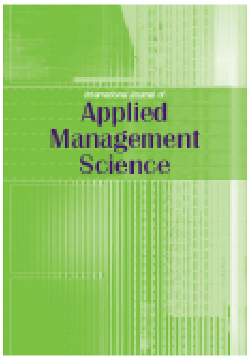

International Journal of Applied Management Science

ISSN online: 1755-8921 - ISSN print: 1755-8913
https://www.inderscience.com/ijams

A note on an EOQ inventory model with varying time-proportional deterioration, time-dependent demand and shortages under the conditions of permissible delay in payment

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DOI: 10.1504/IJAMS.2024.10060567

Article History:

Received:	03 June 2023
Last revised:	01 August 2023
Accepted:	21 September 2023
Published online:	18 January 2024

A note on an EOQ inventory model with varying time-proportional deterioration, time-dependent demand and shortages under the conditions of permissible delay in payment

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Abstract: The objective of the proposed research is to investigate the effect of trade credit policy with shortages within the Economic Order Quantity (EOQ) framework. The inventory model for a deteriorating item is developed with the following characteristics: (1) delay in payments is permitted; (2) demand is deterministic, time-dependent and quadratic function of time; (3) deteriorating items follow a time-proportional deterioration rate; (4) the grace period is known and fixed and (5) shortages are permitted to occur and are completely backlogged. For settling the account, the formulation of the model is derived under two main circumstances: Case 1 – the grace period is less than the shortages period and Case 2 – the grace period is greater than the shortages period of the system. A couple of numerical examples and sensitivity analysis of the optimal cost with respect to different parameters of the model has been carried out.

Keywords: deteriorating items; EOQ; permissible delay in payments; shortages; time-dependent quadratic demand; time-proportional deterioration rate.

Reference to this paper should be made as follows: Swain, P. and Singh, T. (2024) 'A note on an EOQ inventory model with varying time-proportional deterioration, time-dependent demand and shortages under the conditions of permissible delay in payment', *Int. J. Applied Management Science*, Vol. 16, No. 1, pp.89–112.

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1 Introduction

Now-a-days inventory management is a very important and essential assignment for any organisation or company. They call for careful inventory policies to minimise the total cost or to maximise the total profit. The industrial growth has brought about the need for division of management function within an organisation. Therefore, every organisation runs with a number of departments, each performing successfully as a part of the whole job, developing its own policies. For example, the production department wants to have maximum production with minimum cost. This can be achieved by producing only one type of item continuously. The marketing department wants to have maximum inventory so that a customer may be ensured immediate delivery over a wide variety of products. The finance department wants to minimise the unproductive capital investment 'tied up' to it. The personnel department wants to hire the workers for the continuous production of items so that inventory will not be affected. Therefore, to set an inventory policy which serves the interest of a business organisation as a whole and not that of any individual department is an executive-type problem.

The classical Economic Order Quantity (EOQ) model has been used extensively in inventory management for more than seven decades since its commencement due to its simplicity and practical applicability to various cases in practice. It is based on the trade of between two different types of system costs in inventory management: namely, the average ordering cost and the average holding cost. Deterioration is an unavoidable natural phenomenon which prevents the items from being used for its natural purposes. It occurs due to climatic variations, spoilage, evaporation and damage to the item, and obsolescence such as in the case of agricultural products, food stuff, volatile liquids, drugs, fashion goods, electronics components, spare parts of automobiles, high-tech product, among others. The quality and usefulness of deteriorated items reduce with the storage period. Therefore, it becomes very important to study and analyse the effect of deterioration in an inventory model. Consequently, the EOQ inventory model for deteriorating items has caught the interest of many researchers. Numerous research papers have inspected issues associated with deteriorating items for production management, inventory management and supply chain management. Mentioned below is a review of the literature on deteriorating items. Harris (1913) introduced the basic square root formula for the development of the EOQ model for inventory control. This model answered the crucial question (How many items one must order?) in inventory management accurately. Deteriorating items include various categories of items like hardware items; fashion apparels; food items like food stuff, fruits, fish, egg, meat, vegetables, etc.; radioactive substances; volatile substances, chemicals, etc.; drugs and medicines, etc. as they show the loss more in their qualities and quantities. In case of hardware, fashion appeals, etc. the deterioration rate is too low whereas for fruits, food-stuffs, grains, chemicals and radio-active substances, etc. the deterioration rate is remarkably high over the passage of time. For the former case, there is little need for keeping the inventory safe whereas, for the latter case, it is very indispensable for keeping it safe and controlling the inventory. Ghare and Schrader (1963) presented the EOQ inventory system for the decaying items in terms of demand and deterioration mathematically with the help of differential equation

$$\frac{dI(t)}{dt} + \theta(t) I(t) + D(t) = 0, \qquad 0 \leq t \leq T \tag{1}$$

Here $D(t)$, $\theta(t)$, $I(t)$ and T denote the demand rate, the deterioration rate, the inventory level and the length of the replenishment cycle, respectively. In the above inventory model for deteriorating items, it is found that the inventory diminishes due to the constant demand rate. However, in the real-life situations, the demand rate of any item may vary with time. Researchers then developed various types of models with the introduction of time-dependent demand patterns. Silver and Meal (1969) were the first to modify the classical square root formula considering time varying demand. Researchers like Donaldson (1977); Silver (1979); Dave and Patel (1981) and Goyal (1985), etc. made their valuable contributions in the direction of linear trended demand. However, shortages were not allowed to occur in their models. Deb and Chaudhuri (1987), Dave (1989), Goswami and Chaudhuri (1991), Goyal et al. (1992), Giri et al. (1996) and Teng (1996) were the researchers who studied the model of Silver (1979) to incorporate shortages in inventory. The up-to-date records on inventory articles were presented by Goyal and Giri, (2001), Li et al. (2010) and Janssen et al. (2016). Later, Kandemir (2020) presented an algorithm for inventory allocation of common products among many customers in manufacturing system. A series of steps including the cases of backlogs, unfulfilled forecasts and planned safety stocks are found in this algorithm. Agrawal et al. (2022) presented an optimal inventory policy for determining pricing, lot-sizing and promotional expenses for some luxury products like Veblen products.

From the existing literature, it was seen that while dealing with time varying demand pattern, the researchers have considered two types of demand rate, namely linear and exponential. Wee (1995) and Jalan and Chaudhuri (1999) presented their models by assuming exponential time varying demand pattern. A linearly time varying-demand rate represents uniform change in demand rate of the item per unit of time which is rarely seen to occur in real market where as exponentially time-varying demand rate indicates a very rapid change in demand rate of any item which is also unrealistic. Therefore, the most realistic demand pattern among constant, linear, exponential and quadratic demand pattern is the time-dependent demand pattern representing both the accelerated growth and accelerated decline in demand rate. Khanra and Chaudhuri (2003) were the first researcher who studied the no-shortages inventory model for deteriorating items with help of time-dependent quadratic demand rate. Ghosh and Chaudhuri (2006) extended the EOQ model of Khanra and Chaudhuri (2003) over a finite time horizon with shortages in all cycles. Dari and Sani (2020) presented an EPQ (economic production quantity) model for delayed deteriorating items with time-dependent quadratic demand and linear holding cost is assumed to be linearly dependent on time. Their model includes three stages such as production build up period, period before deterioration starts and period after deterioration sets in. In business scenario, a cash discount is an assumption authorised a number of retailers for items in order to encourage consumers to pay within a particular time. The suppliers presenting a cash discount will pass on to it as a sales price deduction, while the customer will submit to the same deduction as a obtain money pay off. It is a business tool for seller in order to encourage purchaser to buy more. A cash discount associated permissible delay sensitivity optimal policy for deteriorating items with Weibull distribution receptive demand and shortages was studied by Tripathi and Pandey (2023).

Generally, constant deterioration is not always perfect for the development of the inventory models for deteriorating items as the process of deterioration varies directly proportional with the time. It is also studied that the varying deterioration rate is suitable for the items whose deterioration starts increasing after storing the items. Covert and Philip (1973) extended Ghare and Schrader's (1963) inventory model by considering the suitable deterioration rate as the two-parameter Weibull distribution deterioration rate instead of constant deterioration rate. Later, Philip (1974) modified Covert and Philip's (1973) inventory model by taking the three-parameter Weibull distribution deterioration rate instead of the two-parameter Weibull distribution deterioration. Singh et al. (2018) presented an EOQ inventory model for deteriorating items with shortages by incorporating constant deterioration and ramp-type of demand rate. A three-cased EOQ optimal inventory model for deteriorating items with the incorporation of the shortages, trapezoidal type demand and three-parameter Weibull distribution deterioration rate was studied by Singh et al. (2021).

In general, it is seen that the customers pay for the items as soon as these are received. But in case of a competitive business environment, suppliers usually offer a grace period to the customers to buy more and earn interest. In other words, suppliers provide an option to their customers to settle their payments after their agreed duration. Also, the customers have to pay interest if they make payments after the expiration of the grace period at a rate agreed upon initially. This delay or grace period that offers an opportunity for the customers to earn interest in the revenues obtained by sales is known as credit period or trade credit period or delay period or permissible delay period. In other words, grace period is an important business tool which is provided to give confidence the customers to purchase more items, to reduce the costs of holding inventory, or to increase the market share. In this context, Goyal (1985) was the first researcher who extended Ghare and Schrader's (1963) inventory model for deteriorating items by incorporating the concept of grace period. Mandal and Phaujder (1989) and Aggarwal and Jaggi (1995) extended Goyal's (1985) inventory model by developing a mathematical model to minimise the average total cost under permissible delay in payments. Jamal et al. (1997) also extended Goyal's (1985) model under the conditions of constant deterioration and shortages. Several valuable contributions in this research field were studied by Chu et al. (1998), Sana and Chaudhuri (2008), Cheng and Wang (2009), Khanra et al. (2011) and Sarkar et al. (2012). In a Vendor Managed Inventory (VMI) system, the supplier is more responsible for deciding the time of arrival of items and its quantity. Shaikh et al. (2022) developed an optimal ordering Vendor Managed Inventory (VMI) system for a deteriorating item with price-dependent demand and shortages under permissible delay in payment. The implementation of the VMI is more suitable than an inventory model without VMI, is stated in their model. Generally, shortages are not allowed occurring at the beginning of the replenishment cycle for many fashionable product or high-tech electronic items. Furthermore, the same condition is required in order to satisfy the customers demand and increase the sales. Inventory models may allow shortages at the end of the replenishment cycle. In this context, a two-warehouse partial backlogging optimal policy for deteriorating items with permissible delay in payment and the condition of inflation was studied by Yang (2022). The classical inventory analysis deals with the items which are produced to meet specified standards. However, some items in a lot may not fulfil such standards and are sold at a discount. Such type of variation in the quality of items may occur due to randomness in production systems. A two-warehouse based inventory model with inventory decisions for imperfect quality and deteriorating items under two-level trade credit was proposed by Tiwari et al. (2022). Recently, Khanra et al. (2013) studied an optimal ordering policy for a deteriorating item under the assumptions of shortages, quadratic demand and delay in payment.

In this paper, an EOQ inventory model for a deteriorating item is studied with time varying time-proportional deterioration rate, time-dependent quadratic demand rate and shortages under permissible delay in payments. The assumption of several demand patterns like constant, linear and exponential demands are not highly suitable for the items like newly launched computer parts, mobile phones, laptops, fashion apparels and cosmetic, etc. in which the time span of these items has a negative impact on demand due to loss of quality of such items. However, quadratic type of time-dependent demand rate is more realistic because it can behave both accelerated growth and retarded growth in demand. The general form of the quadratic demand rate is $D(t) = a + bt + ct^2$. The parameter $c = 0$ represents linear type time-dependent demand and $b = 0$ as well as $c = 0$ simultaneously represent constant demand. The model has been developed under two main circumstances, Case 1: the grace period is less than shortage period in the system and Case 2: the grace period is greater than shortage period of the system. The shortages as well as completely backlogging in this model are permitted to occur. It is also seen that the rate of deterioration changes with respect to time. Therefore, the time-proportional deterioration is suitable for the items whose deterioration rate is directly proportional to the time span of storing the items. The differential calculus technique is used to determine the conditions of the optimal solution. The primary aim of the proposed model is to optimise the optimal total cost and order quantity by determining the shortages period and cycle period. The present model is also valid for the case of newly launched fashionable items like fashion apparels, cosmetic, etc. and electronics items like mobile sim cards, android mobile phones, Laptops, super computers, automobiles and their spare parts. The proposed model is also very much useful to efficient designing and controlling the inventory system in various competitive markets like, pharmaceutical industries, food processing industries and fruits and vegetable markets, etc. The proposed model is analysed and explained with a couple of numerical examples. Finally, a detailed sensitivity analysis is additionally carried out to analyse the impacts of all system parameters.

Apart from Section 1, the paper is organised as follows: In Section 2, mathematical notations and fundamental assumptions involved in the inventory system are introduced. Section 3 provides the mathematical formulation involving two different cases and the solution procedure of obtaining optimal solution. A couple of numerical examples based on two different cases are provided to find the optimal solution and order quantity is given in Section 4. Section 5 provides the sensitivity analysis of a numerical example with respect to key parameters. Finally, conclusions and the directions for future research are pointed out in Section 6.

2 Notations and assumptions

In this section, notations and assumptions are introduced for the mathematical formulation of the model.

2.1 Notations

$I(t)$ The number of units in the inventory at any time t, $t \geq 0$.

t_1 Time when inventory level comes down to zero in year (decision variable).

μ Delay period in settling the account in year.

T Length of the replenishment cycle in year (decision variable).

$\theta(t) = \theta t, \quad 0 < \theta << 1$ Time-proportional deterioration rate.

$D(t) = a + bt + ct^2$ Demand rate at any time t where $a > 0,\ b \neq 0\ \&\ c \neq 0$. It is continuous function of time and is quadratic in nature where the parameters $a,\ b\ \&\ c$ represent the initial rate of demand, the increasing rate of demands and self-changing demand, respectively.

I_0 Initial inventory.

A_0 Replenishment cost of inventory/order (\$/order).

s Shortage cost/item (\$/item unit short).

p Purchase cost per item (\$/item).

h_c Holding cost/item/unit time excluding interest charges (\$/unit/unit time).

I_r Interest charges/year, $I_r \geq I_e$ (\$/year).

I_e Interest earned/year (\$/year).

$Z_1(t_1, T)$ Average total cost function for Case 1.

$Z_2(t_1, T)$ Average total cost function for Case 2.

t_1^* Optimal shortages point of time.

T^* Optimal length of the cycle.

$I_0^*(t_1, T)$ Optimal order quantity.

$Z_1^*(t_1, T)$ Optimal average total cost for Case 1.

$Z_2^*(t_1, T)$ Optimal average total cost for Case 2.

2.2 Assumptions

1 The inventory model deals with a single type of item over a prescribed period of time. There is no scope of repairing or replacement of the deteriorated items during such period.

2 The demand is deterministic, continuous function of time and quadratic in nature.

3 Items follow a variable deterioration rate.

4 The ordering cost is directly proportional to the number of orders placed.

5 The holding cost is directly proportional to size of the inventory as well as time.

6 Shortages in the system are allowed to occur which are completely lost or backlogged.

7 Time horizon of the model is infinite.

8 Grace period is given by the supplier and that facilitates the purchase of items without immediate payment. At the end of the grace period, the account is settled and the customer starts paying for the interest charges on the items in stock.

9 No interest is to be charged after commencement of shortages.

10 No interest is to be earned after permissible delay periods.

11 Replenishment rate is instantaneous on ordering, i.e., the delivery lead time is negligible.

3 The mathematical modelling and the method of solving

In this section, the mathematical formulation of an EOQ inventory model with time-dependent demand, time-proportional deterioration and shortages under delay in payment conditions is developed and its solution procedure is given.

3.1 The mathematical modelling

At the beginning of the ordering cycle, the inventory level $I(t)$ is of I_0 units of items when time $t = 0$. During the period $[0,T]$, the inventory depletes mainly to meet the time-proportional deterioration and time-dependent quadratic demand only. Thus, the rate of change of the inventory level at any time t is governed by the differential equation as follows:

$$\frac{dI(t)}{dt} + \theta(t)I(t) + D(t) = 0, \qquad 0 \le t \le T \tag{2}$$

with the deterioration rate $\theta(t) = \theta t,\ 0 < \theta << 1$, demand rate $D(t) = a + bt + ct^2,\ \ a > 0,$ $b \neq 0$ & $c \neq 0$ and initial conditions $I(t_1) = 0$ and $I(0) = I_0$.

The inventory level decreases due to deterioration rate and demand rate only and become zero at t_1. Shortages occur over the period $[t_1, T]$ and are completely backlogged. The delay period μ is settled by the supplier or whole-seller to the retailer or customer.

The model is derived with two main cases depending on the values of μ, t_1 and T.

The equation (3) is the ordinary linear differential equation of first order and first degree in $I(t)$. Its integrating factor (IF) is

$$IF = e^{\int \theta t dt} = e^{\frac{\theta t^2}{2}} \tag{3}$$

Thus, the solution of equation (4) is obtained by

$$I(t).e^{\frac{\theta t^2}{2}} = \int \left[a + bt + ct^2 \right].e^{\frac{\theta t^2}{2}} dt + k \tag{4}$$

where k is a constant of integration.

Using the conditions $I(t_1) = 0$ and $0 < \theta << 1$ and neglecting the terms containing the higher power of (θ), the equation (5) becomes

$$I(t) = \begin{bmatrix} at_1 + \frac{bt_1^2}{2} + \frac{ct_1^3}{3} + \frac{\theta}{2}\left(\frac{at_1^3}{3} + \frac{bt_1^4}{4} + \frac{ct_1^5}{5} \right) \\ -at - \frac{bt^2}{2} - \frac{ct^3}{3} - \frac{\theta}{2}\left(\frac{at^3}{3} + \frac{bt^4}{4} + \frac{ct^5}{5} \right) \end{bmatrix} e^{-\frac{\theta t^2}{2}}, \quad 0 \le t \le T \tag{5}$$

Further, equation (6) with initial condition $I(0) = I_0$ results

$$I(0) = I_0 = at_1 + \frac{bt_1^2}{2} + \frac{ct_1^3}{3} + \frac{\theta}{2}\left(\frac{at_1^3}{3} + \frac{bt_1^4}{4} + \frac{ct_1^5}{5} \right) \tag{6}$$

The average total cost function in the interval $[0, T]$ consists of the following elements:

1) The ordering cost (*OC*):

$$OC = A_0 \tag{7}$$

2) The holding cost (HC):

The holding cost over $[0, t_1]$ with the condition $0 < \theta << 1$ is given by

$$\begin{aligned} HC = h_c \int_0^{t_1} I(t)\,dt = h_c \int_0^{t_1} & \Bigg[\Bigg[at_1 + \frac{bt_1^2}{2} + \frac{ct_1^3}{3} + \frac{\theta}{2}\left(\frac{at_1^3}{3} + \frac{bt_1^4}{4} + \frac{ct_1^5}{5} \right) \\ & - at - \frac{bt^2}{2} - \frac{ct^3}{3} - \frac{\theta}{2}\left(\frac{at^3}{3} + \frac{bt^4}{4} + \frac{ct^5}{5} \right) \Bigg] e^{-\frac{\theta t^2}{2}} \Bigg] dt \\ & = h_c t_1^2 \left[\frac{a}{2} + \frac{bt_1}{3} + \frac{ct_1^2}{4} + \frac{\theta}{6}\left(\frac{at_1^2}{2} + \frac{2bt_1^3}{5} + \frac{ct_1^4}{3} \right) \right] \end{aligned} \tag{8}$$

3) The shortage cost (SC):

The shortage cost over $[t_1, T]$ with the condition $0 < \theta << 1$ is given by

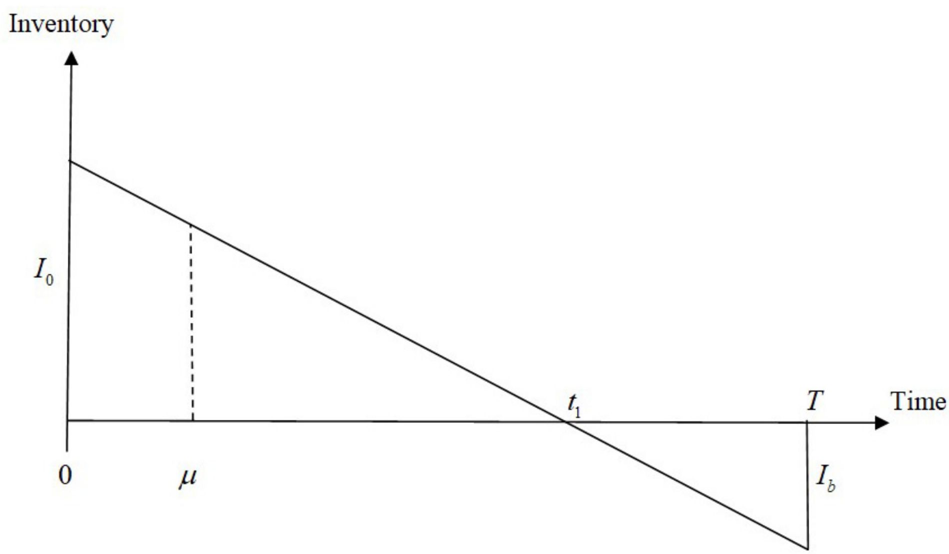
$$\begin{aligned} SC &= -s \int_{t_1}^{T} I(t)\,dt \\ &= -s \begin{bmatrix} \left(at_1 + \frac{bt_1^2}{2} + \frac{ct_1^3}{3} \right)(t - t_1) - \frac{a}{2}\left(T^2 - t_1^2\right) - \frac{b}{6}\left(T^3 - t_1^3\right) - \frac{c}{12}\left(T^4 - t_1^4\right) \\ + \frac{\theta}{2} \begin{bmatrix} \left(\frac{at_1^3}{3} + \frac{bt_1^4}{4} + \frac{ct_1^5}{5} \right)(T - t_1) - \left(at_1 + \frac{bt_1^2}{2} + \frac{ct_1^3}{3} \right)\left(\frac{T^3 - t_1^3}{3} \right) \\ + \frac{a}{6}\left(T^4 - t_1^4\right) + \frac{b}{20}\left(T^5 - t_1^5\right) + \frac{c}{45}\left(T^6 - t_1^6\right) \end{bmatrix} \end{bmatrix} \end{aligned} \tag{9}$$

Based on the grace period μ for settling the account, there are two main possibilities for the computation of interest chargeable and interest earned for the inventory system which are discussed below:

Case 1 $\mu \le t_1 < T$: (grace period is less than the shortages period of the system)

In Figure 1, as the length of the period with positive stock is larger than the grace period, the buyer can use the sales revenue to earn interest at an annual rate I_e in $\left[0, t_1\right]$.

Figure 1 $\mu \le t_1 < T$



Since the time period is larger than the grace period, the buyer can earn the interest in the period [0, T].

Such a case, the earned interest and the interest chargeable are calculated as follows:

The interest earned $\left(IE_1\right)$:

The interest earned over $\left[0, t_1\right]$ is given by

$$\begin{aligned} IE_1 &= pI_e \int_0^{\mu} tD(t)\,dt \\ &= pI_e \left(\frac{a\mu^2}{2} + \frac{b\mu^3}{3} + \frac{c\mu^4}{4} \right) \end{aligned} \tag{10}$$

Moreover, beyond the grace period, the unsold stock is supposed to be financed with an annual rate I_r.

The interest chargeable $\left(IP_1\right)$:

The interest chargeable over $\left[0, t_1\right]$ with condition $0 < \theta << 1$ is given by

$$IP_1 = pI_r\int_{\mu}^{t_1} I(t)dt$$

$$= pI_r\left[\begin{array}{l}\left(at_1+\frac{bt_1^2}{2}+\frac{ct_1^3}{3}\right)(t_1-\mu)-\frac{a}{2}\left(t_1^2-\mu^2\right)-\frac{b}{6}\left(t_1^3-\mu^3\right)-\frac{c}{12}\left(t_1^4-\mu^4\right)\\ +\frac{\theta}{2}\left[\begin{array}{l}\left(\frac{at_1^3}{3}+\frac{bt_1^4}{4}+\frac{ct_1^5}{5}\right)(t_1-\mu)-\left(at_1+\frac{bt_1^2}{2}+\frac{ct_1^3}{3}\right)\left(\frac{t_1^3-\mu^3}{3}\right)\\ +\frac{a}{6}\left(t_1^4-\mu^4\right)+\frac{b}{20}\left(t_1^5-\mu^5\right)+\frac{c}{45}\left(t_1^6-\mu^6\right)\end{array}\right]\end{array}\right] \tag{11}$$

The assessment of the average total cost function per the cycle time for Case 1 comprises the sum of

1 1 the ordering cost
2 the system holding cost (excluding interest charges)
3 the deterioration cost of the items
4 interest charges for the unsold items after grace period minus
5 interest earns from the sales revenue during the grace period.

Thus, using equations (7) to (11), the respective average total cost per unit time $Z_1(t_1,T)$ is given by

$$\begin{aligned}Z_1(t_1,T) &= \frac{1}{T}\left[A_0+HC+SC+IP_1-IE_1\right]\\ &= \frac{A_0}{T}+\frac{h_c t_1^2}{T}\left[\frac{a}{2}+\frac{bt_1}{3}+\frac{ct_1^2}{4}+\frac{\theta}{6}\left(\frac{at_1^2}{2}+\frac{2bt_1^3}{5}+\frac{ct_1^4}{3}\right)\right]\\ &\quad -\frac{s}{T}\left[\begin{array}{l}\left(at_1+\frac{bt_1^2}{2}+\frac{ct_1^3}{3}\right)(t-t_1)-\frac{a}{2}\left(T^2-t_1^2\right)-\frac{b}{6}\left(T^3-t_1^3\right)-\frac{c}{12}\left(T^4-t_1^4\right)\\ +\frac{\theta}{2}\left[\begin{array}{l}\left(\frac{at_1^3}{3}+\frac{bt_1^4}{4}+\frac{ct_1^5}{5}\right)(T-t_1)-\left(at_1+\frac{bt_1^2}{2}+\frac{ct_1^3}{3}\right)\left(\frac{T^3-t_1^3}{3}\right)\\ +\frac{a}{6}\left(T^4-t_1^4\right)+\frac{b}{20}\left(T^5-t_1^5\right)+\frac{c}{45}\left(T^6-t_1^6\right)\end{array}\right]\end{array}\right]\\ &\quad +\frac{pI_r}{T}\left[\begin{array}{l}\left(at_1+\frac{bt_1^2}{2}+\frac{ct_1^3}{3}\right)(t_1-\mu)-\frac{a}{2}\left(t_1^2-\mu^2\right)-\frac{b}{6}\left(t_1^3-\mu^3\right)-\frac{c}{12}\left(t_1^4-\mu^4\right)\\ +\frac{\theta}{2}\left[\begin{array}{l}\left(\frac{at_1^3}{3}+\frac{bt_1^4}{4}+\frac{ct_1^5}{5}\right)(t_1-\mu)-\left(at_1+\frac{bt_1^2}{2}+\frac{ct_1^3}{3}\right)\left(\frac{t_1^3-\mu^3}{3}\right)\\ +\frac{a}{6}\left(t_1^4-\mu^4\right)+\frac{b}{20}\left(t_1^5-\mu^5\right)+\frac{c}{45}\left(t_1^6-\mu^6\right)\end{array}\right]\end{array}\right]\\ &\quad -\frac{pI_e}{T}\left(\frac{a\mu^2}{2}+\frac{b\mu^3}{3}+\frac{c\mu^4}{4}\right)\end{aligned} \tag{12}$$

The main objective of the proposed model is to determine the minimum average cost per unit time satisfying the necessary and sufficient conditions.

As total average cost $Z_1(t_1, T)$ is a function of two decision variables t_1 and T, the necessary conditions for the minimisation of the cost function $Z_1(t_1, T)$ are

$$\frac{\partial Z_1(t_1,T)}{\partial t_1} = \frac{a + bt_1 + ct_1^2}{T}\left[\begin{array}{l} h_c t_1\left(1+\frac{\theta}{3}t_1^2\right) - s\left[T - t_1 + \frac{\theta}{6}\left(3t_1^2 T - 2t_1^3 - T^3\right)\right] \\ +pI_r\left[t_1 - \mu + \frac{\theta}{6}\left(2t_1^3 - 3\mu t_1^2 + \mu^2\right)\right] \end{array}\right] = 0 \tag{13}$$

and

$$\frac{\partial Z_1(t_1,T)}{\partial T} = \frac{s}{T}\left[\begin{array}{l} aT + \frac{bT^2}{2} + \frac{cT^3}{3} - at_1 - \frac{bt_1^2}{2} - \frac{ct_1^3}{3} \\ -\frac{\theta}{2}\left[\frac{at_1^3}{3} + \frac{bt_1^4}{4} + \frac{ct_1^5}{5} - T^2\left(at_1 + \frac{bt_1^2}{2} + \frac{ct_1^3}{3}\right) + \frac{2aT^3}{3} + \frac{bT^4}{4} + \frac{2cT^5}{15}\right] \end{array}\right] \tag{14}$$
$$-\frac{Z_1(t_1,T)}{T^2} = 0$$

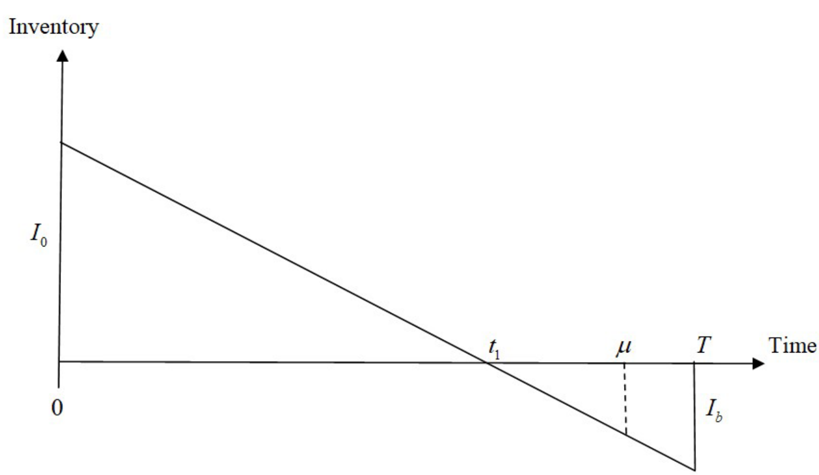
The corresponding sufficient conditions are $\frac{\partial^2 Z_1(t_1,T)}{\partial t_1^2} > 0$, $\frac{\partial^2 Z_1(t_1,T)}{\partial T^2} > 0$ and $\frac{\partial^2 Z_1(t_1,T)}{\partial t_1^2} . \frac{\partial^2 Z_1(t_1,T)}{\partial T^2} - \left(\frac{\partial^2 Z_1(t_1,T)}{\partial t_1 \partial T}\right)^2 > 0$.

(The sufficient conditions are given in Appendix A).

Case 2 $t_1 < \mu < T$: (shortages period is less than the grace period of the system)

In Figure 2, as the length of the replenishment cycle with positive stock is larger than the grace period, the buyer can use the sales revenue to earn interest at an annual rate I_e in $[0, t_1]$.

Figure 2 $t_1 < \mu < T$



Since the length of the replenishment cycle is larger than the grace period, the buyer can earn the interest in the period [0, T].

Such a case, the earned interest is given by

The interest earned $\left(IE_2\right)$:

The interest earned over $\left[0, t_1\right]$ is given by

$$\begin{aligned} IE_2 &= pI_e \int_0^{\mu} tD(t)\,dt \\ &= pI_e \left(\frac{a\mu^2}{2} + \frac{b\mu^3}{3} + \frac{c\mu^4}{4} \right) \end{aligned} \tag{15}$$

Similarly, beyond the grace period, the unsold stock is supposed to be financed with an annual rate I_r.

The interest chargeable $\left(IP_2\right)$:

In this case, the customer of the system pays no interest.

The interest chargeable is given by

$$IP_2 = 0 \tag{16}$$

The assessment of the average total cost per the cycle time for Case 2 comprises the sum of

1 1 the ordering cost
2 the system holding cost (excluding interest charges)
3 the deterioration cost of the items
4 interest charges for the unsold items after grace period minus
5 interest earns from the sales revenue during the grace period.

Thus, using equations (7) to (9) and (15) to (16), the respective average total cost per unit time $Z_2\left(t_1, T\right)$ is given by

$$\begin{aligned} Z_2\left(t_1, T\right) &= \frac{1}{T}\left[A_0 + HC + SC + IP_2 - IE_2\right] \\ &= \frac{A_0}{T} + \frac{h_c t_1^2}{T}\left[\frac{a}{2} + \frac{bt_1}{3} + \frac{ct_1^2}{4} + \frac{\theta}{6}\left(\frac{at_1^2}{2} + \frac{2bt_1^3}{5} + \frac{ct_1^4}{3}\right)\right] \\ &\quad - \frac{s}{T}\left[\begin{array}{l} \left(at_1 + \frac{bt_1^2}{2} + \frac{ct_1^3}{3}\right)\left(t - t_1\right) - \frac{a}{2}\left(T^2 - t_1^2\right) - \frac{b}{6}\left(T^3 - t_1^3\right) - \frac{c}{12}\left(T^4 - t_1^4\right) \\ + \frac{\theta}{2}\left[\begin{array}{l} \left(\frac{at_1^3}{3} + \frac{bt_1^4}{4} + \frac{ct_1^5}{5}\right)\left(T - t_1\right) - \left(at_1 + \frac{bt_1^2}{2} + \frac{ct_1^3}{3}\right)\left(\frac{T^3 - t_1^3}{3}\right) \\ + \frac{a}{6}\left(T^4 - t_1^4\right) + \frac{b}{20}\left(T^5 - t_1^5\right) + \frac{c}{45}\left(T^6 - t_1^6\right) \end{array}\right] \end{array}\right] \\ &\quad - \frac{pI_e}{T}\left[a\mu t_1 + \frac{t_1^2}{2}\left(b\mu - a\right) + \frac{t_1^3}{3}\left(c\mu - \frac{b}{2}\right) - \frac{ct_1^4}{12}\right] \end{aligned} \tag{17}$$

Since, the objective of the inventory system is to obtain the minimum total average cost per unit time, the necessary and sufficient conditions are needed to calculate.

As total average cost $Z_2(t_1,T)$ is a function of two decision variables t_1 and T, the necessary conditions for the minimisation of the cost function $Z_2(t_1,T)$ are

$$\frac{\partial Z_2(t_1,T)}{\partial t_1} = \frac{a+bt_1+ct_1^2}{T}\left[ht_1\left(1+\frac{\theta}{3}t_1^2\right)-s\left[T-t_1+\frac{\theta}{6}\left(3t_1^2T-2t_1^3-T^3\right)\right]\right] -\frac{pI_e}{T}\left[a\mu+t_1(b\mu-a)+t_1^2\left(c\mu-\frac{b}{2}\right)-\frac{ct_1^3}{3}\right]=0 \tag{18}$$

and

$$\frac{\partial Z_2(t_1,T)}{\partial T} = \frac{s}{T}\left[\begin{array}{l} aT+\dfrac{bT^2}{2}+\dfrac{cT^3}{3}-at_1-\dfrac{bt_1^2}{2}-\dfrac{ct_1^3}{3} \\ -\dfrac{\theta}{2}\left[\dfrac{at_1^3}{3}+\dfrac{bt_1^4}{4}+\dfrac{ct_1^5}{5}-T^2\left(at_1+\dfrac{bt_1^2}{2}+\dfrac{ct_1^3}{3}\right)+\dfrac{2aT^3}{3}+\dfrac{bT^4}{4}+\dfrac{2cT^5}{15}\right]\end{array}\right] -\frac{Z_2(t_1,T)}{T^2}=0 \tag{19}$$

The corresponding sufficient conditions are $\frac{\partial^2 Z_2(t_1,T)}{\partial t_1^2}>0$, $\frac{\partial^2 Z_2(t_1,T)}{\partial T^2}>0$ and $\frac{\partial^2 Z_2(t_1,T)}{\partial t_1^2}\cdot\frac{\partial^2 Z_2(t_1,T)}{\partial T^2}-\left(\frac{\partial^2 Z_2(t_1,T)}{\partial t_1\partial T}\right)^2>0$.

(The sufficient conditions are given in Appendix B)

3.2 *Solution procedure for the shortages time, cycle period, the total cost and EOQ: algorithm*

The following steps are required for the execution of numerical examples.

Step 1: Executes 1–3.

1. Determine t_1^* and T^*, the respective optimal values of t_1 and T, from the equations (13) to (14).
2. If $\mu < t_1^* < T^*$, then compute $\frac{\partial^2 Z_1(t_1,T)}{\partial t_1^2}$, $\frac{\partial^2 Z_1(t_1,T)}{\partial T^2}$ and $\frac{\partial^2 Z_1(t_1,T)}{\partial t_1^2}\cdot\frac{\partial^2 Z_1(t_1,T)}{\partial T^2}-\left(\frac{\partial^2 Z_1(t_1,T)}{\partial t_1\partial T}\right)^2$.
3. If $\frac{\partial^2 Z_1(t_1,T)}{\partial t_1^2}>0$, $\frac{\partial^2 Z_1(t_1,T)}{\partial T^2}>0$ and $\frac{\partial^2 Z_1(t_1,T)}{\partial t_1^2}\cdot\frac{\partial^2 Z_1(t_1,T)}{\partial T^2}-\left(\frac{\partial^2 Z_1(t_1,T)}{\partial t_1\partial T}\right)^2>0$, then evaluate $Z_1^*(t_1,T)$ from equation (12) by substituting the value of t_1^* and T^*.

Step 2: Executes 1–3.

1 Determine t_1^* and T^*, the respective optimal values of t_1 and T, from the equations (18) to (19).

2 If $t_1^* < \mu < T^*$, then compute $\frac{\partial^2 Z_2(t_1,T)}{\partial t_1^2}$, $\frac{\partial^2 Z_2(t_1,T)}{\partial T^2}$ and $\frac{\partial^2 Z_2(t_1,T)}{\partial t_1^2}\cdot\frac{\partial^2 Z_2(t_1,T)}{\partial T^2}$ $-\left(\frac{\partial^2 Z_2(t_1,T)}{\partial t_1 \partial T}\right)^2$.

3 If $\frac{\partial^2 Z_2(t_1,T)}{\partial t_1^2} > 0$, $\frac{\partial^2 Z_2(t_1,T)}{\partial T^2} > 0$ and $\frac{\partial^2 Z_2(t_1,T)}{\partial t_1^2}\cdot\frac{\partial^2 Z_2(t_1,T)}{\partial T^2} - \left(\frac{\partial^2 Z_2(t_1,T)}{\partial t_1 \partial T}\right)^2 > 0$, then evaluate $Z_2^*(t_1,T)$ from equation (17) by substituting the value of t_1^* and T^*.

Step 3: Compare $Z_1^*(t_1,T)$ and $Z_2^*(t_1,T)$ as well as find the optimal cost.

Step 4: Compute the respective optimal order quantity from equation (6).

Special cases:

1 By putting $\theta = 0$ in the deterioration rate $\theta(t) = \theta t$, the model reduces to that of Khanra et al. (2013).

2 By putting $t = 1$ in the deterioration rate $\theta(t) = \theta t$ and putting $s = 0$, the model also reduces to that of Khanra et al. (2011).

4 Numerical examples

In this section, a couple of numerical examples are presented to justify the formulated inventory model. The optimal value of shortages point, cycle time, total cost and order quantity is calculated for the given values of system parameters.

Example 1: *Let the parametric values of the system are* $a = 150$ *units/year,* $b = 15$ *units/year,* $c = 3$ *units/year,* $\theta = 0.2$, $K_0 = \$80$ */order,* $I_r = \$0.15$ */year,* $I_e = 0.13$ */year,* $\mu = 0.1$ *year,* $h_c = \$0.12$ */unit/year,* $s = \$0.35$ */year and* $p = \$10$ */unit.*

Solving equations (13) to (14), we have $t_1^* = 0.400312$ year and $T^* = 2.13219$ year provided $\frac{\partial^2 Z_1(t_1,T)}{\partial t_1^2} = 143.949 > 0$, $\frac{\partial^2 Z_1(t_1,T)}{\partial T^2} = 28.149 > 0$ and $\frac{\partial^2 Z_1(t_1,T)}{\partial t_1^2}.\frac{\partial^2 Z_1(t_1,T)}{\partial T^2}$ $-\left(\frac{\partial^2 Z_1(t_1,T)}{\partial T \partial t_1}\right)^2 = 3844.66 > 0$ and the corresponding minimum average cost is $Z_1^*(t_1,T) = 166.717$.

Similarly, solving equations (18) to (19), we have $t_1^* = 0.346895$ year and year provided $\frac{\partial^2 Z_2(t_1,T)}{\partial t_1^2} = 143.516 > 0$, $\frac{\partial^2 Z_2(t_1,T)}{\partial T^2} = 134.46 > 0$ and $\frac{\partial^2 Z_2(t_1,T)}{\partial t_1^2}.\frac{\partial^2 Z_2(t_1,T)}{\partial T^2} - \left(\frac{\partial^2 Z_2(t_1,T)}{\partial T \partial t_1}\right)^2 = 18384.1 > 0$ and the corresponding minimum average cost is $Z_2^*(t_1,T) = 81.7319$.

Hence, the minimum average cost in this case is $Z_2^*(t_1,T) = 81.7319$ where $t_1^* = 0.346895$ year and $T^* = 1.44655$ year. The economic order quantity is given $I_0^*(t_1,T) = 53.193$ units.

Example 2: *Let the parametric values of the system are in proper units:* $a = 150$ *units/year,* $b = 15$ *units/year,* $c = 3$ *units/year,* $\theta = 0.002$, $K_0 = \$80$ */order,* $I_r = \$0.15$ */year,* $I_e = 0.13$ */year,* $\mu = 0.8$ *year,* $h_c = \$0.12$ */unit/year,* $s = 0.35$ */year and* $p = \$10$ */unit.*

Solving equations (13) to (14), we have $t_1^* = 0.993137$ year and $T^* = 2.16382$ year provided $\frac{\partial^2 Z_1(t_1,T)}{\partial t_1^2} = 152.821 > 0$, $\frac{\partial^2 Z_1(t_1,T)}{\partial T^2} = 47.4867 > 0$ and $\frac{\partial^2 Z_1(t_1,T)}{\partial t_1^2}.\frac{\partial^2 Z_1(t_1,T)}{\partial T^2} - \left(\frac{\partial^2 Z_1(t_1,T)}{\partial T \partial t_1}\right)^2 = 6525.23 > 0$ and the corresponding minimum average cost is $Z_1^*(t_1,T) = 160.588$.

Similarly, solving equations (18) to (19), we have $t_1^* = 0.842874$ year and $T^* = 1.14422$ year provided $\frac{\partial^2 Z_2(t_1,T)}{\partial t_1^2} = 177.215 > 0$, $\frac{\partial^2 Z_2(t_1,T)}{\partial T^2} = 16.8177 > 0$ and $\frac{\partial^2 Z_2(t_1,T)}{\partial t_1^2}.\frac{\partial^2 Z_2(t_1,T)}{\partial T^2} - \left(\frac{\partial^2 Z_2(t_1,T)}{\partial T \partial t_1}\right)^2 = 443.0 > 0$ and the corresponding minimum average cost from equation (17) is $Z_2^*(t_1,T) = 20.2551$.

Hence, the minimum average cost in this case is $Z_2^*(t_1,T) = 20.2551$ where $t_1^* = 0.842874$ year and $T^* = 1.14422$ year. The economic order quantity is given $I_0^*(t_1,T) = 132.39$ units.

5 Sensitivity analysis

The sensitivity analysis of the present parameters $a = 150$, $b = 15$, $c = 3$, $\theta = 0.002$, $K_0 = 80$, $I_r = 0.15$, $I_e = 0.13$, $\mu = 0.8$, $h_c = 0.12$, $s = 0.35$ and $p = 10$ of the proposed model is executed by the definite value of one parameter at a time and setting other parameters unchanged. The following key points are noted from Example 1.

1 *From Case 1 and Case 2*: both the shortages period and replenishment period decrease while the optimal cost increases with the ascending values of the parameter a. Here, the shortages period, replenishment period and optimal cost are all moderately sensitive to the parameter a.

2 *From Case 1*: all of the shortage period, replenishment period decrease and the optimal cost decrease with the ascending values of the parameters b, c, I_e & h_c. From Case 2: both the shortage period and replenishment period decrease and the optimal cost increases with the ascending values of the parameter b, c, I_e & h_c. Here, the shortage period, replenishment period and optimal cost are all highly sensitive to the parameters b, c, I_e & h_c.

3 *From Case 1*: the shortage period increases for certain period and then decreases and both the replenishment period and the optimal cost increase with the ascending values of the parameter θ. From Case 2: both the shortages period and optimal cost decrease and the replenishment period increases with the ascending values of the parameter θ. Here, the shortage period, replenishment period and optimal cost are all highly sensitive to the parameter θ.

4 *From Case 1 and Case 2*: all of the shortage period, replenishment period and the optimal cost increase with the ascending values of the parameter K_0 & μ. Here, the shortage period, replenishment period and optimal cost are all moderately sensitive to both the parameter K_0 & μ.

5 *From Case 1*: the shortage period decreases and replenishment period and the optimal cost increase with the ascending values of the parameter I_r. From Case 2: all of the shortage period, replenishment period and the optimal cost remain constant with the ascending values of the parameter I_r. Here, the shortage period, replenishment period and optimal cost are all moderately sensitive to Case 1 and insensitive to Case 2 for the parameter I_r.

6 *From Case 1and Case 2*: both the shortages period and optimal cost increase and the replenishment cost decreases with the ascending values of the parameter s. Here, the shortages period, replenishment period and optimal cost are least sensitive to the parameter s.

7 *From Case 1*: the shortage period decreases and both the replenishment period and the optimal cost increase with the ascending values of the parameter p. From Case 2: both the shortage period and replenishment period decrease and the optimal cost increases with the ascending values of the parameter p. Here, the shortage period, replenishment period and optimal cost are all moderately sensitive to both the parameter p.

From Table 1, it is obvious that the optimal cost is obtained from Case 2 and total average cost diminishes rapidly with the ascending values of parameter μ. This justifies the real life of the market situations.

Table 1 Sensitivity analysis for the shortage period, replenishment time and optimal cost

Parameter	*Change in parameter*	*Case I*			*Case II*			*Optimal cost*	*% Chan-ge in optim-al cost*
		t_1	T	$Z_1(t_1, T)$	t_1	T	$Z_2(t_1, T)$		
a	75	0.414360	2.28076	107.164	0.396931	1.72614	63.6493	63.6493	–22.12
	135	0.402653	2.15577	155.195	0.354216	1.48779	78.5097	78.5097	–3.94
	165	0.398161	2.11089	178.099	0.340382	1.40989	84.5097	84.8069	+3.76
	225	0.391097	2.04310	222.550	0.319983	1.29546	95.9392	95.9392	+17.38
b	7.5	0.410002	2.23271	168.017	0.348087	1.46707	80.6120	80.6120	–1.37
	13.5	0.402079	2.14995	166.912	0.347119	1.45051	81.5123	81.5123	–0.27
	16.5	0.398615	2.11536	166.548	0.346676	1.44266	81.9494	81.9494	+0.27
	22.5	0.392413	2.05548	166.072	0.345860	1.42772	82.2157	82.2157	+0.59
c	1.5	0.403576	2.16518	167.593	0.347435	1.45103	81.5334	81.5334	–0.24
	2.7	0.400938	2.13845	166.881	0.347001	1.44744	81.6924	81.6924	–0.05
	3.3	0.399699	2.12609	166.559	0.346788	1.44567	81.7712	81.7712	+0.05
	4.5	0.397359	2.10302	165.968	0.346369	1.44218	81.9272	81.9272	+0.24
θ	0.10	0.3983520.3995030.4016920.374305	1.92667	158.652	0.350506	1.41780	82.9719	82.9719	+1.52
	0.18		2.07491	164.595	0.347632	1.44031	81.9940	81.9940	+0.32
	0.22		2.20926	169.464	0.346148	1.45309	81.4615	81.4615	–0.33
	0.30		2.43051	171.683	0.343073	1.48278	80.2831	80.2831	–1.77
K_0	40	0.373246	1.88385	135.996	0.301182	1.18065	54.6662	54.6662	–33.11
	72	0.395587	2.08581	161.056	0.339392	1.40154	76.8607	76.8607	–5.95
	88	0.404784	2.17760	172.195	0.353857	1.48890	86.4092	86.4092	+5.72
	120	0.420688	2.35450	192.611	0.377673	1.63861	103.585	103.585	+26.73
I_r	0.075	0.550751	1.91427	127.746	0.346895	1.44655	81.7319	81.7319	0
	0.135	0.420633	2.08858	159.749	0.346895	1.44655	81.7319	81.7319	0
	0.165	0.382820	2.17644	173.466	0.346895	1.44655	81.7319	81.7319	0
	0.225	0.356785	3.24368	232.782	0.346895	1.44655	81.7319	81.7319	0

Table 1 Sensitivity analysis for the shortage period, replenishment time and optimal cost (continued)

Parameter	*Change in parameter*	*Case I*			*Case II*			*Optimal cost*	*% Chan-ge in optim-al cost*
		t_1	T	$Z_1(t_1, T)$	t_1	T	$Z_2(t_1, T)$		
I_e	0.065	0.400593	2.13500	167.058	0.508353	1.50088	77.7605	77.7605	–4.85
	0.117	0.400368	2.13276	166.786	0.368061	1.45357	81.2301	81.2301	–0.61
	0.143	0.400256	2.13163	166.649	0.328831	1.44055	82.1515	82.1515	+0.51
	0.195	0.400030	2.12938	166.376	0.277032	1.42307	83.2869	83.2869	+1.90
μ	0.05		2.04181	158.201	0.308480	1.44530	84.0013	84.0013	+2.78
	0.090.11	0.3524150.3907530.409862	2.11393	165.003	0.339223	1.44645	82.2044	82.2044	+0.58
	0.15	0.448005	2.15057	168.439	0.354560	1.44658	81.2499	81.2499	–0.59
			2.22564	175.429	0.385158	1.44592	79.2278	79.2278	–3.06
h_c	0.060	0.416671	2.16726	170.116	0.360290	1.44982	81.2753	81.2753	–0.55
	0.108	0.403480	2.13893	167.371	0.349490	1.44718	81.6433	81.6433	–0.10
	0.132	0.397194	2.12559	166.077	0.344339	1.44593	81.8192	81.8192	+0.10+0.51
	0.180	0.385190	2.10042	163.631	0.334497	1.44356	82.1555	82.1555	
s	0.175	0.281074	2.94240	131.088	0.258051	1.75326	63.2287	63.2287	–22.63
	0.315	0.382368	2.22462	161.538	0.331568	1.48613	78.7476	78.7476	–3.65
	0.385	0.417205	2.05726	171.488	0.361304	1.41263	84.4765	84.4765	+3.35+14.57
	0.525	0.475646	1.85094	186.910	0.411515	1.31371	93.6460	93.6460	
p	5	0.551277	1.91701	128.075	0.508353	1.50088	77.7605	77.7605	–4.85
	9	0.420696	2.08913	159.817	0.368061	1.45357	81.2301	81.2301	–0.61
	11	0.382768	2.17586	173.397	0.328831	1.44055	82.1515	82.1515	+0.51+1.90
	15	0.356777	3.24528	232.672	0.277032	1.42307	83.2869	83.2869	

Table 2 shows the variation of the optimum values of the parameter μ. From a careful observation of the results from Table 2, it is found that if μ is increased from 0.50 to 0.75, the optimum shortage cost, optimum cycle times and the optimal costs are obtained in Case 2. If μ is increased from 1.00 to 4.03, the optimum shortage cost, optimum cycle times and the optimal costs are obtained in Case 1. If the parameter μ is increased from 4.05 to more, the infeasible solutions are obtained. Hence, the variation of the parameter μ is important and is effective in real-life situations.

Table 2 The variation of μ on optimal policies

Changed value of the parameter μ	*Case I*			*Case II*			*Optimal cost*
	t_1	T	$Z_1(t_1,T)$	t_1	T	$Z_2(t_1,T)$	
0.50	0.764667	2.95124	245.443	0.645763	1.37780	54.574	54.5740
0.75	0.986887	3.02383	273.526	0.81186	1.21121	27.2726	27.2726
1.00	0.941121	1.03275	5.50481	1.00583	1.05559	…	5.50481
1.50	1.51273	2.05406	65.2593	2.09180	2.45401	…	65.2593
2.50	2.63637	3.38803	152.417	2.62446	5.24125	…	152.417
4.00	4.57958	4.60352	10.7648	4.16054	6.36857	…	10.7648
4.01	4.59324	4.60969	7.43832	4.17064	6.37638	…	7.43832
4.02	4.60691	4.61584	4.06570	4.18073	6.38419	…	4.06570
4.03	4.62058	4.62195	0.62776	4.19082	6.3920	…	0.62776
4.04	4.63426	4.62802	…	4.20090	6.39981	…	…
4.05	4.64793	4.63407	…	4.21098	6.40761	…	…
5.00	5.93393	5.02656	…	5.13401	7.14207	…	…

Note: Here '…' indicates no feasible solution.

6 Conclusions and research directions

The objective of the present model is to develop an EOQ inventory model for a deteriorating item varying with time-proportional deterioration, shortages, time-dependent demand rate under permissible delay in payments. The demand pattern considered here is time-dependent and a quadratic function of time. While dealing with time-varying demand patterns, the researchers usually take the linear demand rate which is of the form $D(t) = a + bt, a > 0 \,\&\, b \neq 0$. It implies either steady increase or decrease in the demand rate, which is rarely seen to occur for any item. On the other hand, an exponential demand rate which is of the form $D(t) = ae^{bt}, a > 0 \,\&\, b \neq 0$ implying either exponentially increasing or exponentially decreasing in the demand rate. As it is very high, it is also rarely applicable in the real market. A better alternative would be to consider either accelerated rise or accelerated fall in the demand rate. Accelerated growth in the demand rate is found to occur in case of the state of the art of aircrafts, machines and their spare parts, computers and mobiles. Similarly, the accelerated decline in the demand rate can be found in the case of obsolete aircrafts, machines and their spare parts and computers. The deterioration is time-proportional, i.e., deterioration is a function of time. The general form of the demand rate considered here is $D(t) = a + bt + ct^2$ where $a > 0$, $b \neq 0$ and $c \neq 0$. Here, $c = 0$ and $b = c = 0$ result the time-dependent linear demand pattern and constant demand, respectively. The time-varying deterioration of the inventory system is of the form $\theta(t) = \theta t$ where $0 < \theta << 1$. If $t = 1$, then the time-varying deterioration rate results $\theta(t) = \theta$, which is a constant deterioration rate. The shortages as well as completely backlogging in this model are permitted to occur. To be more precise, the proposed model has been studied for the computation of the shortages time, cycle period, total cost and optimal order quantity under the factors of time-dependent quadratic demand, shortages, time-proportional deterioration and permissible delay in payment. The supplier in the inventory system provides a grace period for encouraging the customers to purchase more quantities and earn interest. Two main cases, namely, Case 1 Grace period is less than the shortages period of the system, $(\mu < t_1 < T)$. Case 2 shortages period is less than the grace period of the system, $(t_1 < \mu < T)$ have been taken into account for the development of the formulation of the proposed model. Finally, models are presented with the assistance of a couple of numerical examples and the sensitivity analysis of all key parameters is discussed. It is found that the model is more sensitive at μ and I_e. The assumption of demand patterns like constant and time-dependent linear are not highly suitable for the items like foodstuffs, food grains, meat, milk, fish, vegetables, fruits, paddy, etc. and newly launched items like computers, mobiles, laptops and automobiles, etc., fashion apparels, cosmetics, etc. in which the age of these items has a negative effect on demand due to loss of quality as well as the quantity of such items. The present model is also valid for the case of newly launched fashionable items like fashion apparels, cosmetic, etc. and electronics items like mobile sim cards, android mobile phones, Laptops, super computers, automobiles and their spare parts. It is also very much useful to efficient designing and controlling the inventory system in various competitive markets like, pharmaceutical industries, food processing industries and fruits and vegetable markets, etc. It is also seen the time-proportional deterioration is suitable for the items whose deterioration rate is directly proportional to the time period of storing the items.

The proposed model which is designed can be extended in several ways such as time varying deterioration rates, two-parameter and three-parameter Weibull distribution deterioration rates, Gamma distribution deterioration rate and time-dependent holding costs. In addition, the demand patterns can also be extended as generalised function of time like cubic demand pattern, time-varying seasonal demand, stock and price-dependent demand and more stochastic demand patterns. In addition, the model can also be generalised by incorporating some more realistic features, such as quantity and cash discount, allowing shortages, inflation rates, etc. Finally, it would be of interest to add warehouse space required.

Acknowledgement

The authors are grateful to the editor-in-chief and the two referees for their constructive comments and suggestions to enhance the presentational quality of this paper.

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Appendix A

$$\frac{\partial^2 Z_1(t_1,T)}{\partial t_1^2} = \frac{b+2ct_1}{T}\left[\begin{array}{l} h\left(t_1+\dfrac{\theta t_1^3}{3}\right)-s\left[T-t_1+\dfrac{\theta}{6}\left(3t_1^2T-2t_1^3-T^3\right)\right] \\ +pI_r\left[t_1-\mu+\dfrac{\theta}{6}\left(2t_1^3-3\mu t_1^2+\mu^2\right)\right] \end{array}\right]$$

$$+\frac{a+bt_1+ct_1^2}{T}\left[\begin{array}{l} h\left(1+\theta t_1^2\right)-s\left[\theta t_1\left(T-t_1\right)-1\right] \\ +pI_r\left[1+\theta t_1\left(t_1-\mu\right)\right] \end{array}\right],$$

$$\frac{\partial^2 Z_1(t_1,T)}{\partial T^2} = \frac{1}{T^2}\left[\begin{array}{l} s\left[\begin{array}{l} 2aT+\dfrac{3bT^2}{2}+\dfrac{4cT^3}{3}-at_1-\dfrac{bt_1^2}{2}-\dfrac{ct_1^3}{3} \\ -\dfrac{\theta}{2}\left[\dfrac{at_1^3}{2}+\dfrac{bt_1^4}{4}+\dfrac{ct_1^5}{5}-3T^2\left(at_1+\dfrac{bt_1^2}{2}+\dfrac{ct_1^3}{3}\right)+\dfrac{8aT^3}{3}+\dfrac{5bT^4}{4}+\dfrac{4cT^5}{5}\right] \end{array}\right] \\ -\dfrac{\partial Z_1(t_1,T)}{\partial T} \end{array}\right]$$

$$-\frac{2}{T^3}\left[\begin{array}{l} s\left[\begin{array}{l} aT^2+\dfrac{bT^3}{2}+\dfrac{cT^4}{3}-T\left(at_1+\dfrac{bt_1^2}{2}+\dfrac{ct_1^3}{3}\right) \\ -\dfrac{\theta}{2}\left[T\left(\dfrac{at_1^3}{3}+\dfrac{bt_1^4}{4}+\dfrac{ct_1^5}{5}\right)-T^3\left(at_1+\dfrac{bt_1^2}{2}+\dfrac{ct_1^3}{3}\right)+\dfrac{2aT^4}{3}+\dfrac{bT^5}{4}+\dfrac{2cT^6}{5}\right] \end{array}\right] \\ -Z_1(t_1,T) \end{array}\right]$$

and

$$\frac{\partial^2 Z_1(t_1,T)}{\partial t_1\partial T} = \left(a+bt_1+ct_1^2\right)\left[\begin{array}{l} \dfrac{s}{T}\left[\dfrac{\theta}{2}\left(T^2-t_1^2\right)-1\right] \\ -\dfrac{1}{T^2}\left[\begin{array}{l} h\left(t_1+\dfrac{\theta t_1^3}{3}\right)-s\left[T-t_1+\dfrac{\theta}{6}\left(3t_1^2T-2t_1^3-T^3\right)\right] \\ +pI_r\left[t_1-\mu+\dfrac{\theta}{6}\left(2t_1^3-3\mu t_1^2+\mu^2\right)\right] \end{array}\right] \end{array}\right].$$

Appendix B

$$\frac{\partial^2 Z_2(t_1,T)}{\partial t_1^2}=\frac{1}{T}\begin{bmatrix}(b+2ct_1)\left[h_c\left(1+\theta t_1^2\right)-s\left[\theta t_1(T-t_1)-1\right]\right]\\ -pI_e\left[b\mu-a+2t_1\left(c\mu-\frac{b}{2}\right)-ct_1^2\right]\end{bmatrix},$$

$$\frac{\partial^2 Z_2(t_1,T)}{\partial T^2}=\frac{1}{T^2}\begin{bmatrix}s\begin{bmatrix}2aT+\frac{3bT^2}{2}+\frac{4cT^3}{3}-at_1-\frac{bt_1^2}{2}-\frac{ct_1^3}{3}\\ -\frac{\theta}{2}\left[\frac{at_1^3}{2}+\frac{bt_1^4}{4}+\frac{ct_1^5}{5}-3T^2\left(at_1+\frac{bt_1^2}{2}+\frac{ct_1^3}{3}\right)+\frac{8aT^3}{3}+\frac{5bT^4}{4}+\frac{4cT^5}{5}\right]\end{bmatrix}\\ -\frac{\partial Z_2(t_1,T)}{\partial T}\end{bmatrix}$$

$$-\frac{2}{T^3}\begin{bmatrix}s\begin{bmatrix}aT^2+\frac{bT^3}{2}+\frac{cT^4}{3}-T\left(at_1+\frac{bt_1^2}{2}+\frac{ct_1^3}{3}\right)\\ -\frac{\theta}{2}\left[T\left(\frac{at_1^3}{3}+\frac{bt_1^4}{4}+\frac{ct_1^5}{5}\right)-T^3\left(at_1+\frac{bt_1^2}{2}+\frac{ct_1^3}{3}\right)+\frac{2aT^4}{3}+\frac{bT^5}{4}+\frac{2cT^6}{5}\right]\end{bmatrix}\\ -Z_2(t_1,T)\end{bmatrix}$$

and

$$\frac{\partial^2 Z_2(t_1,T)}{\partial t_1\partial T}=\frac{s}{T}\left(a+bt_1+ct_1^2\right)\left[\frac{\theta}{2}\left(T^2-t_1^2\right)-1\right]$$

$$-\frac{1}{T^2}\begin{bmatrix}\left(a+bt_1+ct_1^2\right)\left[h\left(t_1+\frac{\theta t_1^3}{3}\right)-s\left[T-t_1+\frac{\theta}{6}\left(3t_1^2T-2t_1^3-T^3\right)\right]\right]\\ -pI_e\left[a\mu+t_1(b\mu-a)+t_1^2\left(c\mu-\frac{b}{2}\right)-\frac{ct_1^3}{3}\right]\end{bmatrix}.$$