2 & k_{22}^2 & \cdots & k_{2N}^2 \\ \vdots & & \ddots & \\ k_{N1}^N & k_{N2}^N & \cdots & k_{NN}^N \end{bmatrix}$$

is strictly diagonally dominant.

For the quadratic game, based on equations (9) and (32), the closed-loop system can be formulated this way:

$$\begin{aligned} &\dot{\mathbf{x}} = \delta\operatorname{diag}\{\bar{h}_i\}\left(K\mathbf{x}+v+\left(\bar{K}\mathbf{y}-\bar{K}\left(1_N\otimes\mathbf{x}\right)\right)\right)-\hat{\mathbf{d}}+\mathbf{d}(t) \\ &\dot{\mathbf{z}} = -\delta\operatorname{diag}\{\bar{h}_i\}\left(K\mathbf{x}+v+\left(\bar{K}\mathbf{y}-\bar{K}\left(1_N\otimes\mathbf{x}\right)\right)\right) \\ &\hat{\mathbf{d}} = \tau\mathbf{h}(\mathbf{x}+\mathbf{z}) \\ &\dot{\mathbf{y}} = --\gamma\bar{\gamma}\left(L\otimes I_{N\times N}+B_0\right)\mathbf{y}-B_0\left(\mathbf{1}_N\otimes\mathbf{x}\right) \end{aligned} \tag{33}$$

where $v=\left[v_1^1,v_2^2,\cdots v_N^N\right]^T$, $\bar{K}=\begin{bmatrix} k_1^T & \mathbf{0}_N^T & \cdots & \mathbf{0}_N^T \\ \mathbf{0}_N^T & k_2^T & \cdots & \mathbf{0}_N^T \\ \vdots & \mathbf{0}_N^T & \ddots & \vdots \\ \mathbf{0}_N^T & \mathbf{0}_N^T & \cdots & k_N^T \end{bmatrix}$, and

$k_i=\left[k_{i1}^i,k_{i2}^i,\cdots,k_{iN}^i\right]^T$.

Remark: Under Assumption 5, it is posited that the Nash equilibrium for the quadratic payoff game is both unique and exists, which is explicitly defined by the equation $\mathbf{x}^*= - K^{-1}v$ (Basar et al., 1999; Frihauf et al., 2012). This singular equilibrium facilitates the derivation of the ensuing non-local convergence results.

Theorem 2: Given that Assumptions 1, 2, and 5 hold, participants with quadratic payoffs update their strategies under equations (5) and (7) and estimate the state using equation (6). Under the influence of disturbances induced by DoS attacks, any pair ($\mathbf{x}$, $\mathbf{y}$) is shown to converge to ($\mathbf{x}^*$, $1_N\otimes\mathbf{x}^*$) as $t\to\infty$, the following conditions are satisfied:

$$\frac{T_p(t_0,t)}{t-t_0}\le\frac{\alpha-\xi}{\alpha+\beta} \tag{34}$$

$\alpha=\dfrac{\phi_1}{2\omega_1}$, $\beta=\dfrac{2\lambda_{\max}(A)\omega_2+\omega_1}{2\omega_1^2}$, in which the constant $\xi\in(0,\alpha)$, and the constant ϕ_1 is given by equation (19) below.

Proof: Define the Lyapunov function

$$V=V_1+V_2+V_3 \tag{35}$$

where, $V_1=\frac{1}{2}\mathbf{e}_d^T\mathbf{e}_d$, $V_2=\mathbf{e}^T\bar{P}\mathbf{e}$, $V_3=\bar{\mathbf{y}}^TP\bar{\mathbf{y}}$, is defined in Theorem 1, $\bar{P}$ is a symmetric positive definite matrix satisfying $P\left(\operatorname{diag}\{\bar{h}_i\}K\right)+\left(\operatorname{diag}\{\bar{h}_i\}K\right)^TP=-\bar{Q}$, and $\dot{V}_1$ is derived in the same manner as in Theorem 1.

$$\begin{aligned} \dot{V}_2 &= \dot{\mathbf{e}}^T\bar{P}\mathbf{e}+\mathbf{e}^T\bar{P}\dot{\mathbf{e}} \\ &=\left(\delta\operatorname{diag}\{\bar{h}_i\}\left(K\mathbf{e}+\bar{K}\bar{\mathbf{y}}+\mathbf{e}_d\right)\right)^T\bar{P}\mathbf{e} \\ &\quad+\mathbf{e}^T\bar{P}\delta\operatorname{diag}\{\bar{h}_i\}\left(K\mathbf{e}+\bar{K}\bar{\mathbf{y}}+\mathbf{e}_d\right) \\ &=\mathbf{e}^T\left(\delta\operatorname{diag}\{\bar{h}_i\}K\right)^T\bar{P}\mathbf{e}+\mathbf{e}^T\bar{P}\delta\operatorname{diag}\{\bar{h}_i\}K\mathbf{e} \\ &\quad+\delta\operatorname{diag}\{\bar{h}_i\}\left(\bar{K}\bar{\mathbf{y}}\right)^T\bar{P}\mathbf{e}+\mathbf{e}^T\bar{P}\delta\operatorname{diag}\{\bar{h}_i\}\left(\bar{K}\bar{\mathbf{y}}+\mathbf{e}_d\right) \\ &\quad+\delta\operatorname{diag}\{\bar{h}_i\}\mathbf{e}_d^T\bar{P}\mathbf{e} \\ &\le-\delta\lambda_{\min}(\bar{Q})\|\mathbf{e}\|^2+\delta\bar{l}_1\|\mathbf{e}\|\|\bar{\mathbf{y}}\|+\delta\bar{l}\,\|\mathbf{e}_d\|\|\mathbf{e}\| \end{aligned} \tag{36}$$

In this context, $\bar{l}_1$ is a positive constant that fulfills the inequality:

$$\left\|2\bar{\mathbf{y}}^TP\left(\mathbf{1}_N\otimes\operatorname{diag}\{\bar{h}_i\}\left(K\mathbf{e}+\bar{K}\bar{\mathbf{y}}-\mathbf{e}_d\right)\right)\right\|\le\bar{l}_2\left(\|\mathbf{e}\|\|\bar{\mathbf{y}}\|+\|\bar{\mathbf{y}}\|^2+\|\bar{\mathbf{y}}\|\|\mathbf{e}_d\|\right),$$

$$\begin{aligned} \dot{V}_3 &= \dot{\bar{\mathbf{y}}}^TP\bar{\mathbf{y}}+\bar{\mathbf{y}}^TP\dot{\bar{\mathbf{y}}} \\ &=\left[-\gamma\bar{\gamma}\left(L\otimes I_{N\times N}+B_0\right)\bar{\mathbf{y}}\right. \\ &\quad\left.+\mathbf{1}_N\otimes\delta\operatorname{diag}\left(K\mathbf{e}+\bar{K}\bar{\mathbf{y}}-\mathbf{e}_d\right)\right]^TP\bar{\mathbf{y}} \\ &\quad+\bar{\mathbf{y}}^TP\left[-\gamma\bar{\gamma}\left(L\otimes I_{N\times N}+B_0\right)\bar{\mathbf{y}}\right. \\ &\quad\left.+\mathbf{1}_N\otimes\delta\operatorname{diag}\left(K\mathbf{e}+\bar{K}\bar{\mathbf{y}}-\mathbf{e}_d\right)\right] \\ &\le-\gamma\bar{\gamma}\lambda_{\min}(Q)\|\bar{\mathbf{y}}\|^2+\delta\bar{l}_2\|\mathbf{e}\|\|\bar{\mathbf{y}}\|+\delta\bar{l}_2\|\bar{\mathbf{y}}\|^2 \\ &\quad+\delta\bar{l}_2\|\mathbf{e}_d\|\|\bar{\mathbf{y}}\| \end{aligned} \tag{37}$$

In this context, $\bar{l}_2$ is a positive constant that fulfills the inequality:

$$\left\|2\bar{\mathbf{y}}^TP\left(\mathbf{1}_N\otimes\operatorname{diag}\{\bar{h}_i\}\left(K\mathbf{e}+\bar{K}\bar{\mathbf{y}}-\mathbf{e}_d\right)\right)\right\|\le\bar{l}_2\left(\|\mathbf{e}\|\|\bar{\mathbf{y}}\|+\|\bar{\mathbf{y}}\|^2+\|\bar{\mathbf{y}}\|\|\mathbf{e}_d\|\right)$$

Case 1: $r\in C_m$

$$\begin{aligned} \dot{V} &= \dot{V}_1+\dot{V}_2+\dot{V}_3 \\ &\le-\tau\lambda_{\min}(\mathbf{h})\|\mathbf{e}_d\|^2+\|\mathbf{e}_d\|\|\dot{\mathbf{d}}(t)\|-\delta\lambda_{\min}(\bar{Q})\|\mathbf{e}\|^2 \\ &\quad+\delta\bar{l}_1\|\mathbf{e}\|\|\bar{\mathbf{y}}\|+\delta\bar{l}_1\|\mathbf{e}_d\|\|\mathbf{e}\|-\gamma\bar{\gamma}\lambda_{\min}(Q)\|\bar{\mathbf{y}}\|^2 \\ &\quad+\delta\bar{l}_2\|\mathbf{e}\|\|\bar{\mathbf{y}}\|+\delta\bar{l}_2\|\bar{\mathbf{y}}\|^2+\delta\bar{l}_2\|\mathbf{e}_d\|\|\bar{\mathbf{y}}\| \\ &\le-\left(\tau\lambda_{\min}(\mathbf{h})-\frac{\varepsilon_1}{2}-\frac{\delta\bar{l}_1\varepsilon_2}{2}-\frac{\delta\bar{l}_2\varepsilon_4}{2}\right)\|\mathbf{e}_d\|^2 \\ &\quad-\left(\delta\lambda_{\min}(\bar{Q})-\frac{\delta\left(\bar{l}_1+\bar{l}_2\right)\varepsilon_3}{2}-\frac{\delta\bar{l}_1}{2\varepsilon_2}\right)\|\mathbf{e}\|^2+\frac{1}{2\varepsilon_1}\|\dot{\mathbf{d}}(t)\|^2 \\ &\quad-\left(\gamma\bar{\gamma}\lambda_{\min}(Q)\delta\bar{l}_2-\frac{\delta\left(\bar{l}_1+\bar{l}_2\right)}{2\varepsilon_3}-\frac{\delta\bar{l}_2}{2\varepsilon_4}\right)\|\bar{\mathbf{y}}\|^2 \end{aligned} \tag{38}$$

where ε_1, ε_2, ε_3 are positive constants that remain to be determined.

For t^3t_0, the inequality $\|\dot{\mathbf{d}}(t)\|\le\sqrt{N}\max\{\bar{d}_i\}$ is consistently satisfied, by choosing ε_1 to be sufficiently large, it is possible to ensure that $\frac{1}{2\varepsilon_3}\|\dot{\mathbf{d}}(t)\|^2$ is sufficiently small.

The selection of ε_2 and ε_3 should satisfy the following condition:

$$\phi_1 = \delta\lambda_{min}\left(\bar{Q}\right) - \frac{\delta\left(\bar{l}_1+\bar{l}_2\right)\varepsilon_3}{2} - \frac{\delta\bar{l}_1}{2\varepsilon_2} > 0 \tag{39}$$

For the already chosen ε_2 and ε_3, there exists a suitable ε_4 such that for all γ the following condition is met:

$$\gamma\bar{\gamma}\lambda_{min}\left(\bar{Q}\right) - \delta\bar{l}_2\frac{\delta\left(\bar{l}_1+\bar{l}_2\right)}{2\varepsilon_3} - \frac{\delta\bar{l}_2}{2\varepsilon_4} > \phi_1 \tag{40}$$

For the already chosen ε_1, ε_2, ε_3, there exists a τ that satisfies:

$$\phi_3 = \tau\lambda_{min}\left(\mathbf{h}\right) - \frac{\varepsilon_1}{2} - \frac{\delta\bar{l}_1\varepsilon_2}{2} - \frac{\delta\bar{l}_2\varepsilon_4}{2} > \phi_1 \tag{41}$$

Upon rearrangement of $\dot{V}$, the following inequality is obtained:

$$\dot{V} \le -\phi_1\left\|\mathbf{E}\right\|^2 + \frac{1}{2\varepsilon_3}\left\|\dot{\mathbf{d}}(t)\right\|^2 \tag{42}$$

When the condition $\left\|\mathbf{E}\right\|^2 \ge \frac{1}{\phi_1\varepsilon_3}\left(\sqrt{N}\max\left\{\bar{d}_i\right\}\right)^2$ is satisfied, it follows that:

$$\dot{V} \le -\phi_1\left\|\mathbf{E}\right\|^2 \tag{43}$$

because of $\omega_1\left\|\mathbf{E}\right\|^2 \le V \le \omega_2\left\|\mathbf{E}\right\|^2$, we can obtain

$$\dot{V} \le \frac{\phi_1}{2\omega_1}V = -\alpha V \tag{44}$$

$$V(t) \le e^{-\alpha(t-t_k)}V(t_k), t\in\left[t_k, t_{k+1}\right) \tag{45}$$

Case 2: $r\in C_p$

$$\begin{aligned}
\dot{V} &= \dot{V}_1 + \dot{V}_2 + \dot{V}_3 \\
&\le -\tau\lambda_{\min}\left(\mathbf{h}\right)\left\|\mathbf{e}_d\right\|^2 + \left\|\mathbf{e}_d\right\|\left\|\dot{\mathbf{d}}\left(t\right)\right\| - \delta\lambda_{\min}\left(\bar{Q}\right)\left\|\mathbf{e}\right\|^2 \\
&\quad +\delta\left(\bar{l}_1+\bar{l}_2\right)\left\|\mathbf{e}\right\|\left\|\bar{\mathbf{y}}\right\| + \delta\bar{l}_1\left\|\mathbf{e}_d\right\|\left\|\mathbf{e}\right\| \\
&\quad +\delta\bar{l}_2\left\|\bar{\mathbf{y}}\right\|^2 + \delta\bar{l}_2\left\|\mathbf{e}_d\right\|\left\|\bar{\mathbf{y}}\right\| \\
&\le -\left(\tau\lambda_{\min}\left(\mathbf{h}\right) - \frac{\varepsilon_1}{2}\right)\left\|\mathbf{e}_d\right\|^2 - \delta\lambda_{\min}\left(\bar{Q}\right)\left\|\mathbf{e}\right\|^2 \\
&\quad +\delta\left(\bar{l}_1+\bar{l}_2\right)\left\|\mathbf{e}\right\|\left\|\bar{\mathbf{y}}\right\| + \delta\bar{l}_1\left\|\mathbf{e}_d\right\|\left\|\mathbf{e}\right\| + \delta\bar{l}_2\left\|\bar{\mathbf{y}}\right\|^2 \\
&\quad +\delta\bar{l}_2\left\|\mathbf{e}_d\right\|\left\|\bar{\mathbf{y}}\right\| + \frac{1}{2\varepsilon_1}\left\|\dot{d}\left(t\right)\right\|^2 \\
&= \mathbf{E}^T A\mathbf{E} + \frac{1}{2\varepsilon_1}\left\|\dot{d}\left(t\right)\right\|^2
\end{aligned} \tag{46}$$

where $A = \begin{bmatrix} -\delta\lambda_{\min}\left(\bar{Q}\right) & \delta\left(\bar{l}_1+\bar{l}_2\right)/2 & \delta\bar{l}_1/2 \\ \delta\left(\bar{l}_1+\bar{l}_2\right)/2 & \delta\bar{l}_2 & \delta\bar{l}_2/2 \\ \delta\bar{l}_1/2 & \delta\bar{l}_2/2 & -\left(\tau\lambda_{\min}\left(\mathbf{h}\right)-\frac{\varepsilon_1}{2}\right) \end{bmatrix}$

then, if $\delta^* = \frac{\lambda_{\min}(\bar{Q})\bar{l}_2\tau\lambda_{\min}(\mathbf{h}) - \frac{\lambda_{\min}(\bar{Q})\bar{l}_2\varepsilon_1}{2} - \frac{\left(\bar{l}_1+\bar{l}_2\right)}{2}\tau\lambda_{\min}(\mathbf{h})\lambda_{\min}(\bar{Q}) + \frac{\left(\bar{l}_1+\bar{l}_2\right)\varepsilon_1}{4}\lambda_{\min}(\bar{Q})}{\frac{\bar{l}_1\bar{l}_2}{4}\left(\lambda_{\min}(\bar{Q})+1\right) + \frac{\bar{l}_2^2}{4}\left(1-\lambda_{\min}(\bar{Q})\right)}$

matrix A is symmetric positive definite.

$$\dot{V} \le \frac{\lambda_{\max}\left(A\right)}{\omega_1}V + \frac{1}{2\varepsilon_1}\left\|\dot{\mathbf{d}}\left(t\right)\right\|^2 \tag{47}$$

When the condition $\left\|\mathbf{E}\right\|^2 \ge \frac{1}{\phi_1\varepsilon_3}\left(\sqrt{N}\max\left\{\bar{d}_i\right\}\right)^2$ is satisfied, the system's behaviour can be characterised by the following inequality:

$$\begin{aligned}
\dot{V} &\le \frac{\lambda_{\max}(A)}{\omega_1}V + \frac{1}{2\varepsilon_3}\left(\sqrt{N}\max\left\{\bar{d}_i\right\}\right)^2 \\
&\le \frac{\lambda_{\max}(A)\omega_2}{\omega_1}\left\|\mathbf{E}\right\|^2 + \frac{1}{2}\left\|\mathbf{E}\right\|^2 \\
&\le \frac{2\lambda_{\max}\left(A\right)\omega_2 + \omega_1}{2\omega_1^2}V = \beta V
\end{aligned} \tag{48}$$

$$V(t) \le e^{\beta(t-t_k)}V\left(t_k\right), t\in\left[t_k, t_{k+1}\right), r\in C_p \tag{49}$$

$$T_m\left(t_k, t\right) = t - t_k - T_p\left(t_k, t\right), r\in C_m \cup C_p \tag{50}$$

$$\dot{V}(t) \le e^{\beta T_p(t_k,t)-\alpha T_m(t_k,t)}V(t_k),\ t\in\left[tk, t_{k+1}\right) \tag{51}$$

For every attack instant $t = t_k$, it holds that $x_i\left(t_k\right)=x_i\left(t_k\right)$, and the player's actions cannot instantaneously jump, which implies that $V(t_k) \le V(t-k)$, then

$$\begin{aligned}
V(t) &\le e^{\beta T_p(t_k,t)-\alpha T_m(t_k,t)}V\left(t_k^-\right) \\
&\le e^{\beta T_p(t_k,t)-\alpha T_m(t_k,t)}V\left(t_{k-1}^-\right) \\
&\le e^{\beta T_p(t_k,t)-\alpha T_m(t_k,t)}V\left(t_{k-0}^-\right) \\
&\le \ldots \le e^{\beta T_p(t_k,t)-\alpha T_m(t_k,t)}V\left(t_0\right)
\end{aligned} \tag{52}$$

If equation (12) in Theorem 1 holds, then we have: βT_p $(t_0, t)-\alpha T_m \le -\xi(t, t_0)$. Consequently, the conclusion is: $V(t) \le e^{-\xi(t, t0)}V(t_0)$, this implies that as $\mathbf{x}(t)\to\mathbf{x}^*$ and $\mathbf{y}(t)\to 1_N\otimes\mathbf{x}$ as $t\to\infty$.

4.2 Nash equilibrium of second-order systems considering stochastic disturbances under DoS attacks

Considering the second-order participants in the game are subject to time-varying disturbances, the specific expression is given by:

$$\begin{aligned}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= u_i + d_i(t), i\in\mathcal{I}
\end{aligned} \tag{53}$$

Here, $v_i\in\mathbb{R}$ represents the velocity-like variable for participant i. The state update of the participants follows equation (53), where the control input is designed as follows:

$$u_i = -\hat{d}_i + \tau_2 h_{i2}\left(vi + \nabla_i f_i\left(\mathbf{y}_i\right)\right), i \in \mathcal{I} \tag{54}$$

In this equation, $\hat{d}_i$ denotes the observed value of the time-varying disturbance $d_i(t)$ for participant i, τ_2 is an adjustable positive parameter, and k_{i2} is a fixed positive parameter. Additionally, $\hat{d}_i$ and y_{ij} are updated through the following differential equations:

$$\begin{aligned} \hat{d}_i &= -\tau_1 h_{i1}\left(\tau_2 h_{i2} z_i + v_i\right) \\ \dot{y}_{ij} &= -\gamma \overline{\gamma}_{ij}\left(\sum_{k=1}^{N} a_{ik}\left(y_{ij} - y_{kj}\right) + a_{ij}\left(y_{ij} - x_j\right)\right) \end{aligned} \tag{55}$$

Here, τ_1 is an adjustable positive parameter, k_{i1} is a fixed positive parameter, and z_i is an auxiliary variable. Additionally, the update of z_i is based on the following formula:

$$\dot{z}_i = v_i + \nabla_i f_i\left(\mathbf{y}_i\right) \tag{56}$$

Based on equations (4), (9), and (10), the closed-loop system is:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\hat{\mathbf{d}} + \tau_2 \mathbf{h}_2(\mathbf{v} + \psi(\mathbf{y})) + \mathbf{d}(t) \\ \dot{\mathbf{z}} &= \mathbf{v} + \psi(\mathbf{y}) \\ \hat{\mathbf{d}} &= \tau_1 \mathbf{h}_1\left(\tau_2 \mathbf{h}_2 \mathbf{z} + \mathbf{v}\right) \\ \dot{\mathbf{y}} &= -\gamma \overline{\gamma}\left(\left(L \otimes I_{N\times N} + B_0\right)\mathbf{y} - B_0\left(\mathbf{1}_N \otimes \mathbf{x}\right)\right) \end{aligned} \tag{57}$$

where $\mathbf{h}_1 = \mathrm{diag}\{h_{i1}\}$, $\mathbf{h}_2 = \mathrm{diag}\{h_{i2}\}$, $\boldsymbol{\gamma} = \mathrm{diag}\left\{\overline{\gamma}_{ij}\right\}$, $B_0 = \mathrm{diag}\{a_{ij}\}$, $z = [z_i]_{vec}$ and $\mathbf{y} = \left[\mathbf{y}_1^T, \mathbf{y}_2^T, \cdots, \mathbf{y}_N^T\right]^T$.

4.2.1 Non-quadratic game

We will next demonstrate that the strategy seeking, which is based on the control input (34), disturbance estimation (37), and state estimation update (33), leads to the actions of the participants converging to the Nash equilibrium of the non-quadratic game.

Theorem 3: Suppose that Assumptions 1–3 hold. In the non-quadratic utility game, participants estimate their states according to equations (53), (54), and (55). For each $\mathbf{x}^*$ satisfying Assumptions 3–4, under DoS attacks and considering disturbances, the pair ($\mathbf{x}^*$, $1_N \otimes \mathbf{x}^*$) is exponentially stable if the following condition is met:

$$\frac{T_p\left(t_0, t\right)}{t - t_0} \leq \frac{\alpha - \xi}{\alpha + \beta} \tag{58}$$

where $\alpha = \frac{\phi_1}{2\omega_1}$, $\beta = \frac{2\lambda_{\max}(A)\omega_2 + \omega_1}{2\omega_1^2}$, and $\xi \in (0, \alpha)$, with the constant ϕ_1 to be specified in equation (69) below.

Proof: We define an auxiliary system as follows:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= \overline{h}_i \frac{\partial f_i(\mathbf{x})}{\partial x_i} - \hat{d}_i + d_i(t), i \in \mathcal{I} \end{aligned} \tag{59}$$

Under assumptions 2 and 3, and for each $\mathbf{x}^*$ satisfying assumption 4, there exists a function $\tilde{M}_1 : G_0 \to R$, where $G_0 = \{\mathbf{e}_v \in R^N \mid \|\mathbf{e}_v\| \leq \rho_0\}$, $\mathbf{e}_v = v + \psi(\mathbf{y})$ for some positive constant ρ_0, it holds that:

$$\begin{aligned} & c_1 \left\|\mathbf{e}_v\right\|^2 \leq \tilde{M}_1\left(\mathbf{e}_v\right) \leq c_2 \left\|\mathbf{e}_v\right\|^2 \\ & \left(\frac{\partial M_1\left(\mathbf{e}_v\right)}{\partial \mathbf{e}_v}\right)^T (\mathbf{v} + \psi(\mathbf{y})) \leq -c_3 \left\|\mathbf{e}_v\right\|^2 \\ & \left\|\frac{\partial M_1\left(\mathbf{e}_v\right)}{\partial \mathbf{e}_v}\right\| \leq c_4 \left\|\mathbf{e}_v\right\| \end{aligned} \tag{60}$$

where c_1, c_2, c_3, and c_4 are some positive constants.

Definition of Lyapunov function

$$V = V_1 + V_2 + V_3 + V_4 \tag{61}$$

where, $V_1 = \frac{1}{2}\mathbf{e}_d^T \mathbf{e}_d$, $V_2 = W_1(\mathbf{e}_v)$, $V_3 = \overline{\mathbf{y}}^T P \overline{\mathbf{y}}$, $V_4 = \frac{1}{2}\mathbf{e}^T \mathbf{e}$, for the second-order participants, the error signal is defined as $\mathbf{E} = \left[\mathbf{e}^T, \overline{\mathbf{y}}^T, \mathbf{e}_v{}^T, \mathbf{e}_d^T\right]$, where $\mathbf{e} = \mathbf{x} - \mathbf{x}^*$, $\overline{\mathbf{y}} = \mathbf{y} - 1_N \otimes \mathbf{x}$, $\mathbf{e}_v = \mathbf{v} + \psi(\mathbf{y})$, $\mathbf{e}_d = \hat{\mathbf{d}} - \mathbf{d}(t)$.

Then, the Lyapunov Function can be derived that $\omega_1\|\mathbf{E}\|^2 \leq V \leq \omega_2\|\mathbf{E}\|^2$, $\omega_1 = \min\left\{\frac{1}{2}, c_1, p_i^{\min}\right\}$, $\omega_2 = \min\left\{\frac{1}{2}, c_2, p_i^{\max}\right\}$, $p_i^{\min} = \min\left\{p_i\right\}$, $p_i^{\max} = \max\left\{p_i\right\}$.

Furthermore, for the second-order system, there exists a domain $\overline{G} = \left\{\mathbf{E} \in R^{N+N^2} \,\middle|\, \|\mathbf{E}\| \leq \rho_1\right\}$, where $\rho_1 > 0$ $\rho_1 > 0$, for $t \in [t_k, t_{k+1})$, $\forall\, k \in Z_+$, it follows that:

$$\begin{aligned} \dot{V}_1 = \mathbf{e}_d^T \dot{\mathbf{e}}_d &= \mathbf{e}_d^T\left(\tau_2 \mathbf{h}_2 \dot{\mathbf{z}} + \dot{\mathbf{v}} - \dot{\mathbf{d}}(t)\right) \\ &= \mathbf{e}_d^T\left(-\tau_1 \mathbf{h}_1 \mathbf{e}_d - \dot{\mathbf{d}}(t)\right) \\ &\leq -\tau_1 \lambda_{\min}\left(\mathbf{h}_1\right)\left\|\mathbf{e}_d\right\|^2 + \left\|\mathbf{e}_d\right\|\left\|\dot{\mathbf{d}}(t)\right\| \end{aligned} \tag{62}$$

$$\begin{aligned} \dot{V}_2 = & \left(\frac{\partial W_1\left(\mathbf{e}_v\right)}{\partial \mathbf{e}_v}\right)^T \left(-\tau_2 \mathbf{h}_2\left(\mathbf{v} + \psi\left(\nabla_i f_i\left(\mathbf{y}_i\right)\right)\right)\right) \\ & -\mathbf{e}_d - \gamma K(\mathbf{y})\overline{\gamma}\left(L \otimes I_{N\times N} + B_0\right)\left(\mathbf{y} - \mathbf{1}_N \otimes \mathbf{x}\right)\Big) \end{aligned} \tag{63}$$

where $K(y) = [k_{ij}]$, if i # j, then $k_{ij} = \mathbf{0}_N^T$, and if $i = j$, then $k_{ij} = \left[\nabla_{i1}^2 f_i\left(\mathbf{y}_i\right), \nabla_{i2}^2 f_i\left(\mathbf{y}_i\right), \cdots, \nabla_{iN}^2 f_i\left(\mathbf{y}_i\right)\right]$, with $\nabla_{i2}^2 f_i\left(\mathbf{y}_i\right) = \frac{\partial^2 f_i(\mathbf{x})}{\partial x_i \partial x_j}|\mathbf{x} = \mathbf{y}_i$. By assumption 4, there exists a positive constant k such that for all $\mathbf{y} \in \mathbb{R}^{N^2}$, it holds that $\left\|K\left(\mathbf{y}\right)\right\|\left\|\overline{\boldsymbol{\gamma}}\left(L \otimes I_{N\times N} + B_0\right)\right\| \leq k$.

Further calculations for $\dot{V}_2$ yield:

$$\begin{aligned}\dot{V}_2 &\le -\tau_2\lambda_{\min}(\mathbf{h}_2)\left(\frac{\partial M_1(\mathbf{e}_v)}{\partial \mathbf{e}_v}\right)^T(\mathbf{v}+\psi(\mathbf{y}))\\ &\quad -\gamma k\left(\frac{\partial M_1(\mathbf{e}_v)}{\partial \mathbf{e}_v}\right)^T(\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x})-\left(\frac{\partial M_1(\mathbf{e}_v)}{\partial \mathbf{e}_v}\right)^T\mathbf{e}_d \\ &\le -c_3\tau_2\lambda_{\min}(\mathbf{h}_2)\|\mathbf{e}_v\|^2+\gamma kc_4\|\mathbf{e}_v\|\|\bar{\mathbf{y}}\|+c_4\|\mathbf{e}_v\|\|\mathbf{e}_d\|\end{aligned} \tag{64}$$

$$\begin{aligned}\dot{V}_3 &= (\dot{\mathbf{y}}-\mathbf{1}_N\otimes\dot{\mathbf{x}})^T P(\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x})+(\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x})^T P(\dot{\mathbf{y}}-\mathbf{1}_N\otimes\dot{\mathbf{x}})\\ &= -\gamma(\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x})^T Q(\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x})+2(\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x})^T\\ &\quad \times P\mathbf{1}_N\otimes\psi(\mathbf{y})-2(\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x})^T P\mathbf{1}_N\otimes(\mathbf{v}+\psi(\mathbf{y}))\\ &\le -\left(\gamma\lambda_{\min}(Q)-2\sqrt{N}\|P\|\max_{i\in v}\{l_i\}\right)\|\bar{\mathbf{y}}\|^2+2\sqrt{N}\|P\|\|\bar{\mathbf{y}}\|\|\mathbf{e}_v\|\\ &\quad +2N\|P\|\max_{i\in v}\{l_i\}\|\bar{\mathbf{y}}\|\|\mathbf{e}\|\end{aligned} \tag{65}$$

$$\begin{aligned}\dot{V}_4 &= (\mathbf{x}-\mathbf{x}^*)^T((\mathbf{v}+\psi(\mathbf{y}))-\psi(\mathbf{y}))\\ &\le -m\|\mathbf{e}\|^2+\|\mathbf{e}\|\|\mathbf{e}_v\|+\max_{i\in v}\{l_i\}\|\mathbf{e}\|\|\bar{\mathbf{y}}\|\end{aligned} \tag{66}$$

Case 1: $r\in C_m$

According to the definition of V, the time derivative of V is given by:

$$\begin{aligned}\dot{V} &= \dot{V}_1+\dot{V}_2+\dot{V}_3+\dot{V}_4\\ &\le -\tau_1\lambda_{\min}(\mathbf{h}_1)\|\mathbf{e}_d\|^2+\|\mathbf{e}_d\|\|\dot{\mathbf{d}}(t)\|-c_3\tau_2\lambda_{\min}(\mathbf{h}_2)\|\mathbf{e}_v\|^2\\ &\quad +\gamma kc_4\|\mathbf{e}_v\|\|\bar{\mathbf{y}}\|+c_4\|\mathbf{e}_v\|\|\mathbf{e}_d\|\\ &\quad -\left(\gamma\lambda_{\min}(Q)-2\sqrt{N}\|P\|\max_{i\in v}\{l_i\}\right)\|\bar{\mathbf{y}}\|^2\\ &\quad +2\sqrt{N}\|P\|\|\bar{\mathbf{y}}\|\|\mathbf{e}_v\|+2N\|P\|\max_{i\in v}\{l_i\}\|\bar{\mathbf{y}}\|\|\mathbf{e}\|\\ &\quad -m\|\mathbf{e}\|^2+\|\mathbf{e}\|\|\mathbf{e}_v\|+\max_{i\in v}\{l_i\}\|\mathbf{e}\|\|\bar{\mathbf{y}}\|\end{aligned} \tag{67}$$

It is noted that for $\mathbf{x}$, $\mathbf{z}\in\mathbb{R}^N$, the inequality $\|\mathbf{x}\|\|\mathbf{z}\|\le\frac{\varepsilon}{2}\|\mathbf{x}\|^2+\frac{1}{2\varepsilon}\|\mathbf{z}\|^2$ holds, where $\varepsilon>0$, thus, $\dot{V}$ can be further organised as follows:

$$\begin{aligned}\dot{V} &\le -\left(\gamma\lambda_{\min}(Q)-2\sqrt{N}\|P\|\max_{i\in v}\{l_i\}-\frac{2\sqrt{N}\|P\|+\gamma kc_4}{2\varepsilon_2}\right.\\ &\qquad \left.-\frac{\left(\max_{i\in v}\{l_i\}+2N\|P\|\max_{i\in v}\{l_i\}\right)\varepsilon_4}{2}\right)\|\bar{\mathbf{y}}\|^2\\ &\quad -\left(\tau_2\lambda_{\min}(\mathbf{h}_2)-\frac{\left(2\sqrt{N}\|P\|+\gamma kc_4\right)\varepsilon_2}{2}-\frac{c_4\varepsilon_1}{2}-\frac{\varepsilon_5}{2}\right)\|\mathbf{e}_v\|^2\\ &\quad -\left(\tau_1\lambda_{\min}(\mathbf{h}_1)-\frac{\varepsilon_3}{2}-\frac{c_4}{2\varepsilon_1}\right)\|\mathbf{e}_d\|^2\\ &\quad -\left(m-\frac{\max_{i\in v}\{l_i\}+2N\|P\|\max_{i\in v}\{l_i\}}{2\varepsilon_4}-\frac{1}{2\varepsilon_5}\right)\|\mathbf{e}\|^2+\frac{1}{2\varepsilon_3}\|\dot{\mathbf{d}}(t)\|^2\end{aligned} \tag{68}$$

where ε_1, ε_2, ε_3, ε_4, ε_5 are positive constants to be determined.

It is noted that for $t\ge t_0$, it always holds that $\|\dot{\mathbf{d}}(t)\|\le\sqrt{N}\max\{\bar{d}_i\}$, by choosing ε_3 sufficiently large, we can ensure that $\frac{1}{2\varepsilon_3}\|\dot{\mathbf{d}}(t)\|^2$ is sufficiently small.

The selection of ε_4 and ε_5 should satisfy

$$\phi_1 = m-\frac{\max_{i\in v}\{l_i\}+2N\|P\|\max_{i\in v}\{l_i\}}{2\varepsilon_3}-\frac{1}{2\varepsilon_5}>0 \tag{69}$$

For the already chosen ε_2 and ε_4, there always exists a γ that satisfies

$$\begin{aligned}\phi_2 &= \gamma\lambda_{\min}(Q)-2\sqrt{N}\|P\|\max_{i\in v}\{l_i\}-\frac{2\sqrt{N}\|P\|+\gamma kc_4}{2\varepsilon_2}\\ &\quad -\frac{\left(\max_{i\in v}\{l_i\}+2N\|P\|\max_{i\in v}\{l_i\}\right)\varepsilon_4}{2}>\phi_1\end{aligned} \tag{70}$$

For the already chosen ε_1, ε_3, there always exists a τ_1 that satisfies

$$\phi_3 = \tau_1\lambda_{\min}(\mathbf{h}_1)-\frac{\varepsilon_3}{2}-\frac{c_4}{2\varepsilon_1}>\phi_1 \tag{71}$$

For the already chosen ε_1, ε_2, ε_5 and γ, there always exists a τ_2 that satisfies

$$\phi_4 = \tau_2\lambda_{\min}(\mathbf{h}_2)-\frac{\left(2\sqrt{N}\|P\|+\gamma kc_4\right)\varepsilon_2}{2}-\frac{c_4,\varepsilon_1}{2}-\frac{\varepsilon_5}{2}>\phi_1 \tag{72}$$

After organising $\dot{V}$, we ultimately obtain

$$\dot{V}\le-\phi_1\|\mathbf{E}\|^2+\frac{1}{2\varepsilon_3}\|\dot{\mathbf{d}}(t)\|^2 \tag{73}$$

When $\|\mathbf{E}\|^2\ge\frac{1}{\phi_1\varepsilon_3}\left(\sqrt{N}_{\max}\{\bar{d}_i\}\right)^2$, it follows that

$$\dot{V}\le-\phi_1\|\mathbf{E}\|^2 \tag{74}$$

because of $\omega_1\|\mathbf{E}\|^2\le\omega_2\|\mathbf{E}\|^2$, we can obtain

$$\dot{V}\le-\frac{\phi_1}{2\omega_1}V=-\alpha V \tag{75}$$

$$V(t)\le e^{-\alpha(t-t_k)}V(t_k),\ t\in[t_k,t_{k+1}) \tag{76}$$

Case 2: $r\in C_p$

$$\begin{aligned}\dot V &= \dot V_1+\dot V_2+\dot V_3+\dot V_4\\ &\le -\left(\tau_1\lambda_{\min}(\mathbf{h}_1)-\frac{\varepsilon_3}{2}\right)\|\mathbf{e}_d\|^2-c_3\tau_2\lambda_{\min}(\mathbf{h}_2)\|\mathbf{e}_v\|^2\\ &+\left(\gamma kc_4+2\sqrt{N}\|P\|\right)\|\mathbf{e}_v\|\|\bar{\mathbf{y}}\|+c_4\|\mathbf{e}_v\|\|\mathbf{e}_d\|\\ &2\sqrt{N}\|P\|\max_{i\in v}\{l_i\}\|\bar{\mathbf{y}}\|^2+(2N\|P\|+1)\max_{i\in V}\{l_i\}\|\bar{\mathbf{y}}\|\|\mathbf{e}\|\\ &-m\|\mathbf{e}\|^2+\|\mathbf{e}\|\|\mathbf{e}_v\|+\frac{1}{2\varepsilon_3}\|\dot{\mathbf{d}}(t)\|^2\end{aligned}\tag{77}$$

where

$$B=\begin{bmatrix}-m & \frac{(2N\|P\|+1)\max_{i\in v}\{l_i\}}{2} & \frac{1}{2} & 0\\ \frac{(2N\|P\|+1)\max_{i\in v}\{l_i\}}{2} & 2\sqrt{N}\|P\|\max_{i\in v}\{l_i\} & \frac{(2\sqrt{N}\|P\|+\gamma kc_4)}{2} & 0\\ \frac{1}{2} & \frac{(2\sqrt{N}\|P\|+\gamma kc_4)}{2} & -c_3\tau_2\lambda_{\min}(\mathbf{h}_2) & \frac{c_4}{2}\\ 0 & 0 & \frac{c_4}{2} & \frac{\tau_1\lambda_{\min}(\mathbf{h}_1)-\varepsilon_3}{2}\end{bmatrix}$$

$$\det(B)=-2m\sqrt{N}\|P\|\max_{i\in v}\{l_i\}\times\left(c_3\tau_2\lambda_{\min}(h_2)\left(\tau_1\lambda_{\min}(h_1)-\frac{\varepsilon_3}{2}\right)-\frac{c_4^2}{4}\right)$$

By appropriately selecting τ_1 and τ_2, it is possible to ensure that matrix B is symmetric positive definite.

$$\dot V\le\frac{\lambda_{\max}(A)}{\omega_1}V+\frac{1}{2\varepsilon_3}\|\dot d(t)\|^2\tag{78}$$

when $\|\mathbf{E}\|^2\ge\frac{1}{\varepsilon_3}\left(\sqrt{N}\max\{\bar d_i\}\right)^2$,

$$\begin{aligned}\dot V&\le\frac{\lambda_{\max}(B)}{\omega_1}V+\frac{1}{2\varepsilon_3}\left(\sqrt{N}\max\{\bar d_i\}\right)^2\\ &\le\frac{\lambda_{\max}(B)\omega_2}{\omega_1}\|\mathbf{E}\|^2+\frac{1}{2}\|\mathbf{E}\|^2\\ &\le\left(\frac{\lambda_{\max}(A)\omega_2+\omega_1}{2\omega_1^1}\right)V=\beta V\end{aligned}\tag{79}$$

$$V(t)\le e^{\beta(t-t_k)}V(t_k),t\in[t_k,t_{k+1})\tag{80}$$

$$T_m(t_k,t)=t-t_k-T_p(t_k,t),r\in C_m\cup C_p\tag{81}$$

$$V(t)\le e^{\beta T_p(t_k,t)-\alpha T_m(t_k,t)}V(t_k)\ t\in[t_k,t_{k+1})\tag{82}$$

At each attack instant $t=t_k$, it is evident that $x_i(t_k)=x_i(t_k^-)$, because the player's actions cannot instantaneously jump, which implies that $V(t_k)\le V(t-k)$, then

$$\begin{aligned}V(t)&\le e^{\beta T_p(tk,t)-\alpha T_m(tk,t)}V(t_k^-)\\ &\le e^{\beta T_p(tk,t)-\alpha T_m(tk,t)}V(t_{k-1}^-)\\ &\le e^{\beta T_p(tk,t)-\alpha T_m(tk,t)}V(t_{k-1}^-)\\ &\le\ldots\le e^{\beta T_p(tk,t)-\alpha T_m(tk,t)}V(t_0)\end{aligned}\tag{83}$$

If Theorem 3 (58) holds, then $\beta T_p(t_0,t)-\alpha T_m(t_0,t)\le-\xi(t_0,t)$, which implies that $\mathbf{x}(t)\to\mathbf{x}^*$ and $\mathbf{y}(t)\to 1_N\otimes\mathbf{x}$ as $t\to\infty$.

4.2.2 Quadratic game

Utilising the objective function from equation (31), the closed-loop system can be expressed as:

$$\begin{aligned}\dot{\mathbf{x}}&=\mathbf{v}\\ \dot{\mathbf{v}}&=-\hat{\mathbf{d}}+\tau_2\mathbf{h}_2\left(\mathbf{v}+K\mathbf{x}+v+\left(\bar K\mathbf{y}-\bar K(1_N\otimes\mathbf{x})\right)\right)+\mathbf{d}(t)\\ \dot{\mathbf{z}}&=\mathbf{v}+\psi\left(K\mathbf{x}+v+\left(\bar K\mathbf{y}-\bar K(1_N\otimes\mathbf{x})\right)\right)\\ \dot{\hat{\mathbf{d}}}&=\tau_1\mathbf{h}_1(\tau_2\mathbf{h}_2\mathbf{z}+\mathbf{v})\\ \dot{\mathbf{y}}&=-\gamma\bar\gamma\left((L\otimes I_{N\times N}+B_0)\mathbf{y}-B_0(\mathbf{1}_N\otimes\mathbf{x})\right)\end{aligned}\tag{84}$$

Theorem 4: Under Assumptions 1–3, the Nash equilibrium of the game under DoS attacks can be achieved through strategy (53), (54), and (55). If the following condition is satisfied:

$$\frac{T_p(t0,t)}{t-t_0}\le\frac{\alpha-\xi}{\alpha+\beta}\tag{85}$$

where $\alpha=\frac{\phi_1}{2\omega_1}$, $\beta=\frac{2\lambda_{\max}(A)\omega_2+\omega_1}{2\omega_1^2}$, and the constant $\xi\in(0,\alpha)$, the constant ϕ_1 to be specified in equation (92) below.

Proof: Construct a Lyapunov function:

$$V=V_1+V_2+V_3+V_4$$

where $V_1=\frac{1}{2}\mathbf{e}_d^T\mathbf{e}_d$, $V_2=\frac{1}{2}\left(\mathbf{v}+\psi\left(H\mathbf{e}+\bar H\bar{\mathbf{y}}\right)\right)^T\left(\mathbf{v}+\psi\left(H\mathbf{e}+\bar H\bar{\mathbf{y}}\right)\right)$, $V_3=\bar{\mathbf{y}}^TP\mathbf{y}$, $V_4=\frac{1}{2}\mathbf{e}^T\mathbf{e}$

then, it follows that $\dot V_1$ is under Theorem 3.

$$\begin{aligned}\dot V_2&=\left(\mathbf{v}+\psi\left(K\mathbf{e}+\bar K\right)\right)^T\left(-\tau_2\mathbf{h}_2\left(\mathbf{v}+\psi\left(K\mathbf{e}+\bar K\bar{\mathbf{y}}\right)\right)\right.\\ &\left.-\mathbf{e}_d-\gamma K(\mathbf{y})\bar\gamma(L\otimes I_{N\times N}+B_0)(\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x})\right)\end{aligned}\tag{86}$$

where $K(\mathbf{y})=[k_{ij}]$, if i # j, then $k_{ij}=\mathbf{0}_N^T$, and if $i=j$, then $k_{ij}=\left[\nabla_{i1}^2f_i(\mathbf{y}_i),\nabla_{i2}^2f_i(\mathbf{y}_i),\cdots,\nabla_{iN}^2f_i(\mathbf{y}_i)\right]$, with $\nabla_{i2}^2f_i(\mathbf{y}_i)=\frac{\partial^2f_i(\mathbf{x})}{\partial x_i\partial x_j}|\mathbf{x}=\mathbf{y}_i$. By assumption 4, there exists $h>0$ such that for all $\mathbf{y}\in\mathbb{R}^{N^2}$, it holds that $\|K(\mathbf{y})\|\|\bar\gamma(L\otimes I_{N\times N}+B_0)\|\le k$.

Further calculations for $\dot V^2$ yield:

$$\begin{aligned}\dot{V}_2 &\le -\tau_2\lambda_{\min}(\mathbf{h}_2)\left\|\mathbf{v}+\left[K\mathbf{e}+\bar{K}\bar{\mathbf{y}}\right]_{vec}\right\|^2 \\ &\quad -\gamma k\left\|\mathbf{v}+\psi\left(K\mathbf{e}+\bar{K}\bar{\mathbf{y}}\right)\right\|\left\|\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x}\right\| \\ &\quad +\left\|\mathbf{v}+\psi\left(K\mathbf{e}+\bar{K}\bar{\mathbf{y}}\right)\right\|\left\|\mathbf{e}_d\right\| \\ &= -\tau_2\lambda_{\min}(\mathbf{h}_2)\|\mathbf{e}_v\|^2-\gamma k\|\mathbf{e}_v\|\|\bar{\mathbf{y}}\|+\|\mathbf{e}_v\|\|\mathbf{e}_d\|\end{aligned} \tag{87}$$

$$\begin{aligned}\dot{V}_3 &= \left(\dot{\mathbf{y}}-\mathbf{1}_N\otimes\dot{\mathbf{x}}\right)^T P\left(\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x}\right) \\ &\quad +\left(\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x}\right)^T P\left(\dot{\mathbf{y}}-\mathbf{1}_N\otimes\dot{\mathbf{x}}\right) \\ &= -\gamma\left(\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x}\right)^T Q\left(\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x}\right) \\ &\quad +2\left(\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x}\right)^T P\mathbf{1}_N\otimes\psi\left(K\mathbf{e}+\bar{K}\bar{\mathbf{y}}\right) \\ &\quad -2\left(\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x}\right)^T P\mathbf{1}_N\otimes\left(\mathbf{v}+\psi\left(K\mathbf{e}+\bar{K}\bar{\mathbf{y}}\right)\right) \\ &\le -\left(\gamma\lambda_{\min}(Q)-2\sqrt{N}\|P\|\max_{i\in v}\{l_i\}\right)\left\|\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x}\right\|^2 \\ &\quad +2\sqrt{N}\|P\|\left\|\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x}\right\|\left\|\mathbf{v}+\psi\left(K\mathbf{e}+\bar{K}\bar{\mathbf{y}}\right)\right\| \\ &\quad +2N\|P\|\max_{i\in v}\{l_i\}\left\|\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x}\right\|\left\|\mathbf{x}-\mathbf{x}^*\right\| \\ &= -\left(\gamma\lambda_{\min}(Q)-2\sqrt{N}\|P\|\max_{i\in v}\{l_i\}\right)\|\bar{\mathbf{y}}\|^2 \\ &\quad +2\sqrt{N}\|P\|\|\bar{\mathbf{y}}\|\|\mathbf{e}_v\|+2N\|P\|\max_{i\in v}\{l_i\}\|\bar{\mathbf{y}}\|\|\mathbf{e}\|\end{aligned} \tag{88}$$

$$\begin{aligned}\dot{V}_4 &= \left(\mathbf{x}-\mathbf{x}^*\right)^T\left(\left(\mathbf{v}+\psi(\mathbf{y})\right)-\psi(\mathbf{y})\right) \\ &\le -m\left\|\mathbf{x}-\mathbf{x}^*\right\|^2+\left\|\mathbf{x}-\mathbf{x}^*\right\|\left\|\mathbf{v}+\psi(\mathbf{y})\right\| \\ &\quad +\max_{i\in v}\{l_i\}\left\|\mathbf{x}-\mathbf{x}^*\right\|\left\|\mathbf{y}-\mathbf{1}_N\otimes\mathbf{x}\right\| \\ &= -m\|\mathbf{e}\|^2+\|\mathbf{e}\|\|\mathbf{e}_v\|+\max_{i\in v}\{l_i\}\|\mathbf{e}\|\|\bar{\mathbf{y}}\|\end{aligned} \tag{89}$$

Case 1: $r\in C_m$

According to the definition of V, the time derivative of V is given by:

$$\begin{aligned}\dot{V} &= \dot{V}_1+\dot{V}_2+\dot{V}_3+\dot{V}_4 \\ &\le -\tau_1\lambda_{\min}(\mathbf{h}_1)\|\mathbf{e}_d\|^2+\|\mathbf{e}_d\|\|\dot{\mathbf{d}}(t)\|-\tau_2\lambda_{\min}(\mathbf{h}_2)\|\mathbf{e}_v\|^2 \\ &\quad +\left(2\sqrt{N}\|P\|-\gamma k\right)\|\mathbf{e}_v\|\|\bar{\mathbf{y}}\|+\|\mathbf{e}_v\|\|\mathbf{e}_d\| \\ &\quad -\left(\gamma\lambda_{\min}(Q)-2\sqrt{N}\|P\|\max_{i\in v}\{l_i\}\right)\|\bar{\mathbf{y}}\|^2 \\ &\quad +\left(2N\|P\|\max_{i\in v}\{l_i\}+\max_{i\in v}\{l_i\}\right)\|\bar{\mathbf{y}}\|\|\mathbf{e}\|-m\|\mathbf{e}\|^2+\|\mathbf{e}\|\|\mathbf{e}_v\|\end{aligned} \tag{90}$$

It is noted that for $\mathbf{x}$, $\mathbf{z}\in\mathbb{R}^N$, the inequality $\|\mathbf{x}\|\|\mathbf{z}\|\le\frac{\varepsilon}{2}\|\mathbf{x}\|^2+\frac{1}{2\varepsilon}\|\mathbf{z}\|^2$ holds, where $\varepsilon>0$.

Thus, $\dot{V}$ can be further organised as follows:

$$\begin{aligned}\dot{V} &\le -\left(\gamma\lambda_{\min}(Q)-2\sqrt{N}\|P\|\max_{i\in v}\{l_i\}-\frac{\left(2\sqrt{N}\|P\|-\gamma k\right)\varepsilon_1}{2}\right. \\ &\qquad \left.-\frac{(1+2N\|P\|)\max_{i\in v}\{l_i\}\varepsilon_4}{2}\right)\|\bar{\mathbf{y}}\|^2 \\ &\quad -\left(\tau_2\lambda_{\min}(\mathbf{h}_2)-\frac{\left(2\sqrt{N}\|P\|+\gamma h\right)}{2\varepsilon_1}-\frac{\varepsilon_2}{2}-\frac{1}{2\varepsilon_5}\right)\|\mathbf{e}_v\|^2 \\ &\quad -\left(\tau_1\lambda_{\min}(\mathbf{h}_1)-\frac{\varepsilon_3}{2}-\frac{1}{2\varepsilon_2}\right)\|\mathbf{e}_d\|^2 \\ &\quad -\left(m-\frac{\left(1+2N\|P\|\right)\max_{i\in v}\{l_i\}}{2\varepsilon_4}-\frac{\varepsilon_5}{2}\right)\|\mathbf{e}\|^2+\frac{1}{2\varepsilon_3}\|\dot{\mathbf{d}}(t)\|^2\end{aligned} \tag{91}$$

Among them, ε_1, ε_2, ε_3 are positive constants to be determined.

It is noted that for $t\ge t_0$, it always holds that $\|\dot{\mathbf{d}}(t)\|\le\sqrt{N}\max\{\bar{d}_i\}$, by choosing ε_3 sufficiently large, we can ensure that $\frac{1}{2\varepsilon_3}\|\dot{\mathbf{d}}(t)\|^2$ is sufficiently small.

For the selection of ε_4, ε_5 the following condition should be satisfied:

$$\phi_1 = m-\frac{\left(1+2N\|P\|\right)\max_{i\in n}\{l_i\}}{2\varepsilon_4}\frac{\varepsilon_5}{2}>0 \tag{92}$$

For the already chosen ε_4, there always exists a γ that satisfies:

$$\begin{aligned}\phi_2 &= \gamma\lambda_{\min}(Q)-2\sqrt{N}\|P\|\max_{i\in v}\{l_i\} \\ &\quad -\frac{\left(2\sqrt{N}\|P\|-\gamma k\right)\varepsilon_1}{2}-\frac{\left(1+2N\|P\|\right)\max_{i\in n}\{l_i\}\varepsilon_4}{2}>\phi_1\end{aligned} \tag{93}$$

For the already chosen ε_3, there always exists a τ_1 that satisfies:

$$\phi_3 = \tau_1\lambda_{\min}(\mathbf{h}_1)-\frac{\varepsilon_3}{2}-\frac{1}{2\varepsilon_2}>\phi_1 \tag{94}$$

For the already chosen ε_2 and γ, there exists a τ_2 that satisfies:

$$\phi_4 = \tau_2\lambda_{\min}(\mathbf{h}_2)-\frac{\left(2\sqrt{N}\|P\|+\gamma k\right)}{2\varepsilon_1}-\frac{\varepsilon_2}{2}-\frac{1}{2\varepsilon_5}>\phi_1 \tag{95}$$

After organising $\dot{V}$, we ultimately obtain

$$\dot{V}\le-\phi_1\|\mathbf{E}\|^2+\frac{1}{2\varepsilon_3}\|\dot{\mathbf{d}}(t)\|^2 \tag{96}$$

When $\|\mathbf{E}\|^2\ge\left(\sqrt{N}\max\{\bar{d}_i\}\right)^2$, it follows that

$$\dot{V}\le-\phi_1\|\mathbf{E}\|^2 \tag{97}$$

We have $\omega_1 \|\mathbf{E}\|^2 \le V \le \omega_2 \|\mathbf{E}\|^2$

$$\dot{V} \le \frac{-\phi_1}{2\omega_1} V = -\alpha V \tag{98}$$

$$V(t) \le e^{-\alpha(t-t_k)} V(t_k),\, t \in [t_k, t_{k+1}) \tag{99}$$

Case 2: $r \in C_p$

$$\begin{aligned}
\dot{V} &= \dot{V}_1 + \dot{V}_2 + \dot{V}_3 + \dot{V}_4 \\
&\le -\tau_1 \lambda_{\min}(\mathbf{h}_1) \|\mathbf{e}_d\|^2 + \|\mathbf{e}_d\| \|\dot{\mathbf{d}}(t)\| - \tau_2 \lambda_{\min}(\mathbf{h}_2) \|\mathbf{e}_v\|^2 \\
&\quad + \left(2\sqrt{N}\|P\| - \gamma k\right) \|\mathbf{e}_v\| \|\bar{\mathbf{y}}\| + \|\mathbf{e}_v\| \|\mathbf{e}_d\| \\
&\quad + 2\sqrt{N}\|P\| \max_{i\in v}\{l_i\} \|\bar{\mathbf{y}}\|^2 + (2N\|P\|+1) \max_{i\in v}\{l_i\} \|\bar{\mathbf{y}}\| \|\mathbf{e}\| \\
&\quad - m\|\mathbf{e}\|^2 + \|\mathbf{e}\| \|\mathbf{e}_v\| \\
&\le -\left(\tau_1 \lambda_{\min}(\mathbf{h}_1) - \frac{\varepsilon_3}{2}\right) \|\mathbf{e}_d\|^2 + \|\mathbf{e}_d\| \|\dot{\mathbf{d}}(t)\| \\
&\quad - \tau_2 \lambda_{\min}(\mathbf{h}_2) \|\mathbf{e}_v\|^2 + \left(2\sqrt{N}\|P\| - \gamma k\right) \|\mathbf{e}_v\| \|\bar{\mathbf{y}}\| \\
&\quad + \|\mathbf{e}_v\| \|\mathbf{e}_d\| + 2\sqrt{N}\|P\| \max_{i\in v}\{l_i\} \|\bar{\mathbf{y}}\|^2 \\
&\quad + (2N\|P\|+1) \max_{i\in v}\{l_i\} \|\bar{\mathbf{y}}\| \|\mathbf{e}\| - m\|\mathbf{e}\|^2 \\
&\quad + \|\mathbf{e}\| \|\mathbf{e}_v\| + \frac{1}{2\varepsilon_3} \|\dot{\mathbf{d}}(t)\|^2 \\
&= \mathbf{E}^T A \mathbf{E} + \frac{1}{2\varepsilon_3} \|\dot{\mathbf{d}}(t)\|^2
\end{aligned} \tag{100}$$

where

$$B = \begin{bmatrix}
-m & \frac{(2N+1)\|P\| \max_{i\in v}\{l_i\}}{2} & \frac{1}{2} & 0 \\
\frac{(2N+1)\|P\| \max_{i\in v}\{l_i\}}{2} & 2\sqrt{N}\|P\| \max_{i\in v}\{l_i\} & \frac{\left(2\sqrt{N}\|P\| - \gamma k\right)}{2} & 0 \\
\frac{1}{2} & \frac{\left(2\sqrt{N}\|P\| - \gamma k\right)}{2} & -\tau_2 \lambda_{\min}(\mathbf{h}_2) & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & -\left(\tau_1 \lambda_{\min}(\mathbf{h}_1) - \frac{\varepsilon_3}{2}\right)
\end{bmatrix}$$

Similarly to Theorem 2, by appropriately selecting τ_1 and τ_2, it is possible to ensure that matrix B is symmetric positive definite.

$$\dot{V} \le \frac{\lambda_{\max}(A)}{\omega_1} V + \frac{1}{2\varepsilon_3} \|\dot{d}(t)\|^2 \tag{101}$$

when $\|\mathbf{E}\|^2 \ge \frac{1}{\varepsilon_3}\left(\sqrt{N} \max\{\bar{d}_i\}\right)^2$, it follows that

$$\begin{aligned}
\dot{V} &\le \frac{\lambda_{\max}(B)}{\omega_1} V + \frac{1}{2\varepsilon_3}\left(\sqrt{N} \max\{\bar{d}_i\}\right)^2 \\
&\le \frac{\lambda_{\max}(B)\,\omega_2}{\omega_1} \|\mathbf{E}\|^2 + \frac{1}{1} \|\mathbf{E}\|^2 \\
&\le \left(\frac{\lambda_{\max}(A)\,\omega_2 + \omega_1}{2\omega_1^2}\right) V = \beta V
\end{aligned} \tag{102}$$

$$V(t) \le e^{\beta(t-t_k)} V(t_k)\; t \in [t_k, t_{k+1}) \tag{103}$$

$$T_m(t_k, t) = t - t_k - T_p(t_k, t)\; r \in C_m \cup C_p \tag{104}$$

$$V(t) \le e^{\beta(t-t_k) - \alpha T_m(t, t_k)} V(t_k)\; t \in [t_k, t_{k+1}) \tag{105}$$

At each attack instant t = t_k, it is evident that $x_i(t_k) = x_i(t_k^-)$, because the player's actions cannot instantaneously jump, which implies that $V(t_k) \le x_i(t_k)$, then

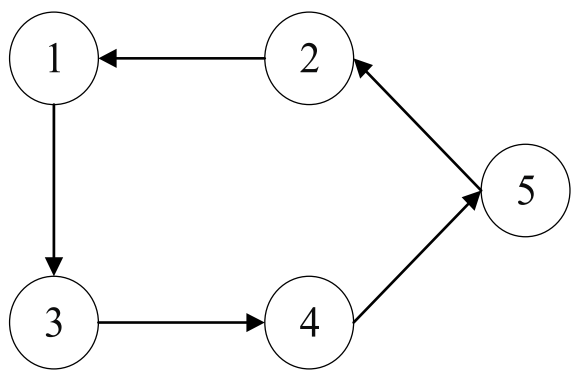
$$\begin{aligned}
V(t) &\le e^{\beta T_p(t_k, t) - \alpha T_m(t_k, t)} V(t_k^-) \\
&\le e^{\beta T_p(t_k, t) - \alpha T_m(t_k, t)} V(t_{k-1}) \\
&\le e^{\beta T_p(t_k, t) - \alpha T_m(t_k, t)} V(t_{k-1}^-) \\
&\le \cdots \le e^{\beta T_p(t_k, t) - \alpha T_m(t_k, t)} V(t_0)
\end{aligned} \tag{106}$$

If Theorem 4 (55) holds, then $\beta T_p(t_0, t) - \alpha T_m(t_0, t) \le -\xi(t - t_0)$, which implies that $\mathbf{x}(t) \to \mathbf{x}^*$ and $\mathbf{y}(t) \to \mathbf{1}_N \otimes \mathbf{x}$ as $t \to \infty$.

5 Simulation examples

In this section, we simulate online games for first-order and second-order systems to verify the strategies proposed in the previous sections.

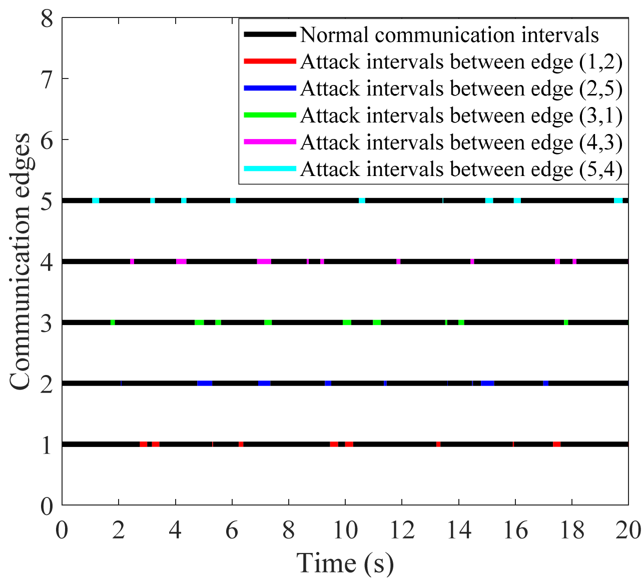
Figure 1 Communication graph among agents



5.1 First-order system

Consider a first-order system consisting of five agents. They interact through the communication topology shown in Figure 1, denoted as $\mathcal{G}$. Figure 2 illustrates the times at which the agents are subjected to attacks.

Figure 2 Times of attacks on agents (see online version for colours)



5.1.1 *Non-quadratic game*

The cost function for each agent is:

$$f_1(\mathbf{x}) = -x_1^3 - 2x_1x_2 - 2$$

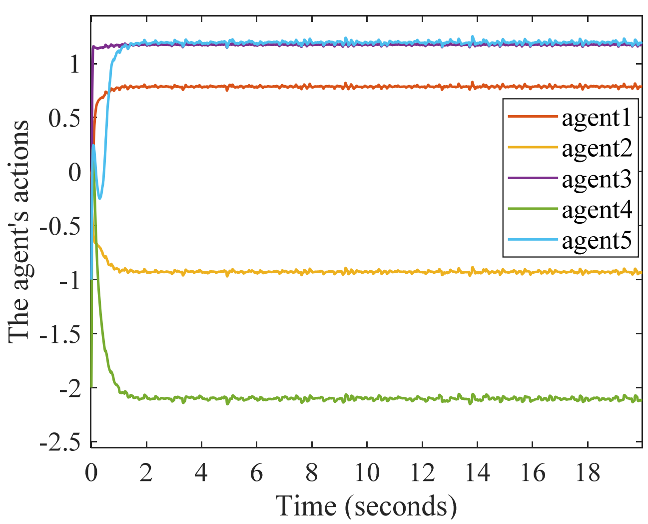
$$f_2(\mathbf{x}) = -\frac{3}{2}x_2^2 - x_1x_2 - 2x_2$$

$$f_3(\mathbf{x}) = -x_3^3 - (x_1 + x_2)x_3 + 4x_3$$

$$f_4(\mathbf{x}) = -\frac{1}{2}x_4^2 - x_3x_4 + x_2x_4$$

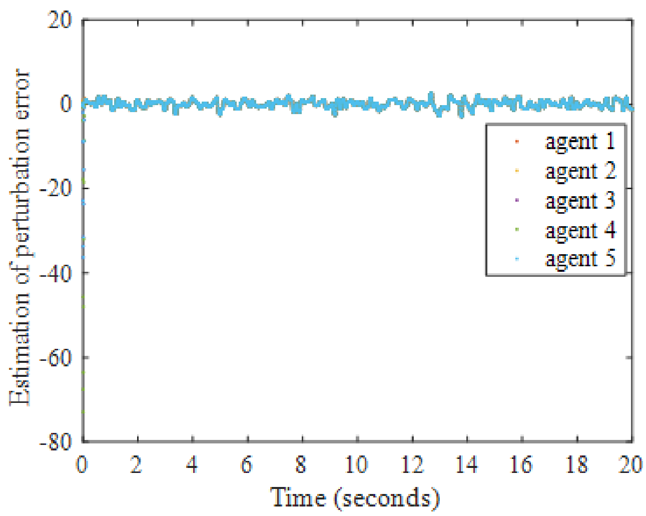
$$f_5(\mathbf{x}) = -\frac{5}{32}x_5^4 - (x_1 + x_2 + x_3 + x_4)x_5$$

Figure 3 The agent's actions (see online version for colours)



Their Nash equilibrium is $\mathbf{x}^* = [1, -1, 2, -3, 1.17]^T$. The initial values are chosen as $\mathbf{x}(0) = [0, 0, -1, -2, -1]$ and $y_i(0) = [0, 0, 0, 0, 0]^T$, $\forall i \in \{1, \ldots, 5\}$. The parameters for the closed-loop system (8) are given as $\omega_0 = 100$, $\gamma = 10$. The time-varying disturbance is set as random noise, and the simulation time interval is [0*s*, 20*s*], the attack times are shown in Figure 2. The simulation results of the agents' actions are depicted in Figure 3, from which it can be observed that they converge to their respective Nash equilibria. Figure 4 illustrates that, even in the presence of DoS attacks, the estimation of disturbances has achieved the desired effect.

Figure 4 Estimation of perturbation error (see online version for colours)



5.1.2 *Quadratic game*

The cost function for each agent is:

$$f_1(\mathbf{x}) = -x_1^2 - 2x_1x_2 - 2$$

$$f_2(\mathbf{x}) = -\frac{3}{2}x_2^2 - x_1x_2 - 2x_2$$

$$f_3(\mathbf{x}) = -x_3^2 - (x_1 + x_2)x_3 + 4x_3$$

$$f_4(\mathbf{x}) = -\frac{1}{2}x_4^2 - x_3x_4 + x_2x_4$$

$$f_5(\mathbf{x}) = -\frac{5}{32}x_5^4 - (x_1 + x_2 + x_3 + x_4)x_5$$

Their Nash equilibrium is $\mathbf{x}^* = [1, -1, 2, -3, 1.17]^T$. The initial values are chosen as $\mathbf{x}(0) = [0, 0, -1, -2, -1]$ and $y_i(0) = [0, 0, 0, 0, 0]^T$, $\forall i \in \{1, \ldots, 5\}$. The parameters for the closed-loop system (8) are given as $\omega_0 = 100$, $\gamma = 10$.

The time-varying disturbance is set as random noise, and the simulation period is [0*s*, 20*s*], the attack times are shown in Figure 2. The simulation results of the agents' actions are depicted in Figure 3, from which it can be observed that they converge to their respective Nash equilibria. Figure 4 illustrates that, even in the presence of DoS attacks, the estimation of disturbances has achieved the desired effect.

Figure 5 The agent's actions (see online version for colours)

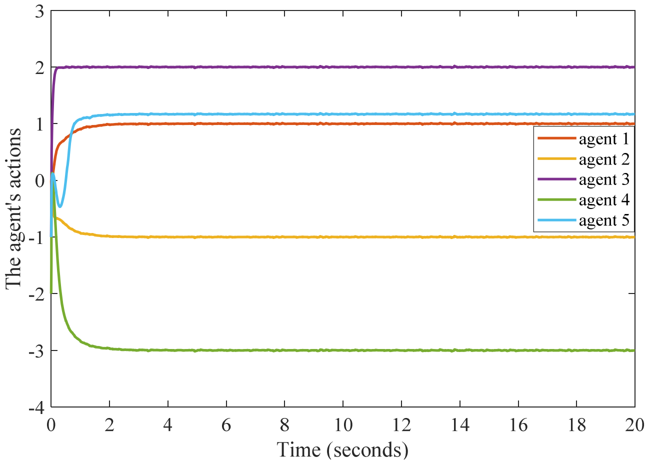
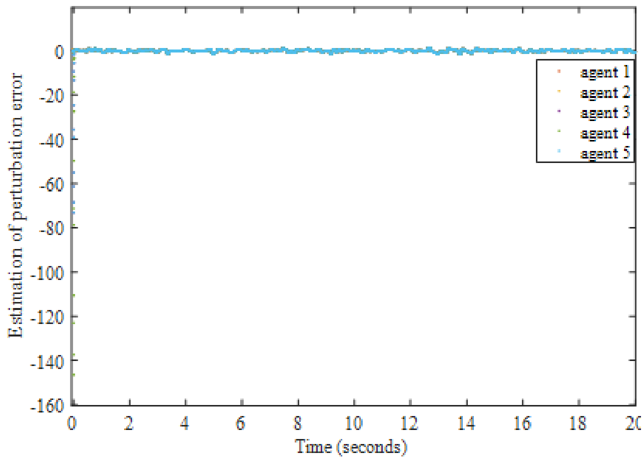


Figure 6 Estimation of perturbation error (see online version for colours)



5.2 Second-order system

Consider a second-order system comprising five agents that interact with each other through the communication topology, $\mathcal{G}$ as illustrated in Figure 6. Figure 6 represents the moment when the agents are under attack.

Figure 7 Communication graph among agents

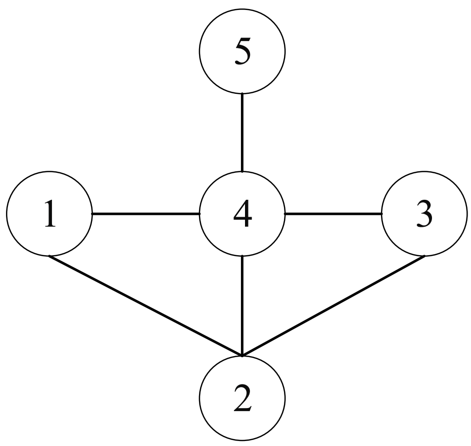
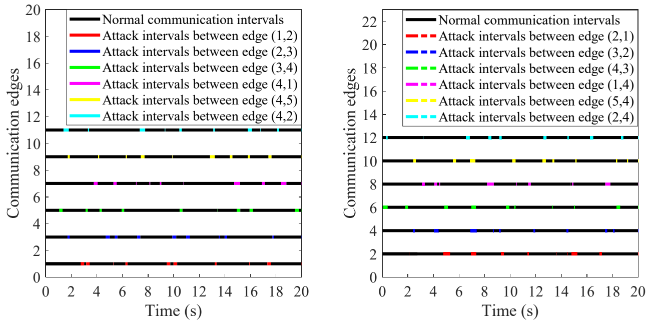


Figure 8 Times of attacks on agents (see online version for colours)



5.2.1 Non-quadratic game

The cost function for each agent is:

$$f_1(\mathbf{x}) = \left\| x_1 - 1_2 \right\|^2 + \left(x_{11} + x_{21}\right)^2 + \left(x_{12} + x_{42}\right)^3$$

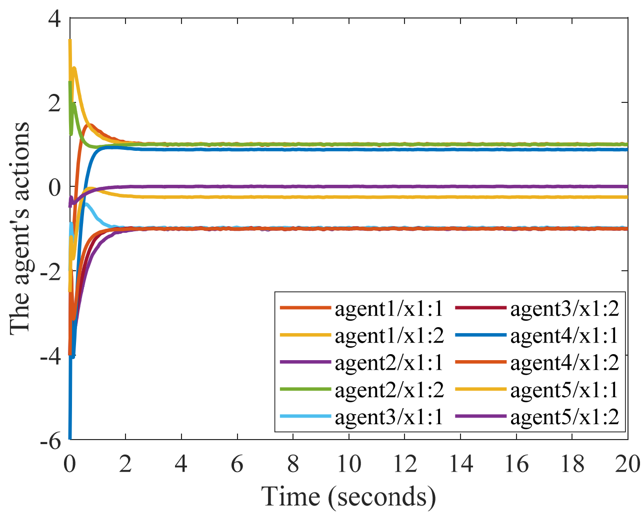
$$f_2(\mathbf{x}) = \left(x_{21} + 1\right)^2 + \left(x_{22} - 1\right)^2 + \left(x_{12} + x_{32}\right)^3 + \left(x_{22} - 1 - x_{52}\right)^2$$

$$f_3(\mathbf{x}) = \left\| x_3 + \mathbf{1}_2 \right\|^2 + \left(x_{32} + x_{22}\right)^2 + \left(x_{31} + x_{41}\right)^2 + \left\| x_5 \right\|^3$$

$$f_4(\mathbf{x}) = \left(x_{41} - 1\right)^2 + \left(x_{42} + 1\right)^2 + \left(x_{41} - 1 - x_{51}\right)^2 + \left(x_{42} + 1 - x_{52}\right)^2$$

In the given system, let $x_i = [x_{i1}, x_{i2}]^T$, $\mathbf{1}_2 = [1, 1]^T$, and $\mathbf{x} = \left[x_1^T, x_2^T, x_3^T, x_4^T, x_5^T\right]^T$. Through computation, the Nash equilibrium point can be determined as $\mathbf{x}^* = [1, 1,–1, 1–, 1,–1, 1,–0, 0]^T$. The parameters for the closed-loop system are given as $\omega_0 = 100$, $\gamma = 30$, $\tau = 30$, the initial values of the variables are set as: $\mathbf{x}(0) = [–2.5, 3.5,–4, 2.5,–2, –4,–2.5, 0, 0.5]^T$, $\mathbf{v}(0) = \mathbf{0}_{10}$, $\mathbf{z}(0) = \mathbf{0}_{10}$, $y(0) = \mathbf{0}_{10}$

Figure 9 The agent's actions (see online version for colours)



The simulation time span is $\mathcal{G}$, with the attack moments as shown in Figure 8, satisfying condition (39). The simulation results of the agents' actions are depicted in Figure 9, indicating that they converge to their respective Nash equilibria. Figure 10 shows that in the presence of DoS attacks, the estimation error of disturbances quickly reaches an ideal state. Figure 11 illustrates the motion trajectories of the agents. Therefore, the proposed control strategy has been numerically verified.

Figure 10 Estimation of perturbation error (see online version for colours)

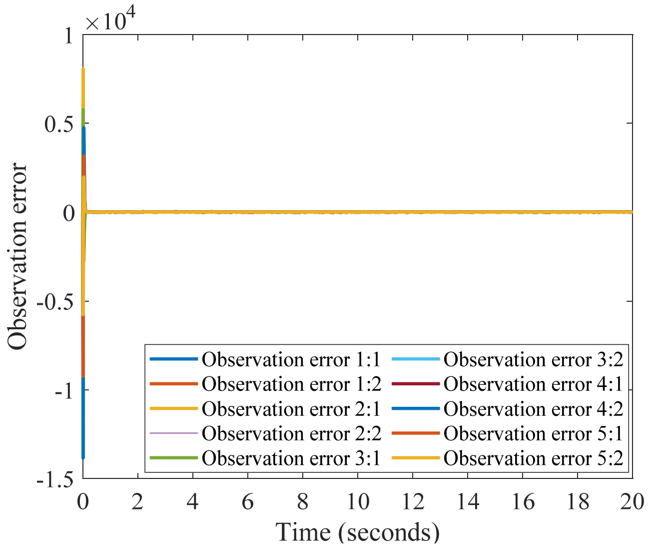
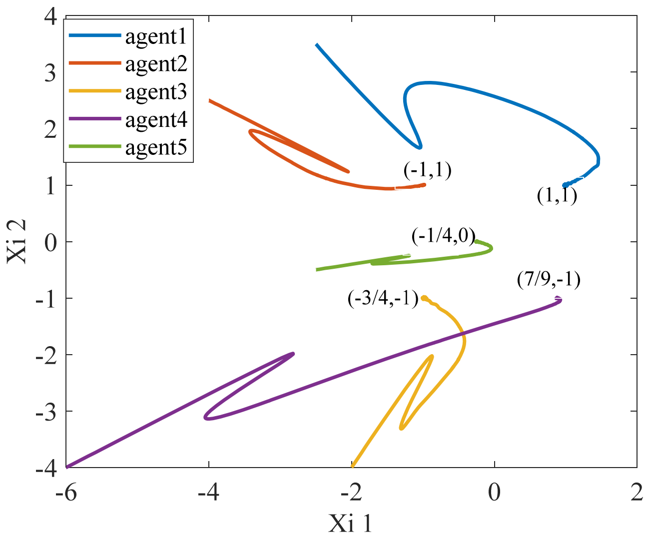


Figure 11 The motion trajectory of intelligent agents (see online version for colours)



5.2.2 *Quadratic game*

The cubic term in the aforementioned non-quadratic utility function is transformed into a quadratic term, resulting in the following utility function:

$$f_1(\mathbf{x}) = \|x_1 - 1_2\|^2 + (x_{11} + x_{21})^2 + (x_{12} + x_{42})^2$$

$$f_2(\mathbf{x}) = (x_{21} + 1)^2 + (x_{22} - 1)^2 + (x_{12} + x_{32})^2 + (x_{22} - 1 - x_{52})^2$$

$$f_3(\mathbf{x}) = \|x_3 + \mathbf{1}_2\|^2 + (x_{32} + x_{22})^2 + (x_{31} + x_{41})^2 + \|x_5\|^2$$

$$f_4(\mathbf{x}) = (x_{41} - 1)^2 + (x_{42} + 1)^2 + (x_{41} - 1 - x_{51})^2 + (x_{42} + 1 - x_{52})^2$$

$$f_5(\mathbf{x}) = \|x_5\|^2 + (x_{51} + 1 + x_{21})^2$$

The relevant parameters are set under those used in non-quadratic game scenarios. The simulation outcomes are depicted in Figures 12, 13, and 14.

Figure 12 The agent's actions (see online version for colours)

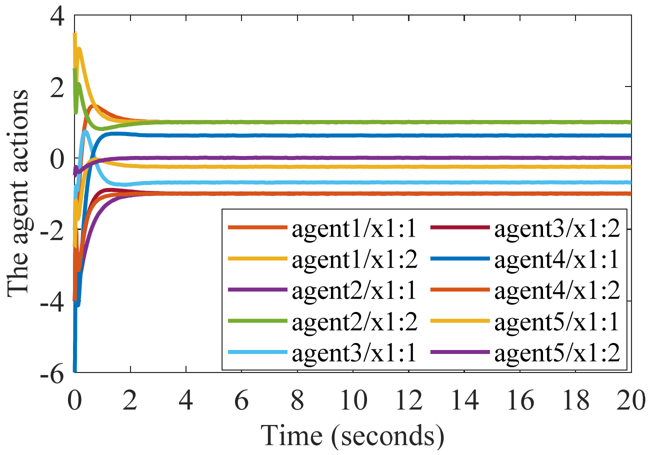


Figure 13 Estimation of perturbation error (see online version for colours)

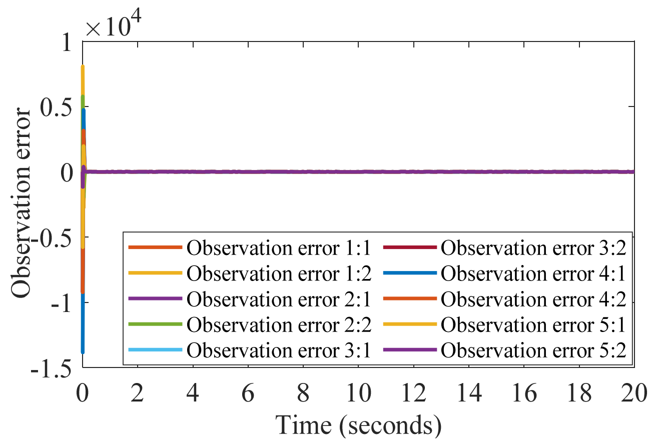
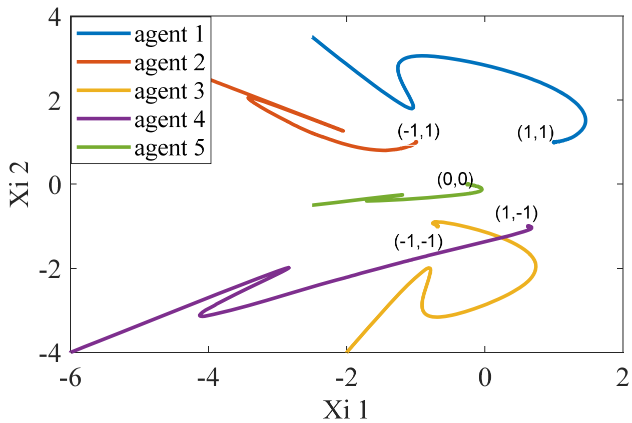


Figure 14 The motion trajectory of intelligent agents (see online version for colours)



6 Conclusions

In addressing the Nash equilibrium-seeking problem under DoS attacks, we have considered stochastic disturbances and designed disturbance observers and control strategies for first- and second-order systems. Furthermore, we have examined both non-quadratic and quadratic utility games. Simulations have revealed that, under identical disturbance parameters, the second-order system is more effective at suppressing disturbances than the first-order system. This study references illustrative examples from the literature, including those presented by Ye et al. (2017), and conducts simulation validation focusing on a system comprising five agents. While the methodology can be expanded to encompass more agents, its applicability within larger swarms or collectives warrants further investigation. This is attributed to the intricate dynamics of complex system behaviour emergence and the theories of phase transitions, which constitute significant areas for future research endeavors. Subsequent studies may contemplate incorporating event-driven mechanisms to optimise system efficiency and reduce associated costs.

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