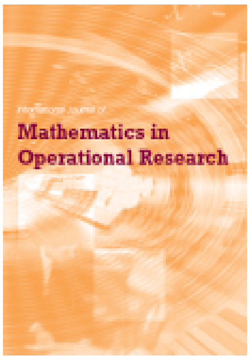

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Abstract: The analytic hierarchy process (AHP) is a decision-making method, which has as its greatest criticism the rank reversal effect. This paper formulates the fourth step of the AHP (synthesis) as a 'well-posed' mathematical problem. A theorem guarantees the existence of the square condensed original formulation for the AHP. This means that any decision problem modelled by AHP with a different number of alternatives and criteria can be condensed into a model with an equal number of alternatives and criteria without loss of condensed information. This condensed formulation can be better conditioned than the original rectangular formulation of the AHP. The square condensed equivalent formulation is also a 'well-posed' mathematical problem. The concepts are applied to two practical cases from the literature, and sensitivity analysis is performed. Four classical matrix norms are reformulated to obtain theoretical bounds for the error estimate closer to actual error.

Keywords: multiple criteria analysis; rank reversal; linear systems of equations; sensitivity analysis.

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1 Introduction

The analytic hierarchy process (AHP) is a well-known and widely used decision-making method (Pereira and Bamel, 2023; Yu and Hong, 2022). Applications of AHP can be found in different areas of application and in different contexts, including the combined use with different techniques or methods (Aulakh et al., 2022; van Hecke, 2021; Martínez-Gómez, 2018; Tooranloo et al., 2018). The original method consists of four steps: modelling, valuation, prioritisation and synthesis. The synthesis step can be written as

$$\mathbf{X}_{A\times 1}^{ORG} = \mathbf{G}_{A\times C}\mathbf{B}_{C\times 1}, \tag{1}$$

where $\mathbf{G}_{A\times C}$ is the matrix whose columns correspond to the priorities of the alternative with respect to each criterion, $\mathbf{B}_{C\times 1}$ is the criteria priority vector, $\mathbf{X}_{A\times 1}$ is the overall priorities for the alternatives, C is the number of criteria, and A represents the number of alternatives. As presented in Alvarez et al. (2021) the synthesis step of the original AHP can be formulated as a linear system of equations (2),

$$\mathbf{C}_{C\times A}\mathbf{X}_{A\times 1}^{EQV} = \mathbf{D}_{C\times 1}, \tag{2}$$

where $\mathbf{D}_{C\times 1} = \mathbf{C}_{C\times A}\mathbf{G}_{A\times C}\mathbf{B}_{C\times 1}$, and the matrix $\mathbf{C}_{C\times A}$ will be chosen appropriately depending on $\mathbf{G}_{A\times C}$. Since $\mathbf{G}_{A\times C}$ contains the 'main information' of the system under analysis regarding modelling, valuation and prioritisation, then two immediate choices for $\mathbf{C}_{C\times A}$ are: the transposed matrix $\mathbf{C}_{C\times A} = \mathbf{G}_{C\times A}^{T}$ and the pseudo-inverse matrix $\mathbf{C}_{C\times A} = \mathbf{G}_{C\times A}^{\dagger}$. Thus, the equivalent linear system (2) retains the 'main information' of the original synthesis (1). In this way, the solution of equivalent linear system (2) should be the original synthesis (1), or in other words, the original synthesis (1) can be derived from the solution of an equivalent synthesis (2).

However, it is possible to notice that in some cases, the equivalent synthesis of the AHP (2) can be an 'ill-posed' mathematical problem because its solution is not unique. Specifically, in cases where $\mathbf{G}_{A\times C}$ is deficient rank or if $C < A$ (Alvarez et al., 2021). Here, a 'well-posed' problem must be understood as one that satisfies the Hadamard's definition, which establishes that the equivalent formulation (2) has to verify three conditions: existence of solution, uniqueness of the solution, and solution stability under small changes in the input data. Otherwise the problem is said to be 'ill-posed'. Furthermore, the formulation is said to be an 'ill-conditioned' problem if the third condition is violated. The uniqueness of the solution for any decision-making method is important, because this property guarantees a unique choice as the best possible. That is, if there is no uniqueness of the solution, then there will not be a unique 'best alternative' to choose from.

Saaty himself recognised the need to guarantee the uniqueness of the solution when he makes the following three statements. "There are people who have made it an obsession to find ways to avoid rank reversal in every decision and wish to alter the synthesis of the AHP away from normalization or idealization. They are likely to obtain outcomes that are not compatible with what the real outcome of a decision should be, because in decision making we also want uniqueness of the answer we get" (Saaty, 2004). "It is essential that a credible decision theory yields unique answers for the alternatives of a decision, perhaps not only in terms of ranks, but also in terms of priorities" (Saaty and Hu, 1998). "In multicriteria processes, different methods may each produce a ranking of the alternatives of a decision that is different than another method. Such variability in ranking violates the uniqueness requirement mentioned above and is, therefore, unacceptable" (Saaty and Hu, 1998). In this last sentence, Saaty warns about the importance that different multicriteria decision methods should produce similar answers to the same problem.

On the other hand, even though the equivalent formulation of the AHP is an 'ill-posed' mathematical problem, it is important to note that this does not prevent the existence of practical cases where $C < A$ or $\mathbf{G}_{A\times C}$ is deficient rank. These cases can be understood in the context of decision making as follows. First, when $C < A$ the number of criteria considered in the analysis is insufficient to establish a prioritisation of alternatives as reliably as when $C \geq A$. This is because in terms of linear systems the number of unknowns is greater than the number of equations, and consequently there will be infinite solutions. In these cases, it is advisable to introduce new linearly independent (LI) criteria in the analysis, or to condensate the alternatives until obtaining $C = A$. Second, when $\mathbf{G}_{A\times C}$ is deficient rank it means that there is a linear dependence (LD) between the columns or rows of $\mathbf{G}_{A\times C}$. Therefore, the predefined set of criteria and alternatives and/or the judgements made do not allow prioritisation of alternatives as reliably as when $\mathbf{G}_{A\times C}$ is full rank. Here the word reliability is related to the uniqueness and stability of the solution. In other words, having less reliability means that $\mathbf{X}^{ORG}_{A\times 1}$ and $\mathbf{X}^{EQV}_{A\times 1}$ are not unique or are more sensitive to uncertainty for this set of criteria, alternatives and judgements. Deficient rank can occur for two reasons related to Axiom 3 in Saaty (1991). The first reason would be inaccurate and/or inconsistent judgements. The second reason would be the existence of interdependence between the alternatives or criteria, even if the judgements are coherent and consistent.

Axiom 3 establishes 'outer' and 'inner' dependence between the various levels of a hierarchy (alternatives and criteria) (Neves et al., 2022; Saaty, 1986). This axiom is assumed to be verified in most applications of the AHP. However, as far as the authors know, there is no mathematical way to verify this axiom in a practical application of the method. On the other hand, the analysis performed in Alvarez et al. (2021) indicates a mathematical way of verifying whether there is a linear dependence between the criteria and/or alternatives. The test would be to ensure that the matrix $\mathbf{G}_{A\times C}$ used in the synthesis is full rank. Furthermore, even if the matrix $\mathbf{G}_{A\times C}$ is full rank, we can seek greater stability for the method. Thus, the following question arises. What other constraint must be required for the synthesis of the original AHP to be as reliable as possible? In other words, is there any way to write the synthesis of the original AHP as a 'well-posed' mathematical problem? That is, a synthesis with unique solution $\mathbf{X}^{ORG}_{A\times 1}$ as stable as possible. We believe that this challenge can be overcome if the matrix $\mathbf{G}_{A\times C}$ from synthesis is condensed into a full rank square matrix. In Alvarez et al. (2021), a square equivalent formulation for the AHP is proposed as

$$\mathbf{M}_{A\times C}\mathbf{C}_{C\times A}\mathbf{X}^{SEQV}_{A\times 1} = \mathbf{M}_{A\times C}\mathbf{C}_{C\times A}\mathbf{G}_{A\times C}\mathbf{B}_{C\times 1}, \tag{3}$$

where matrices $\mathbf{C}_{C\times A}$ and $\mathbf{M}_{A\times C}$ must be chosen according to $\mathbf{G}_{A\times C}$. The square matrix $\mathbf{M}_{A\times C}\mathbf{C}_{C\times A}$ is singular if $\mathbf{G}_{A\times C}$ is deficient rank or if $C < A$. So, in these cases the square equivalent formulation of the AHP (3) is an 'ill-posed' problem.

This paper continues the theoretical development started in Alvarez et al. (2021). The main goal here is to reformulate the synthesis of the original AHP in such a way that it is always a 'well-posed' mathematical problem. Thus, the formulation of the synthesis of the AHP as a 'well-posed' mathematical problem is presented in the next Section, where a square condensed original and equivalent formulation for the AHP are obtained. Subsequently, existence of the solution for these formulations is shown. The following section presents the sensitivity analysis via condition number, showing that in some cases it is possible to improve the conditioning of these square condensed formulations. In addition, the four matrix norms used in de Almeida et al. (2021) are reformulated to obtain theoretical bounds for the error estimate closer to actual error. The notation used will be the same as shown in Alvarez et al. (2021) and de Almeida et al. (2021).

2 Formulation of the original AHP as a 'well-posed' mathematical problem

Theorem 1 guarantees the existence of a condensed square formulation with a unique solution that is as stable as possible for the original AHP. It must be said that this square formulation is not unique, but the solution of all square formulations coincides with the original synthesis (1). In the case that $C > A$ and $\mathbf{G}_{A\times C}$ is full rank there are $N = \frac{C!}{(C-K)!K!}$ combinations of columns that can be condensed, where $K = rank(\mathbf{G}_{A\times C}) = A$ and $C!$ is the factorial of C. From these N combinations it is necessary to remove those that produce a singular square matrix ($det(\mathbf{G}_{A\times A}) = 0$). These condensed columns (criteria) are not removed from the analysis, they are only grouped into a smaller number in order to guarantee the uniqueness and stability of the solution. Thus, we obtain a reduced square formulation with the minimum number of necessary criteria that preserves the original synthesis (1). Similarly, in the case that $C < A$ and $\mathbf{G}_{A\times C}$ is full rank ($K = C$) there are $N = \frac{A!}{(A-K)!K!}$ combinations of rows that can be condensed. Furthermore, if $\mathbf{G}_{A\times C}$ is deficient rank the number N may be smaller than the one calculated above, as it may be necessary to condense both columns and rows. So suppose there are a number of different non-singular N square matrices. Since all these N square formulations have the same unique solution, then for the problem to be the 'well-posed' possible, the condensed matrix with the lowest condition number should be chosen. That is, the square matrix that generates the most stable formulation should be chosen.

Theorem 1: Let $\mathbf{G}_{A\times C}$, $\mathbf{B}_{C\times 1}$ and $\mathbf{X}^{ORG}_{A\times 1}$ be the inputs and outputs of the original AHP satisfying equation (1). Let $K = rank(\mathbf{G}_{A\times C})$ be the rank of matrix $\mathbf{G}_{A\times C}$. Then, the two statements below are verified.

1 There are N non-singular condensed square matrix $\mathbf{H}_{K\times K}$ and vector $\mathbf{F}_{K\times 1}$, such that the condensed synthesis ${}^{AUX}\hat{\mathbf{X}}^{ORG}_{K\times 1} = \mathbf{H}_{K\times K}\mathbf{F}_{K\times 1}$ has the same solution as $\mathbf{X}^{ORG}_{A\times 1}$.

2 The condensed equivalent synthesis $\mathbf{H}^{-1}_{K\times K}\ {}^{AUX}\hat{\mathbf{X}}^{ORG}_{K\times 1} = \mathbf{F}_{K\times 1}$ with the lowest condition number is the best possible 'well-posed' mathematical problem for these input data $\mathbf{G}_{A\times C}$ and $\mathbf{B}_{C\times 1}$.

Proof: Proof of existence (item 1) is done using Propositions 1 and 2 in Alvarez et al. (2021) as needed. Suppose in the worst case, where the matrix $\mathbf{G}_{A\times C}$ is deficient rank $K < \min\{C, A\}$. Therefore, it will be necessary to condense columns and rows. First, the columns will be condensed until the number of LI columns is equal to K using the Proposition 1 in Alvarez et al. (2021). The original synthesis (1) can be rewritten as a linear combination of the column vectors of $\mathbf{G}_{A\times C}$, where $\mathbf{Z}^j_{A\times 1} = [\mathbf{g}_{1j}\ \mathbf{g}_{2j} \cdots \mathbf{g}_{Aj}]^T$ $\forall j = \{1, 2, \ldots, C\}$.

$$\mathbf{X}^{ORG}_{A\times 1} = \begin{bmatrix} \mathbf{g}_{11} & \mathbf{g}_{12} & \cdots & \mathbf{g}_{1C} \\ \mathbf{g}_{21} & \mathbf{g}_{22} & \cdots & \mathbf{g}_{2C} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{g}_{A1} & \mathbf{g}_{A2} & \cdots & \mathbf{g}_{AC} \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_C \end{bmatrix} = \mathbf{b}_1\mathbf{Z}^1_{A\times 1} + \mathbf{b}_2\mathbf{Z}^2_{A\times 1} + \ldots + \mathbf{b}_C\mathbf{Z}^C_{A\times 1}. \quad (4)$$

Let $n = C - K$ be the number of LD column vectors in $\mathbf{G}_{A\times C}$ with $n \in \{2, 3, \ldots, C-1\}$. For simplicity it will be assumed that they correspond to the first n columns and are ordered in sequence. Then, using Proposition 1 in Alvarez et al. (2021), $(n-1)$ columns can be factored (condensed) as follows:

$$\begin{aligned} \mathbf{X}^{ORG}_{A\times 1} &= \underbrace{(\mathbf{b}_1 + \mathbf{b}_2 + \ldots + \mathbf{b}_n)}_{factoring} \mathbf{Z}^n_{A\times 1} + \mathbf{b}_{n+1}\mathbf{Z}^{n+1}_{A\times 1} + \ldots + \mathbf{b}_C\mathbf{Z}^C_{A\times 1} \\ &= \begin{bmatrix} \mathbf{g}_{1n} & \mathbf{g}_{1(n+1)} & \cdots & \mathbf{g}_{1C} \\ \mathbf{g}_{2n} & \mathbf{g}_{2(n+1)} & \cdots & \mathbf{g}_{2C} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{g}_{An} & \mathbf{g}_{A(n+1)} & \cdots & \mathbf{g}_{AC} \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{n} \mathbf{b}_j \\ \mathbf{b}_{n+1} \\ \vdots \\ \mathbf{b}_C \end{bmatrix} = \tilde{\mathbf{G}}_{A\times K}\mathbf{F}_{K\times 1}, \end{aligned} \quad (5)$$

where for simplicity it is assumed that the first n columns of $\mathbf{G}_{A\times C}$ are equal.

Now, the rows will be condensed until the number of LI rows is equal to K. Let $m = A - K$ be the number of LD row vectors in $\mathbf{G}_{A\times C}$ with $m \in \{2, 3, \ldots, A-1\}$. For simplicity it is assumed that the first m rows are ordered sequentially. Then, using Proposition 2 in Alvarez et al. (2021), $(m-1)$ rows can be condensed as follows:

$$\begin{aligned} \mathbf{X}^{ORG}_{A\times 1} &= \tilde{\mathbf{G}}_{A\times K}\mathbf{F}_{K\times 1} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_{m-1} \\ \mathbf{x}_m \\ \vdots \\ \mathbf{x}_A \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_{m-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{x}_m \\ \vdots \\ \mathbf{x}_A \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{g}_{mn} & \cdots & \mathbf{g}_{mC} \\ \vdots & \cdots & \vdots \\ \mathbf{g}_{mn} & \cdots & \mathbf{g}_{mC} \\ 0 & \ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{n} \mathbf{b}_j \\ \mathbf{b}_{n+1} \\ \vdots \\ \mathbf{b}_C \end{bmatrix} + \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \ldots & 0 \\ \mathbf{g}_{mn} & \cdots & \mathbf{g}_{mC} \\ \vdots & \vdots & \vdots \\ \mathbf{g}_{An} & \cdots & \mathbf{g}_{AC} \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{n} \mathbf{b}_j \\ \mathbf{b}_{n+1} \\ \vdots \\ \mathbf{b}_C \end{bmatrix}, \end{aligned} \quad (6)$$

where it is assumed that the first m rows of $\mathbf{G}_{A\times C}$ are equal. Create an auxiliary system like the following:

$$
\begin{aligned}
{}^{AUX}\hat{\mathbf{X}}^{ORG}_{K\times 1} &= \begin{bmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \\ \vdots \\ \hat{\mathbf{x}}_K \end{bmatrix} = \begin{bmatrix} m\mathbf{g}_{mn} & m\mathbf{g}_{m(n+1)} & \cdots & m\mathbf{g}_{mC} \\ \mathbf{g}_{(m+1)n} & \mathbf{g}_{(m+1)(n+1)} & \cdots & \mathbf{g}_{(m+1)C} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{g}_{An} & \mathbf{g}_{A(n+1)} & \cdots & \mathbf{g}_{AC} \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{n} \mathbf{b}_j \\ \mathbf{b}_{n+1} \\ \vdots \\ \mathbf{b}_C \end{bmatrix} \\
&= \mathbf{H}_{K\times K}\mathbf{F}_{K\times 1}.
\end{aligned} \tag{7}
$$

The condensed alternatives were grouped in $\hat{\mathbf{x}}_1$. In this way, the auxiliary vector ${}^{AUX}\hat{\mathbf{X}}^{ORG}_{K\times 1}$ contains the number of necessary alternatives that guarantee the uniqueness of the solution. Note that ${}^{AUX}\hat{\mathbf{X}}^{ORG}_{K\times 1}$ is normalised because $\sum_{i=1}^{K} \hat{\mathbf{x}}_i = 1$, and $\mathbf{x}_1 = \cdots = \mathbf{x}_m = (1/m)\hat{\mathbf{x}}_1$. Thus, $\mathbf{X}^{ORG}_{A\times 1}$ is recovered as

$$
\mathbf{X}^{ORG}_{A\times 1} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_{m-1} \\ \mathbf{x}_m \\ \mathbf{x}_{m+1} \\ \vdots \\ \mathbf{x}_A \end{bmatrix} = \begin{bmatrix} (1/m)\hat{\mathbf{x}}_1 \\ \vdots \\ (1/m)\hat{\mathbf{x}}_1 \\ (1/m)\hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \\ \vdots \\ \hat{\mathbf{x}}_K \end{bmatrix}, \tag{8}
$$

after solving the condensed formulation (7). Note that the columns of $\mathbf{H}_{K\times K}$ are normalised in the context of the AHP, that is, $\sum_{i=1}^{K} \mathbf{g}_{ij} = 1\ \forall j$. In addition, a system similar to (7) is obtained if the rows are condensed before the columns.

Let M be the number of singular matrices $\mathbf{H}_{K\times K}$, then the number of non-singular condensed square matrices $\mathbf{H}_{K\times K}$ is $N = \frac{C!}{(C-K)!K!} + \frac{A!}{(A-K)!K!} - M$. Particular cases described above are when $\mathbf{G}_{A\times C}$ is full rank. First, if $K = \min\{C, A\} = A$ then it is only necessary to condense the columns (criteria). Second, if $K = \min\{C, A\} = C$ then it is only necessary to condense the rows (alternatives).

In addition, if the LD columns or LD rows are not equal, but a linear combination of the LI columns or LI rows, the condensation methodology remains valid. That is, consider the first $(n-1)$ columns linear combination of the last K LI columns as follows:

$$
\begin{aligned}
&\mathbf{Z}^1_{A\times 1} = \alpha^1_1 \mathbf{Z}^n_{A\times 1} + \alpha^1_2 \mathbf{Z}^{n+1}_{A\times 1} + \cdots + \alpha^1_K \mathbf{Z}^C_{A\times 1}, \\
&\vdots \quad = \qquad\qquad \vdots, \\
&\mathbf{Z}^{n-1}_{A\times 1} = \alpha^{n-1}_1 \mathbf{Z}^n_{A\times 1} + \alpha^{n-1}_2 \mathbf{Z}^{n+1}_{A\times 1} + \cdots + \alpha^{n-1}_K \mathbf{Z}^C_{A\times 1},
\end{aligned} \tag{9}
$$

where α^j_i is the i^{th} coefficient of the linear combination that generates $\mathbf{Z}^j_{A\times 1}$ with $j = 1, \ldots, (n-1)$ and $i = 1, \ldots, K$. Thus, the first $(n-1)$ columns can be condensed as follows:

$$\begin{aligned}
\mathbf{X}^{ORG}_{A\times 1} &= \mathbf{b}_1\mathbf{Z}^1_{A\times 1} + \mathbf{b}_2\mathbf{Z}^2_{A\times 1} + \ldots + \mathbf{b}_{n-1}\mathbf{Z}^{n-1}_{A\times 1} + \mathbf{b}_n\mathbf{Z}^n_{A\times 1} + \ldots + \mathbf{b}_C\mathbf{Z}^C_{A\times 1} \\
&= \underbrace{\left(\mathbf{b}_1\alpha^1_1 + \mathbf{b}_2\alpha^2_1 + \ldots + \mathbf{b}_{n-1}\alpha^{n-1}_1 + \mathbf{b}_n\right)}_{factoring}\mathbf{Z}^n_{A\times 1} \\
&\quad + \underbrace{\left(\mathbf{b}_1\alpha^1_2 + \mathbf{b}_2\alpha^2_2 + \ldots + \mathbf{b}_{n-1}\alpha^{n-1}_2 + \mathbf{b}_{n+1}\right)}_{factoring}\mathbf{Z}^{n+1}_{A\times 1} + \ldots \\
&\quad + \underbrace{\left(\mathbf{b}_1\alpha^1_K + \mathbf{b}_2\alpha^2_K + \ldots + \mathbf{b}_{n-1}\alpha^{n-1}_K + \mathbf{b}_C\right)}_{factoring}\mathbf{Z}^C_{A\times 1} \qquad (10) \\
&= \begin{bmatrix} \mathbf{g}_{1n} & \mathbf{g}_{1(n+1)} & \cdots & \mathbf{g}_{1C} \\ \mathbf{g}_{2n} & \mathbf{g}_{2(n+1)} & \cdots & \mathbf{g}_{2C} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{g}_{An} & \mathbf{g}_{A(n+1)} & \cdots & \mathbf{g}_{AC} \end{bmatrix}
\begin{bmatrix} \sum_{j=1}^{n-1}\mathbf{b}_j\alpha^j_1 + \mathbf{b}_n \\ \sum_{j=1}^{n-1}\mathbf{b}_j\alpha^j_2 + \mathbf{b}_{n+1} \\ \vdots \\ \sum_{j=1}^{n-1}\mathbf{b}_j\alpha^j_K + \mathbf{b}_C \end{bmatrix} = \tilde{\mathbf{G}}_{A\times K}\mathbf{F}_{K\times 1}.
\end{aligned}$$

If it is necessary to condense $(m-1)$ rows, and these rows are linear combination of the last K LI rows as follows:

$$\begin{aligned}
\mathbf{Y}^1_{1\times K} &= \beta^1_1\mathbf{Y}^m_{1\times K} + \beta^1_2\mathbf{Y}^{m+1}_{1\times K} + \cdots + \beta^1_K\mathbf{Y}^A_{1\times K}, \\
\vdots \quad &= \qquad \vdots, \qquad\qquad (11) \\
\mathbf{Y}^{m-1}_{1\times K} &= \beta^{m-1}_1\mathbf{Y}^m_{1\times K} + \beta^{m-1}_2\mathbf{Y}^{m+1}_{1\times K} + \cdots + \beta^{m-1}_K\mathbf{Y}^A_{1\times K},
\end{aligned}$$

where β^j_i is the i^{th} coefficient of the linear combination that generates $\mathbf{Y}^j_{1\times K}$ with $j = 1, \ldots, (m-1)$ and $i = 1, \ldots, K$. Then

$$\begin{aligned}
\mathbf{X}^{ORG}_{A\times 1} &= \tilde{\mathbf{G}}_{A\times K}\mathbf{F}_{K\times 1} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_{m-1} \\ \mathbf{x}_m \\ \vdots \\ \mathbf{x}_A \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_{m-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{x}_m \\ \vdots \\ \mathbf{x}_A \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{g}_{1n} & \cdots & \mathbf{g}_{1C} \\ \vdots & \ldots & \vdots \\ \mathbf{g}_{(m-1)n} & \cdots & \mathbf{g}_{(m-1)C} \\ 0 & \ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \ldots & 0 \end{bmatrix}
\begin{bmatrix} \sum_{j=1}^{n-1}\mathbf{b}_j\alpha^j_1 + \mathbf{b}_n \\ \sum_{j=1}^{n-1}\mathbf{b}_j\alpha^j_2 + \mathbf{b}_{n+1} \\ \vdots \\ \sum_{j=1}^{n-1}\mathbf{b}_j\alpha^j_K + \mathbf{b}_C \end{bmatrix} \qquad (12)
\end{aligned}$$

$$
+\begin{bmatrix} 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 0 \\ \mathbf{g}_{mn} & \cdots & \mathbf{g}_{mC} \\ \vdots & \vdots & \vdots \\ \mathbf{g}_{An} & \cdots & \mathbf{g}_{AC} \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{n-1} \mathbf{b}_j \alpha_1^j + \mathbf{b}_n \\ \sum_{j=1}^{n-1} \mathbf{b}_j \alpha_2^j + \mathbf{b}_{n+1} \\ \vdots \\ \sum_{j=1}^{n-1} \mathbf{b}_j \alpha_K^j + \mathbf{b}_C \end{bmatrix}.
$$

Create the auxiliary system

$$
{}^{AUX}\hat{\mathbf{X}}_{K\times 1}^{ORG} = \begin{bmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \\ \vdots \\ \hat{\mathbf{x}}_K \end{bmatrix} = \begin{bmatrix} (1+\sum_{j=1}^{m-1} \beta_1^j)\mathbf{g}_{mn} & \dots & (1+\sum_{j=1}^{m-1} \beta_1^j)\mathbf{g}_{mC} \\ (1+\sum_{j=1}^{m-1} \beta_2^j)\mathbf{g}_{(m+1)n} & \dots & (1+\sum_{j=1}^{m-1} \beta_2^j)\mathbf{g}_{(m+1)C} \\ \vdots & \vdots & \vdots \\ (1+\sum_{j=1}^{m-1} \beta_K^j)\mathbf{g}_{An} & \dots & (1+\sum_{j=1}^{m-1} \beta_K^j)\mathbf{g}_{AC} \end{bmatrix}
$$

$$
\times \begin{bmatrix} \sum_{j=1}^{n-1} \mathbf{b}_j \alpha_1^j + \mathbf{b}_n \\ \sum_{j=1}^{n-1} \mathbf{b}_j \alpha_2^j + \mathbf{b}_{n+1} \\ \vdots \\ \sum_{j=1}^{n-1} \mathbf{b}_j \alpha_K^j + \mathbf{b}_C \end{bmatrix} = \mathbf{H}_{K\times K}\mathbf{F}_{K\times 1}, \tag{13}
$$

and $\mathbf{X}_{A\times 1}^{ORG}$ is recovered as

$$
\mathbf{X}_{A\times 1}^{ORG} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_{m-1} \\ \mathbf{x}_m \\ \vdots \\ \mathbf{x}_A \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_{m-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{x}_m \\ \vdots \\ \mathbf{x}_A \end{bmatrix}, \tag{14}
$$

where

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{x}_m \\ \vdots \\ \mathbf{x}_A \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ [1/(1+\sum_{j=1}^{m-1}\beta_1^j)]\hat{\mathbf{x}}_1 \\ [1/(1+\sum_{j=1}^{m-1}\beta_2^j)]\hat{\mathbf{x}}_2 \\ \vdots \\ [1/(1+\sum_{j=1}^{m-1}\beta_K^j)]\hat{\mathbf{x}}_K \end{bmatrix} \tag{15}$$

after solving the condensed system (13) and

$$\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_{m-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \beta_1^1\mathbf{x}_m + \ldots + \beta_K^1\mathbf{x}_A \\ \vdots \\ \beta_1^{m-1}\mathbf{x}_m + \ldots + \beta_K^{m-1}\mathbf{x}_A \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{16}$$

Proof of item 2: equations (7) or (13) define N condensed square synthesis of the AHP with the same solution ${}^{AUX}\hat{\mathbf{X}}^{ORG}_{K\times 1}$. For every non-singular matrix $\mathbf{H}_{K\times K}$ there is a unique inverse $\mathbf{H}^{-1}_{K\times K}$. Thus, for each condensed square synthesis (7) or (13) there corresponds a condensed square equivalent synthesis determined by

$$\mathbf{H}^{-1}_{K\times K}\ {}^{AUX}\hat{\mathbf{X}}^{ORG}_{K\times 1} = \mathbf{F}_{K\times 1}, \tag{17}$$

whose unique solution is ${}^{AUX}\hat{\mathbf{X}}^{ORG}_{K\times 1} = \mathbf{H}_{K\times K}\mathbf{F}_{K\times 1}$. In other words, $\mathbf{H}^{-1}_{K\times K}\ {}^{AUX}\hat{\mathbf{X}}^{ORG}_{K\times 1} = \mathbf{F}_{K\times 1}$ if and only if ${}^{AUX}\hat{\mathbf{X}}^{ORG}_{K\times 1} = \mathbf{H}_{K\times K}\mathbf{F}_{K\times 1}$.

Let $cond_m(\mathbf{H}_{K\times K}) = \|\mathbf{H}_{K\times K}\|_m\|\mathbf{H}^{-1}_{K\times K}\|_m$ be the condition number of $\mathbf{H}_{K\times K}$, where $cond(\circ)_m \in \mathbb{R}$ and $\|\circ\|_m$ represents some norm for matrices. If among the N non-singular matrix $\mathbf{H}_{K\times K}$, the one with the lowest condition number is chosen, then we obtain the best possible 'well-posed' equivalent synthesis (17), since it has a unique solution and is the most stable for these input data $\mathbf{G}_{A\times C}$ and $\mathbf{B}_{C\times 1}$. □

Therefore, equations (7) or (13) and (17) must be considered the condensed square original and equivalent synthesis of the AHP, which are a 'well-posed' mathematical problem. However, these formulations also had the rank reversal effect (Alvarez et al., 2021; de Almeida et al., 2021). Note that once $\mathbf{G}_{A\times C}$ and $\mathbf{B}_{C\times 1}$ are constructed it is simple to obtain $\mathbf{H}_{K\times K}$ and $\mathbf{F}_{K\times 1}$. For these reasons, it is suggested to incorporate the procedure for obtaining $\mathbf{H}_{K\times K}$ and $\mathbf{F}_{K\times 1}$ as a post-processing after completing the original synthesis of the AHP. In this way the synthesis will be more robust and complete. That is, the synthesis process performed with equation (7) or (13) will always be a 'well-posed' mathematical problem, while the original process defined by equation (1) is an 'ill-posed' mathematical problem if $\mathbf{G}_{A\times C}$ is deficient rank or if $C < A$.

3 Matrix norms appropriate for sensitivity analysis via condition number

As presented in Alvarez et al. (2021) and de Almeida et al. (2021), the sensitivity analysis of the condensed square equivalent synthesis (17) follows as

$$\frac{\|\Delta^{AUX}\hat{\mathbf{X}}^{ORG}_{K\times 1}\|_m}{\|^{AUX}\hat{\mathbf{X}}^{ORG}_{K\times 1}\|_m} \leq cond_m(\mathbf{H}_{K\times K})\frac{\|\Delta\mathbf{F}_{K\times 1}\|_m}{\|\mathbf{F}_{K\times 1}\|_m} = \|\text{Thr. error } \mathbf{F}\|_m, \tag{18}$$

$$\frac{\|\Delta^{AUX}\hat{\mathbf{X}}^{ORG}_{K\times 1}\|_m}{\|^{AUX}\hat{\mathbf{X}}^{ORG}_{K\times 1} + \Delta^{AUX}\hat{\mathbf{X}}^{ORG}_{K\times 1}\|_m} \leq cond_m(\mathbf{H}_{K\times K})\frac{\|\Delta\mathbf{H}_{K\times K}\|_m}{\|\mathbf{H}_{K\times K}\|_m} = \|\text{Thr. error } \mathbf{H}\|_m, \tag{19}$$

where perturbations $\Delta\mathbf{F}_{K\times 1}$ and $\Delta\mathbf{H}_{K\times K}$ can be generated by the critical element or the critical column vector, and $\| \circ \|_m$ represents some norm for matrices. The condition number is defined as $cond_m(\mathbf{H}_{K\times K}) = \|\mathbf{H}_{K\times K}\|_m \|\mathbf{H}^{-1}_{K\times K}\|_m$ for square matrix and $cond_m(\mathbf{G}_{A\times C}) = \|\mathbf{G}_{A\times C}\|_m \|\mathbf{G}^{\dagger}_{A\times C}\|_m$ for rectangular matrix, and is considered a measure of the stability of the linear system solution. Equations (18) and (19) establish theoretical upper bounds for the relative error of the solution when considering separate uncertainties in $\mathbf{F}_{K\times 1}$ or $\mathbf{H}_{K\times K}$ (Meyer and Stewart, 2023; Hager, 2021; Golub and van Loan, 2013). However, uncertainties in judgements made by the experts can imply uncertainties in $\mathbf{F}_{K\times 1}$ and $\mathbf{H}_{K\times K}$ simultaneously. In this case, the theoretical upper bounds will be addressed in future work.

Table 1 Four reformulated matrix norms: Euclidean, maximum absolute column sum, maximum absolute row sum and Frobenius

Classical norm	*Weighted norm*
$\|\mathbf{C}_{C,A}\|_2 = \sum_{j=1}^{A}\sqrt{\sum_{i=1}^{C}\|\mathbf{c}_{ij}\|^2}$	$\|\mathbf{C}_{C,A}\|_{w2} = \frac{1}{AC}\|\mathbf{C}_{C,A}\|_2$
$\|\mathbf{C}_{C,A}\|_1 = \max_{1\leq i\leq C}\left\{\sum_{j=1}^{A}\|\mathbf{c}_{ij}\|\right\}$	$\|\mathbf{C}_{C,A}\|_{w1} = \frac{1}{A}\|\mathbf{C}_{C,A}\|_1$
$\|\mathbf{C}_{C,A}\|_\infty = \max_{1\leq j\leq A}\left\{\sum_{i=1}^{C}\|\mathbf{c}_{ij}\|\right\}$	$\|\mathbf{C}_{C,A}\|_{w\infty} = \frac{1}{C}\|\mathbf{C}_{C,A}\|_\infty$
$\|\mathbf{C}_{C,A}\|_F = \sqrt{\sum_{i=1}^{C}\sum_{j=1}^{A}\|\mathbf{c}_{ij}\|^2}$	$\|\mathbf{C}_{C,A}\|_{wF} = \frac{1}{AC}\|\mathbf{C}_{C,A}\|_F$
$\|\mathbf{C}_{C,A}\|_{\max} = \max_{1\leq i\leq C, 1\leq j\leq A}\{\|\mathbf{c}_{ij}\|\}$	$\|\mathbf{C}_{C,A}\|_{w\max} = \|\mathbf{C}_{C,A}\|_{\max}$

Matrix norms $\| \circ \|_m$ that generate less overestimated error estimates are always of great practical value. Five classical matrix norms were used to determine the relative error in de Almeida et al. (2021). It is well known that in finite dimensional space all these matrix norms are mathematically equivalent (Demmel, 1997). Here, it is suggested to reformulate four of these matrix norms as shown in Table 1. Note that the logic of reformulating matrix norms consists of weighting. Thus, the weighted norms $\| \circ \|_{wm}$ can be understood as mean values when compared to the $\| \circ \|_{\max}$. Since $\| \circ \|_{wm} \leq \| \circ \|_{\max}$ for any matrix $\mathbf{C}_{C,A}$, therefore these weighted norms should provide error estimates closer to the actual error, and the $\| \circ \|_{\max}$ norm yields the theoretical upper bound for the relative error of the solution.

4 Two examples of application of this methodology

Two applications of the AHP found in the literature were chosen: $C > A$ presented in Wedley and Choo (2009) and $C < A$ presented in Bagla and Gupta (2011). The case $C = A$ will not be presented because no application with deficient rank was found in the literature that justifies the need to condense rows or columns.

4.1 Case $C > A$ presented in Wedley and Choo (2009)

The original synthesis of the AHP performed by Wedley and Choo (2009) is written as

$$\mathbf{X}^{ORG}_{3\times1} = \mathbf{G}_{3\times5}\mathbf{B}_{5\times1} = \begin{bmatrix} 0.421 & 0.274 & 0.180 & 0.516 & 0.358 \\ 0.368 & 0.373 & 0.403 & 0.355 & 0.224 \\ 0.211 & 0.353 & 0.417 & 0.129 & 0.418 \end{bmatrix} \begin{bmatrix} 0.300 \\ 0.150 \\ 0.200 \\ 0.100 \\ 0.250 \end{bmatrix}. \tag{20}$$

In this case, $A = 3$, $C = 5$ and $\mathbf{G}_{3\times5}$ is full rank ($K = 3$). Equation (2) defines an equivalent synthesis for AHP. If matrix $\mathbf{C}_{C\times A}$ is chosen as $\mathbf{C}_{C\times A} = \mathbf{G}^T_{C\times A}$, then the equivalent synthesis is

$$\mathbf{G}^T_{5\times3}\mathbf{X}^{EQV}_{3\times1} = \mathbf{G}^T_{5\times3}\mathbf{G}_{3\times5}\mathbf{B}_{5\times1}. \tag{21}$$

Since $\mathbf{G}_{3\times5}$ is full rank the equivalent synthesis has a unique solution $\mathbf{X}^{EQV}_{3\times1} = \mathbf{X}^{ORG}_{3\times1}$ (Alvarez et al., 2021). Matrices $\mathbf{G}_{3\times5}$ and $\mathbf{G}^T_{5\times3}$ are 'well-conditioned' because $cond_2(\mathbf{G}_{3\times5}) = 7.5810 = cond_2(\mathbf{G}^T_{5\times3})$. Therefore, in this case the equivalent synthesis is a 'well-posed' mathematical problem, and the only thing that could be improved with the condensation process described in Section 2 would be the stability of the solution $\mathbf{X}^{EQV}_{3\times1}$.

Since $K = 3 = A$ and $C = 5$ there are two LD columns that if condensed generate ten different square condensed matrices ${}^l\mathbf{H}_{3\times3}$, where $l = 1, \ldots, 10$. By Theorem 1 choosing columns 1 and 2 ($l = 1$) to condense is obtained

$${}^1\mathbf{X}^{CND}_{3\times1} = \underbrace{\begin{bmatrix} 0.180 & 0.516 & 0.358 \\ 0.403 & 0.355 & 0.224 \\ 0.417 & 0.129 & 0.418 \end{bmatrix}}_{{}^1\mathbf{H}_{3\times3}} \underbrace{\begin{bmatrix} \alpha^1_1 & \alpha^1_2 & 1 & 0 & 0 \\ \alpha^2_1 & \alpha^2_2 & 0 & 1 & 0 \\ \alpha^3_1 & \alpha^3_2 & 0 & 0 & 1 \end{bmatrix}}_{{}^1\alpha_{3\times5}} \begin{bmatrix} 0.300 \\ 0.150 \\ 0.200 \\ 0.100 \\ 0.250 \end{bmatrix}, \tag{22}$$

where the matrix ${}^1\alpha_{3\times5}$ is formed by the coefficients α^j_i of the linear combination of columns 1 and 2 in terms of columns 3, 4 and 5. Thus, the following condensed synthesis is obtained

$${}^1\mathbf{X}^{CND}_{3\times1} = \underbrace{\begin{bmatrix} 0.180 & 0.516 & 0.358 \\ 0.403 & 0.355 & 0.224 \\ 0.417 & 0.129 & 0.418 \end{bmatrix}}_{{}^1\mathbf{H}_{3\times3}} \underbrace{\begin{bmatrix} 0.281 & 0.670 & 1 & 0 & 0 \\ 0.715 & 0.222 & 0 & 1 & 0 \\ 0.004 & 0.108 & 0 & 0 & 1 \end{bmatrix}}_{{}^1\alpha_{3\times5}} \begin{bmatrix} 0.30 \\ 0.15 \\ 0.20 \\ 0.10 \\ 0.25 \end{bmatrix}$$

$$= \ {}^{1}\mathbf{H}_{3\times3}\ {}^{1}\mathbf{F}_{3\times1} = \begin{bmatrix} 0.180\ 0.516\ 0.358 \\ 0.403\ 0.355\ 0.224 \\ 0.417\ 0.129\ 0.418 \end{bmatrix} \begin{bmatrix} 0.385 \\ 0.348 \\ 0.267 \end{bmatrix} = \mathbf{X}^{ORG}_{3\times1}, \qquad (23)$$

and $cond({}^{1}\mathbf{H}_{3\times3}) < cond(\mathbf{G}_{3\times5})$. In this case, the critical column vector $\mathbf{Z}^{1,crt}_{3\times1}$ was condensed, and the condensation led to a lower condition number.

Another possible choice is to condense columns 2 and 5 ($l = 7$) of the matrix $\mathbf{G}_{3\times5}$. Proceeding in a similar way, one obtains

$${}^{7}\mathbf{X}^{CND}_{3\times1} = \ {}^{7}\mathbf{H}_{3\times3}\ {}^{7}\mathbf{F}_{3\times1} = \begin{bmatrix} 0.421\ 0.180\ 0.516 \\ 0.368\ 0.403\ 0.355 \\ 0.211\ 0.417\ 0.129 \end{bmatrix} \begin{bmatrix} 71.83749 \\ -19.80078 \\ -51.03671 \end{bmatrix} = \mathbf{X}^{ORG}_{3\times1}, \qquad (24)$$

and $cond({}^{7}\mathbf{H}_{3\times3}) >> cond(\mathbf{G}_{3\times5})$. Although this condensation leads to a worse condition number, all condensations yield the same solution ${}^{l}\mathbf{X}^{CND}_{3\times1} = \mathbf{X}^{ORG}_{3\times1}$. In Table 2, it can be seen that only two condensations improve the conditioning of the model, and the condensed matrix ${}^{6}\mathbf{H}_{3\times3}$ guarantees a unique solution with better stability of the solution. However, matrices $\mathbf{G}_{3\times5}$, ${}^{1}\mathbf{H}_{3\times3}$ and ${}^{6}\mathbf{H}_{3\times3}$ have a very similar condition number, and therefore equivalent stability for the sensitivity analysis. This shows that condensation does not always allow to obtain better condition number when $\mathbf{G}_{A\times C}$ is full rank. Although this seems to occur whenever $\mathbf{G}_{A\times C}$ is deficient rank. On the other hand, condensing the critical column vector $\mathbf{Z}^{1,crt}_{3\times1}$ does not guarantee a better condition number, since it was observed an improvement for ${}^{1}\mathbf{H}_{3\times3}$ and worsening for ${}^{2}\mathbf{H}_{3\times3}$, ${}^{3}\mathbf{H}_{3\times3}$ and ${}^{4}\mathbf{H}_{3\times3}$.

Table 2 Condition number of condensed synthesis, $cond_2(\mathbf{G}_{3\times5}) = 7.5810$

${}^{l}\mathbf{H}_{3\times3}$	Condensed columns	$cond_2({}^{l}\mathbf{H}_{3\times3})$
${}^{1}\mathbf{H}_{3\times3}$	$\{1,2\}$	6.2383
${}^{2}\mathbf{H}_{3\times3}$	$\{1,3\}$	7.8850
${}^{3}\mathbf{H}_{3\times3}$	$\{1,4\}$	16.1601
${}^{4}\mathbf{H}_{3\times3}$	$\{1,5\}$	59.2164
${}^{5}\mathbf{H}_{3\times3}$	$\{2,3\}$	18.4287
${}^{6}\mathbf{H}_{3\times3}$	$\{2,4\}$	6.1182
${}^{7}\mathbf{H}_{3\times3}$	$\{2,5\}$	1726.2543
${}^{8}\mathbf{H}_{3\times3}$	$\{3,4\}$	8.6033
${}^{9}\mathbf{H}_{3\times3}$	$\{3,5\}$	168.8825
${}^{10}\mathbf{H}_{3\times3}$	$\{4,5\}$	58.5912

The sensitivity analysis presented in Figure 1 indicates that minor perturbations in ${}^{6}\mathbf{H}_{3\times3}$ are necessary to cause the rank reversal when compared to perturbations in $\mathbf{G}_{3\times5}$. Absolute and positive perturbations were performed on element $\mathbf{h}^{crt}_{21} \in {}^{6}\mathbf{H}_{3\times3}$ and $\mathbf{g}^{crt}_{21} \in \mathbf{G}_{3\times5}$. Both are critical elements in their respective matrices. Figure 2 indicates that smaller perturbations in $\mathbf{B}_{5\times1}$ are needed to cause rank reversal when compared to perturbations in ${}^{6}\mathbf{F}_{3\times1}$. Absolute and positive perturbations were performed on element $\mathbf{b}_3$ (critical criterion) of the original and condensed square synthesis. Details on how perturbations are performed and how to build the sensitivity charts can be found in de Almeida et al. (2021). The vertical lines *rank reversal (1)* or *rank reversal (2)* and *bound of infeasible* delimit the region where the rank reversal occurs (Alvarez et al., 2021; de Almeida et al., 2021). *Rank reversal (1)* is the smallest perturbation that causes the rank reversal when only the critical element is perturbed. *Rank reversal (2)* is the smallest perturbation that causes the rank reversal when all components of the critical column vector are perturbed. *Bound of infeasible* is the limit value of viable perturbations of the critical element.

Figure 1 Sensitivity analysis perturbing $\mathbf{G}_{3\times5}$ in original AHP and ${}^6\mathbf{H}_{3\times3}$ in condensed square AHP ($C > A$ and $l = 6$)

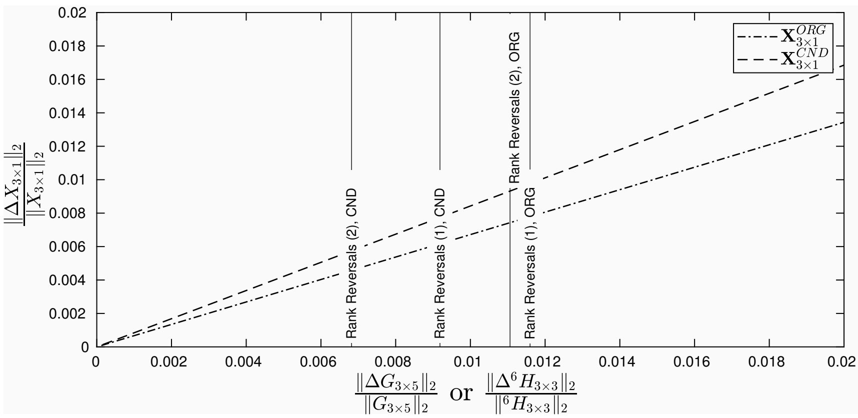
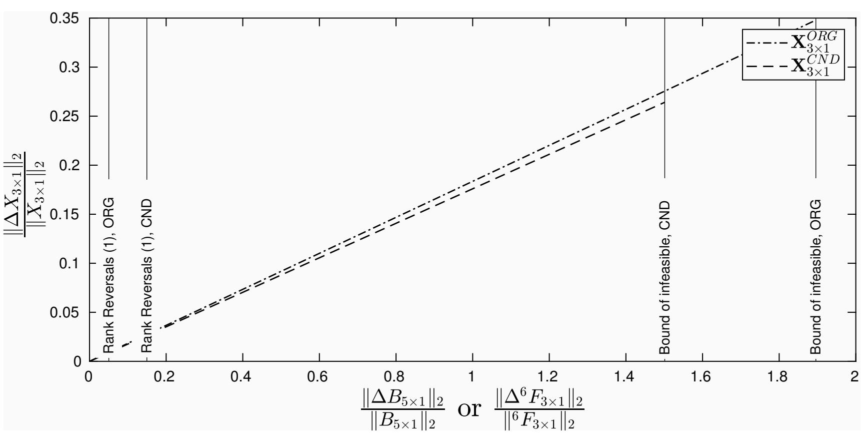


Figure 2 Sensitivity analysis perturbing $\mathbf{B}_{3\times1}$ in original AHP and ${}^6\mathbf{F}_{3\times1}$ in condensed square AHP ($C > A$ and $l = 6$)



The final step in our analysis is to verify numerically whether weighted norms provide error estimates closer to the actual error than classical norms. Following the development described in Section 3, the theoretical limit of the relative error of the solution is plotted for $\mathbf{G}_{3\times5}$ and ${}^1\mathbf{H}_{3\times3}$. The classical matrix norms $\| \circ \|_m$ were used to determine the relative error in Figures 3 and 5, while the weighted matrix norms $\| \circ \|_{wm}$ were used to determine the relative error in Figures 4 and 6.

Figure 3 Sensitivity analysis for $\mathbf{G}_{3\times5}$ using classical norms $\|\circ\|_m$ with absolute and positive perturbations on element $\mathbf{g}_{21}^{crt} \in \mathbf{G}_{3\times5}$

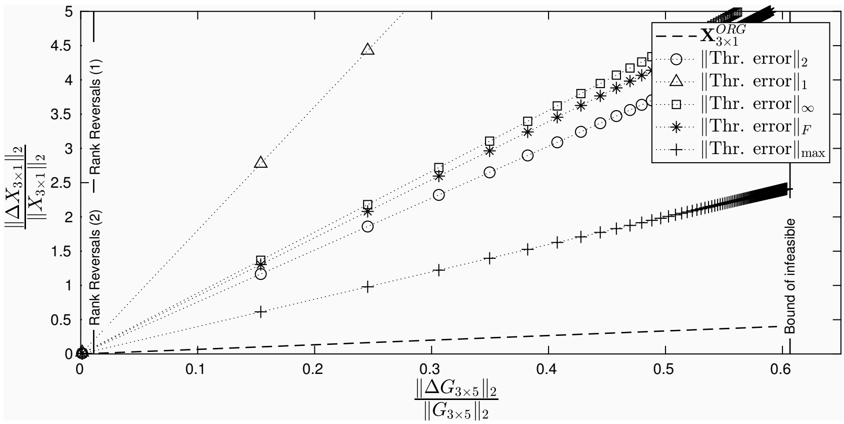


Figure 4 Sensitivity analysis for $\mathbf{G}_{3\times5}$ using weighted norms $\|\circ\|_{wm}$ with absolute and positive perturbations on element $\mathbf{g}_{21}^{crt} \in \mathbf{G}_{3\times5}$

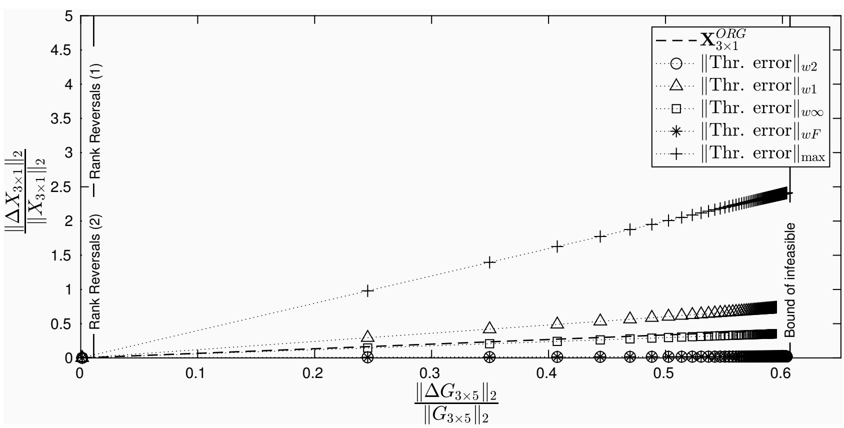


Figure 5 Sensitivity analysis for ${}^1\mathbf{H}_{3\times3}$ using classical norms $\|\circ\|_m$ with absolute and negative perturbations on element $\mathbf{h}_{11}^{crt} \in^1 \mathbf{H}_{3\times3}$

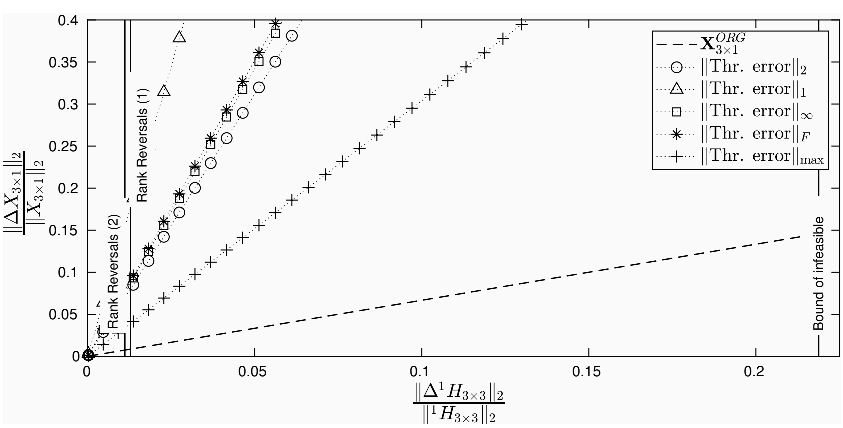
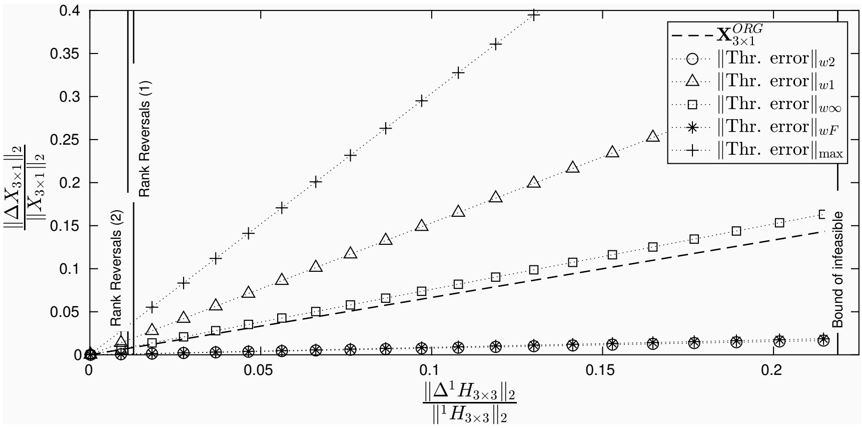


Figure 6 Sensitivity analysis for ${}^1\mathbf{H}_{3\times 3}$ using weighted norms $\|\circ\|_{wm}$ with absolute and negative perturbations on element $\mathbf{h}_{11}^{crt} \in^1 \mathbf{H}_{3\times 3}$



4.2 *Case $C < A$ presented in Bagla and Gupta (2011)*

The original synthesis of the AHP performed by Bagla and Gupta (2011) is written as

$$\mathbf{X}_{9\times1}^{ORG} = \mathbf{G}_{9\times4}\mathbf{B}_{4\times1} = \begin{bmatrix} 0.0195 & 0.0581 & 0.0314 & 0.0787 \\ 0.3083 & 0.2664 & 0.2820 & 0.3263 \\ 0.1534 & 0.0581 & 0.0175 & 0.1591 \\ 0.0195 & 0.0178 & 0.0689 & 0.0390 \\ 0.0195 & 0.1287 & 0.0314 & 0.0198 \\ 0.1534 & 0.1287 & 0.0689 & 0.0198 \\ 0.0195 & 0.0178 & 0.0689 & 0.0390 \\ 0.1534 & 0.0581 & 0.1488 & 0.1591 \\ 0.1534 & 0.2664 & 0.2820 & 0.1591 \end{bmatrix} \begin{bmatrix} 0.5650 \\ 0.1175 \\ 0.2622 \\ 0.0553 \end{bmatrix}. \tag{25}$$

In this case, $A = 9$, $C = 4$ and $\mathbf{G}_{9\times4}$ is full rank ($K = 4$). An equivalent synthesis for AHP is

$$\mathbf{G}_{4\times9}^{T}\mathbf{X}_{9\times1}^{EQV} = \mathbf{G}_{4\times9}^{T}\mathbf{G}_{9\times4}\mathbf{B}_{4\times1}. \tag{26}$$

Although $\mathbf{G}_{9\times4}$ is full rank, this equivalent synthesis does not have a unique solution, on the contrary it has infinite solutions (Alvarez et al., 2021). Matrices $\mathbf{G}_{9\times4}$ and $\mathbf{G}_{4\times9}^{T}$ are 'well-conditioned' because $cond_2(\mathbf{G}_{9\times4}) = 9.3448 = cond_2(\mathbf{G}_{4\times9}^{T})$. Therefore, this equivalent synthesis is an 'ill-posed' mathematical problem, and the condensation process described in Section 2 must be performed to make the equivalent synthesis a 'well-posed' mathematical problem.

Since $K = 4 = C$ and $A = 9$ there are five LD rows that if condensed generate 126 different condensed square matrices ${}^l\mathbf{H}_{4\times4}$, where $l = 1, \ldots, 126$. By Theorem 1 choosing rows 1, 2, 4, 8 and 9 ($l = 25$) to condense is obtained

$$\mathbf{X}^{ORG}_{9\times 1} = \underbrace{\begin{bmatrix} \beta_1^1 & \beta_2^1 & \beta_3^1 & \beta_4^1 \\ \beta_1^2 & \beta_2^2 & \beta_3^2 & \beta_4^2 \\ 1 & 0 & 0 & 0 \\ \beta_1^4 & \beta_2^4 & \beta_3^4 & \beta_4^4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \beta_1^8 & \beta_2^8 & \beta_3^8 & \beta_4^8 \\ \beta_1^9 & \beta_2^9 & \beta_3^9 & \beta_4^9 \end{bmatrix}}_{^{25}\beta_{9\times 4}} \underbrace{\begin{bmatrix} 0.1534 & 0.0581 & 0.0175 & 0.1591 \\ 0.0195 & 0.1287 & 0.0315 & 0.0199 \\ 0.1534 & 0.1287 & 0.0689 & 0.0199 \\ 0.0195 & 0.0178 & 0.0689 & 0.0390 \end{bmatrix}}_{^{25}\mathbf{G}^{LI}_{4\times 4}} \underbrace{\begin{bmatrix} 0.5650 \\ 0.1175 \\ 0.2622 \\ 0.0553 \end{bmatrix}}_{\mathbf{B}_{4\times 1}}$$

$$= \underbrace{\begin{bmatrix} 0.3530 & 0.5875 & -0.3587 & 0.4575 \\ 1.1526 & 0.7551 & 0.3682 & 3.0886 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5389 & -0.3134 & 0.2582 & 1.9082 \\ 0.0328 & 1.2074 & 0.4171 & 3.1174 \end{bmatrix}}_{^{25}\beta_{9\times 4}} \quad (27)$$

$$\times \underbrace{\begin{bmatrix} 0.1534 & 0.0581 & 0.0175 & 0.1591 \\ 0.0195 & 0.1287 & 0.0315 & 0.0199 \\ 0.1534 & 0.1287 & 0.0689 & 0.0199 \\ 0.0195 & 0.0178 & 0.0689 & 0.0390 \end{bmatrix}}_{^{25}\mathbf{G}^{LI}_{4\times 4}} \underbrace{\begin{bmatrix} 0.5650 \\ 0.1175 \\ 0.2622 \\ 0.0553 \end{bmatrix}}_{\mathbf{B}_{4\times 1}},$$

where the matrix $^{25}\mathbf{G}^{LI}_{4\times 4}$ is formed by K linear independent rows of a matrix $\mathbf{G}_{9\times 4}$ (not condensed) and the matrix $^{25}\beta_{9\times 4}$ is formed by the coefficients β_i^j of the linear combination of rows 1, 2, 4, 8 e 9 in terms of rows 3, 5, 6 e 7. Note that $\mathbf{G}_{9\times 4} = {}^{l}\beta_{9\times 4}\,{}^{l}\mathbf{G}^{LI}_{4\times 4}$ for $l = 1, \ldots, 126$. Thus, the condensed square synthesis is obtained by auxiliary system

$$\begin{aligned} {}^{25}\mathbf{X}^{CND}_{4\times 1} &= \hat{\mathbf{X}}^{AUX}_{4\times 1} \\ &= \begin{bmatrix} \hat{\mathbf{x}}_3 \\ \hat{\mathbf{x}}_5 \\ \hat{\mathbf{x}}_6 \\ \hat{\mathbf{x}}_7 \end{bmatrix} = \begin{bmatrix} 0.3289 \\ 0.1149 \\ 0.2038 \\ 0.3524 \end{bmatrix} = \begin{bmatrix} 0.4721 & 0.1787 & 0.0538 & 0.4897 \\ 0.0632 & 0.4167 & 0.1018 & 0.0643 \\ 0.2585 & 0.2169 & 0.1161 & 0.0334 \\ 0.2063 & 0.1877 & 0.7283 & 0.4126 \end{bmatrix} \begin{bmatrix} 0.5650 \\ 0.1175 \\ 0.2622 \\ 0.0553 \end{bmatrix} \\ &= \begin{bmatrix} (1+\sum_j \beta_1^j)\mathbf{g}_{31} & (1+\sum_j \beta_1^j)\mathbf{g}_{32} & (1+\sum_j \beta_1^j)\mathbf{g}_{33} & (1+\sum_j \beta_1^j)\mathbf{g}_{34} \\ (1+\sum_j \beta_2^j)\mathbf{g}_{51} & (1+\sum_j \beta_2^j)\mathbf{g}_{52} & (1+\sum_j \beta_2^j)\mathbf{g}_{53} & (1+\sum_j \beta_2^j)\mathbf{g}_{54} \\ (1+\sum_j \beta_3^j)\mathbf{g}_{61} & (1+\sum_j \beta_3^j)\mathbf{g}_{62} & (1+\sum_j \beta_3^j)\mathbf{g}_{63} & (1+\sum_j \beta_3^j)\mathbf{g}_{64} \\ (1+\sum_j \beta_4^j)\mathbf{g}_{71} & (1+\sum_j \beta_4^j)\mathbf{g}_{72} & (1+\sum_j \beta_4^j)\mathbf{g}_{73} & (1+\sum_j \beta_4^j)\mathbf{g}_{74} \end{bmatrix} \end{aligned} \quad (28)$$

$$\times \begin{bmatrix} 0.5650 \\ 0.1175 \\ 0.2622 \\ 0.0553 \end{bmatrix},$$

and $cond({}^{25}\mathbf{H}_{4\times4}) < cond(\mathbf{G}_{9\times4})$. In this way, a condensed square equivalent synthesis to equation (28) is ${}^{25}\mathbf{H}_{4\times4}^{-1}\hat{\mathbf{X}}_{4\times1}^{AUX} = \mathbf{B}_{4\times1}$, which is a 'well-posed' mathematical problem. The solution of the original AHP is recovered as

$$\mathbf{X}_{9\times1}^{ORG} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \\ \mathbf{x}_5 \\ \mathbf{x}_6 \\ \mathbf{x}_7 \\ \mathbf{x}_8 \\ \mathbf{x}_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{x}_3 \\ 0 \\ \mathbf{x}_5 \\ \mathbf{x}_6 \\ \mathbf{x}_7 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ 0 \\ \mathbf{x}_4 \\ 0 \\ 0 \\ 0 \\ \mathbf{x}_8 \\ \mathbf{x}_9 \end{bmatrix}, \tag{29}$$

where

$$\begin{bmatrix} \mathbf{x}_3 \\ \mathbf{x}_5 \\ \mathbf{x}_6 \\ \mathbf{x}_7 \end{bmatrix} = \begin{bmatrix} \frac{1}{1+\beta_1^1+\beta_1^2+\beta_1^4+\beta_1^8+\beta_1^9}\hat{\mathbf{x}}_3 \\ \frac{1}{1+\beta_2^1+\beta_2^2+\beta_2^4+\beta_2^8+\beta_2^9}\hat{\mathbf{x}}_5 \\ \frac{1}{1+\beta_3^1+\beta_3^2+\beta_3^4+\beta_3^8+\beta_3^9}\hat{\mathbf{x}}_6 \\ \frac{1}{1+\beta_4^1+\beta_4^2+\beta_4^4+\beta_4^8+\beta_4^9}\hat{\mathbf{x}}_7 \end{bmatrix}, \tag{30}$$

after solving the condensed system (28) and

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_4 \\ \mathbf{x}_8 \\ \mathbf{x}_9 \end{bmatrix} = \begin{bmatrix} \beta_1^1\mathbf{x}_3+\beta_2^1\mathbf{x}_5+\beta_3^1\mathbf{x}_6+\beta_4^1\mathbf{x}_7 \\ \beta_1^2\mathbf{x}_3+\beta_2^2\mathbf{x}_5+\beta_3^2\mathbf{x}_6+\beta_4^2\mathbf{x}_7 \\ \beta_1^4\mathbf{x}_3+\beta_2^4\mathbf{x}_5+\beta_3^4\mathbf{x}_6+\beta_4^4\mathbf{x}_7 \\ \beta_1^8\mathbf{x}_3+\beta_2^8\mathbf{x}_5+\beta_3^8\mathbf{x}_6+\beta_4^8\mathbf{x}_7 \\ \beta_1^9\mathbf{x}_3+\beta_2^9\mathbf{x}_5+\beta_3^9\mathbf{x}_6+\beta_4^9\mathbf{x}_7 \end{bmatrix} \tag{31}$$

Another possible choice is to condense rows 1, 2, 4, 6 and 7 of the matrix $\mathbf{G}_{9\times4}$, which will be denoted as ${}^{20}\mathbf{H}_{4\times4}$. Proceeding in a similar way, it is possible to obtain

$$\begin{aligned} \hat{\mathbf{X}}_{4\times1}^{AUX} &= {}^{20}\mathbf{H}_{4\times4}\mathbf{B}_{4\times1} \\ &= \begin{bmatrix} -6.8365 & -2.5881 & -0.7790 & -7.0917 \\ 1.8648 & 12.3028 & 3.0057 & 1.8970 \\ 14.0543 & 5.3205 & 13.6340 & 14.5791 \\ -8.0826 & -14.0352 & -14.8607 & -8.3844 \end{bmatrix} \begin{bmatrix} 0.5650 \\ 0.1175 \\ 0.2622 \\ 0.0553 \end{bmatrix}, \end{aligned} \tag{32}$$

and $cond({}^{20}\mathbf{H}_{4\times4}) >> cond(\mathbf{G}_{9\times4})$. In this case, the condensation led to a worse condition number, but all condensations yield the same solution $\mathbf{X}_{A\times1}^{ORG}$. For reasons of space, only a few condensations have been shown. In Table 3, it can be seen that some condensations improve a little the conditioning of the model (l = 25, 35, 72, 78, 95, 96, 104), some get a little worse the conditioning of the model (l = 55, 65, 90, 105) and other get a lot worse the conditioning of the model (l = 6, 118). This last situation occurs because the rows 4 and 7 of $\mathbf{G}_{9\times4}$ are the same, so if these are not condensed, ${}^{l}\mathbf{H}_{4\times4}$ becomes deficient rank.

Table 3 Condition number of condensed synthesis, $cond_2(\mathbf{G}_{9\times4}) = 9.3448$

${}^{l}\mathbf{H}_{4\times4}$	*Condensed columns*	$cond_2({}^{l}\mathbf{H}_{4\times4})$
${}^{4}\mathbf{H}_{4\times4}$	$\{1,2,3,4,8\}$	227.7186
${}^{6}\mathbf{H}_{4\times4}$	$\{1,2,3,5,6\}$	1.5753×10^{16}
${}^{20}\mathbf{H}_{4\times4}$	$\{1,2,4,6,7\}$	2808.2904
${}^{25}\mathbf{H}_{4\times4}$	$\{1,2,4,8,9\}$	7.1400
${}^{35}\mathbf{H}_{4\times4}$	$\{1,2,7,8,9\}$	7.1400
${}^{55}\mathbf{H}_{4\times4}$	$\{1,3,7,8,9\}$	28.3810
${}^{65}\mathbf{H}_{4\times4}$	$\{1,4,7,8,9\}$	56.2397
${}^{72}\mathbf{H}_{4\times4}$	$\{2,3,4,5,7\}$	7.6550
${}^{78}\mathbf{H}_{4\times4}$	$\{2,3,4,7,8\}$	6.0834
${}^{90}\mathbf{H}_{4\times4}$	$\{2,3,7,8,9\}$	10.4497
${}^{95}\mathbf{H}_{4\times4}$	$\{2,4,5,7,9\}$	8.8626
${}^{96}\mathbf{H}_{4\times4}$	$\{2,4,5,8,9\}$	8.5252
${}^{104}\mathbf{H}_{4\times4}$	$\{2,5,7,8,9\}$	8.5252
${}^{105}\mathbf{H}_{4\times4}$	$\{2,6,7,8,9\}$	9.3493
${}^{118}\mathbf{H}_{4\times4}$	$\{3,5,6,8,9\}$	6.3048×10^{17}

The sensitivity analysis presented in Figure 7 indicates that minor perturbations in $\mathbf{G}_{9\times4}$ are necessary to cause the rank reversal when compared to perturbations in ${}^{25}\mathbf{H}_{4\times4}$, since the vertical line *rank reversal (1)* of the original synthesis is further to the left than the vertical line *rank reversal (1)* of the condensed synthesis. Absolute and negative perturbations were performed on element $\mathbf{h}_{21} \in {}^{25}\mathbf{H}_{4\times4}$ and $\mathbf{g}_{51} \in \mathbf{G}_{9\times4}$. Both are not critical elements in their respective matrices (critical element $\mathbf{g}_{44}^{crt}$ was condensed in this choice), but were randomly selected ensuring similarity in the perturbations. Figure 8 indicates that similar perturbations on $\mathbf{B}_{4\times1}$ and ${}^{25}\mathbf{F}_{4\times1}$ are needed to cause rank reversal. Absolute and negative perturbations were performed on element $\mathbf{b}_2$ (critical criterion) of the original and condensed square synthesis.

To verify numerically whether weighted norms provide error estimates closer to the actual error than classical norms, the theoretical limit of the relative error of the solution is plotted for $\mathbf{G}_{9\times4}$ and ${}^{25}\mathbf{H}_{4\times4}$. The classical matrix norms $\|\circ\|_m$ were used to determine the relative error in Figures 9 and 11, while the weighted matrix norms $\|\circ\|_{wm}$ were used to determine the relative error in Figures 10 and 12. *Rank reversal (1)* is at $\frac{\|\Delta\mathbf{G}_{9\times4}\|_2}{\|\mathbf{G}_{9\times4}\|_2} \approx 0$ for perturbations in $\mathbf{g}_{44}^{crt} \in \mathbf{G}_{9\times4}$, as can be seen in Figures 9 and 10. This means that the original synthesis (25) is very sensitive to small perturbations in the critical element. However, Figures 11 and 12 show that the condensed formulation (28) is more stable to perturbations in the critical element.

Figure 7 Sensitivity analysis perturbing $\mathbf{G}_{9\times4}$ in original AHP and ${}^{25}\mathbf{H}_{4\times4}$ in condensed square AHP ($C < A$ and $l = 25$)

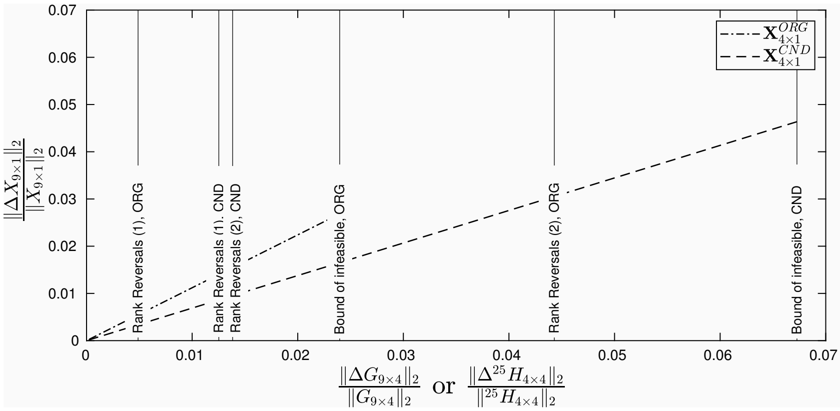


Figure 8 Sensitivity analysis perturbing $\mathbf{B}_{4\times1}$ in original AHP and ${}^{25}\mathbf{F}_{4\times1}$ in condensed square AHP ($C < A$ and $l = 25$)

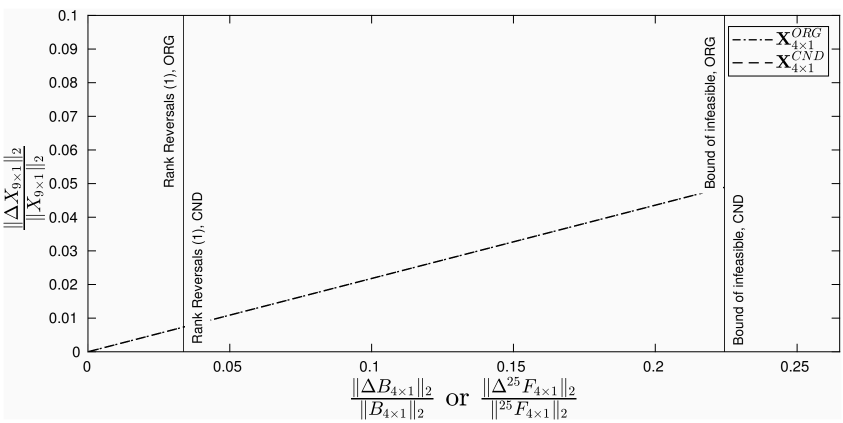


Figure 9 Sensitivity analysis for $\mathbf{G}_{9\times4}$ using classical norms $\| \circ \|_m$ with absolute and positive perturbations on element $\mathbf{g}^{crt}_{44} \in \mathbf{G}_{9\times4}$

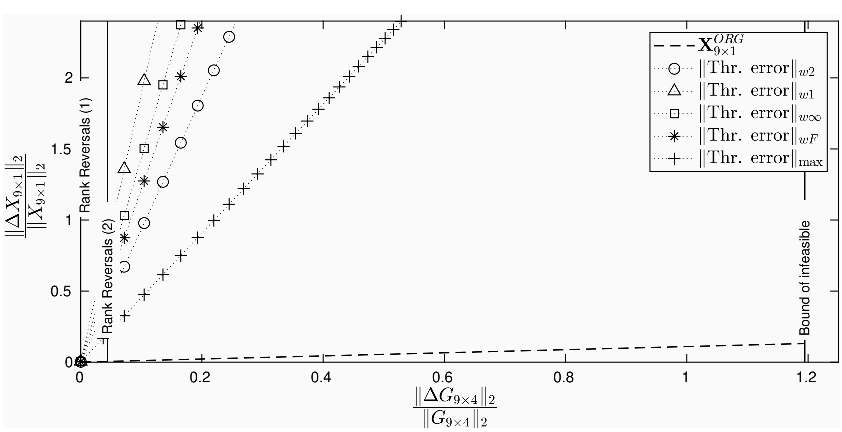


Figure 10 Sensitivity analysis for $\mathbf{G}_{9\times4}$ using weighted norms $\|\circ\|_{wm}$ with absolute and positive perturbations on element $\mathbf{g}_{44}^{crt} \in \mathbf{G}_{9\times4}$

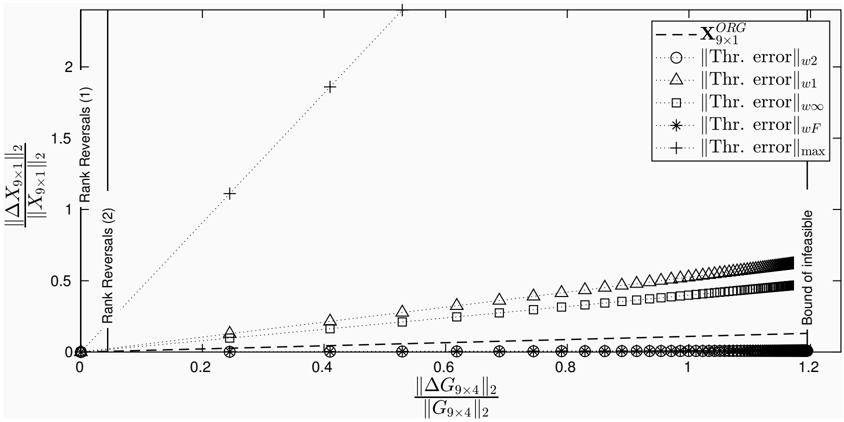


Figure 11 Sensitivity analysis for ${}^{25}\mathbf{H}_{4\times4}$ using classical norms $\|\circ\|_m$ with absolute and negative perturbations on element $\mathbf{h}_{21}^{crt} \in {}^{25}\mathbf{H}_{4\times4}$

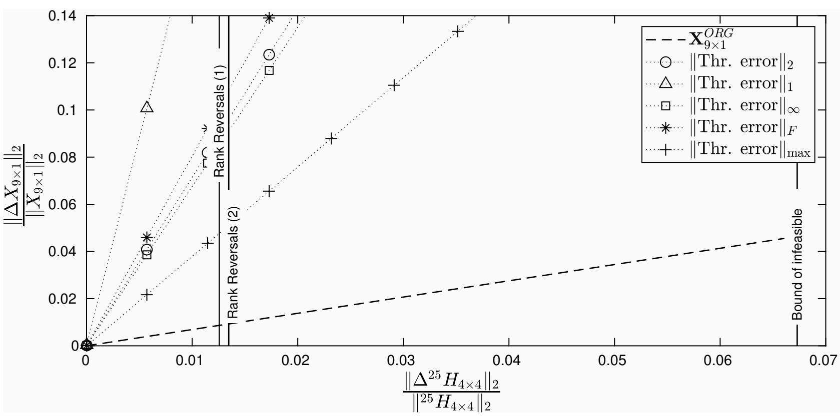
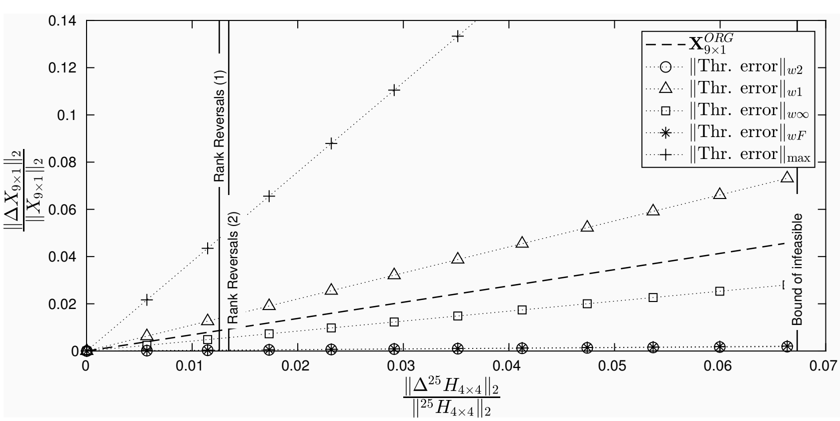


Figure 12 Sensitivity analysis for ${}^{25}\mathbf{H}_{4\times4}$ using weighted norms $\|\circ\|_{wm}$ with absolute and negative perturbations on element $\mathbf{h}_{21}^{crt} \in {}^{25}\mathbf{H}_{4\times4}$



5 Conclusions

The contribution of the paper is twofold:

1 the synthesis of the AHP as a ‘well-posed’ mathematical problem

2 matrix norms appropriate for sensitivity analysis via condition number.

The synthesis of the AHP as a ‘well-posed’ mathematical problem pretends to ensure uniqueness of the solution and better conditioning of the method by a condensation process. When the number of criteria is greater than the number of alternatives in the modelling and the matrix $\mathbf{G}_{A\times C}$ is full rank, the condensed square original and equivalent synthesis of the AHP are a ‘well-posed’ mathematical problem. Then, for both formulations there is always a unique choice as the best possible, where this choice is the same for both formulations, and the condensation process is not necessary. In these cases, the condensation process would only improve the stability of the solution if the condition number of the system is reduced ($cond_2(^l\mathbf{H}_{K\times K}) < cond_2(\mathbf{G}_{A\times C})$ for some l). However, when the number of criteria is less than the number of alternatives in the modelling (even if the matrix $\mathbf{G}_{A\times C}$ is full rank) or the matrix $\mathbf{G}_{A\times C}$ is deficient rank, the equivalent synthesis will have infinite solutions, and therefore it will be an ‘ill-posed’ mathematical problem. In these cases the condensation process is necessary to make the equivalent synthesis a ‘well-posed’ mathematical problem with guarantees a unique solution. But, since the condition number of the condensed matrices $^l\mathbf{H}_{K\times K}$ is not always less than the condition number of the original matrix $\mathbf{G}_{A\times C}$, there is no guarantee of a solution with better stability for any condensed square matrix. It should be stressed that in condensed formulations criteria or alternatives are not removed from the analysis. They are only condensed to ensure uniqueness of the solution and better conditioning of the method.

On the other hand, the sensitivity analysis via condition number used weighted norms pretends to obtain error estimates closer to the actual error, unlike the classical matrix norms used in de Almeida et al. (2021). The weighted norms $\| \circ \|_{wm}$ can be understood as mean values when compared to the $\| \circ \|_{\max}$, and $\| \circ \|_{wm} \leq \| \circ \|_{\max}$ for any matrix $\mathbf{C}_{C,A}$. This shows the importance of choosing the appropriate norm to carry out a theoretical sensitivity analysis closer to the actual application cases. In addition, as the sensitivity analysis via condition number shows, the rank reversal effect is still present in square and rectangular formulations. In Alvarez et al. (2021) and de Almeida et al. (2021) it was stated that it is impossible to eliminate the rank reversal effect, since it is inherent to every linear system. Even so, the condensed square original (7) and equivalent (17) synthesis can be less sensitive to rank reversal than the rectangular original (1) and equivalent (2) synthesis.

In future work, other steps of the AHP (modelling, valuation and/or prioritisation) should be analysed to ensure greater control over the rank reversal effect. In this way, a relationship between the number of alternatives/criteria, the experts’ judgements in the pairwise comparison matrices, the $cond\,(\mathbf{G}_{A,C})$, the weighted norms $\| \circ \|_{wm}$ and the rank reversal can be constructed. Furthermore, it will be intended to find theoretical upper bounds when there are simultaneously uncertainties in $\mathbf{F}_{K\times 1}$ and $\mathbf{H}_{K\times K}$.

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