H_∞ stabilisation of discrete time-delayed systems with anti-windup approach

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Abstract: This paper is devoted to analysing the stability of a discrete time-delayed system subjected to saturation. To ensure the asymptotic stability of the closed-loop system, an anti-windup compensator has been designed where the anti-windup gains are calculated via linear matrix inequality (LMI) technique. The accomplishment of the system has been investigated using H_∞ technique to tackle external interference. By employing Wirtinger inequality with reciprocal convex inequality, delay-dependent solution of the system ensures the asymptotic stability of the system. An optimisation methodology is given to maximise the basin of attraction. Numerical illustrations prove the efficacy of the proposed criterion.

Keywords: discrete-time system; time delay; actuator saturation; asymptotic stability; anti-windup; H_∞.

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1 Introduction

Saturation is a familiar phenomenon discussed in the real world. Most dynamical systems suffer from the problem of actuator saturation nonlinearities. Actuator degrades the performance of the system resulting in instability of the system. The popular techniques used to design controllers to mitigate the effect of saturation are:

1 scheduled controllers (Wang and Sun, 2022)

2 constrained model predictive control (Yu et al., 2021)

3 anti-windup compensators (Moreno-Valenzuela, 2022).

Anti-windup is a classical technique to deal with saturation. Scheduled controllers are used in aerospace industry, model predictive controllers in chemical industries while anti-windup compensators are employed in control systems with actuator saturation.

The basic concept of anti-windup strategy is to modify a pre-designed controller to weaken the impact of saturation on a system. This is done by introducing an additional feedback loop in the existing control system to minimise the issues originated by saturation (Ofodile et al., 2021). Primitive research on anti-windup design decreases the effect of saturation using direct way by minimising the difference between actuator output and input (Astrom and Rundqwist, 1989; Franklin et al., 2010). Thereafter, indirect methods were used to diminish saturation effect by improving closed-loop performance and stability of the system (Kothare et al., 1994; Kapoor et al., 1998; Kothare and Morari, 1999; Cao et al., 2002; Hu et al., 2002). Various techniques for synthesis of anti-windup compensators have been developed and are widely used in industries (Wen et al., 2022).

Time delayed systems have emerged as a topic of keen attention among the researchers (Sun and Mao, 2019; Wu et al., 2020; Chang et al., 2021; Venkatesh et al., 2021, 2022; Mao et al., 2022; Swarnakar, 2022; Agrawal et al., 2023) during past three decades. Delays are deliberately introduced in physical system model due to unmodelled inertia of system components, measurement, computational delays, transport and transmission lags. In realistic system design, time-delays should be taken into account in various applications such as neural network, long transmission lines in pneumatic systems, control in congestion analysis, echo cancellation, multipath propagation in mobile communication, thermal process, chemical process and many more (Chen and Wang, 2021; Yang et al., 2021, 2022).

The consequence of external interference is unavoidable in practical systems. Lot of literature is available that discussed the stability issues (Pal and Negi, 2018; Wang and Sun, 2022). To curtail the consequences of disturbance and perturbation, H_∞ technique emerges to be most appropriate strategy for discrete systems (Du et al., 2021; Yong et al., 2021). A state feedback H_∞ controller has been synthesised for uncertain discrete delayed system described by triple Lyapunov Krasovskii functional (LKF) with actuator saturation and external interference (Pal and Negi, 2018).

Many papers deal with development of appropriate Lyapunov functions to analyse the stability of discrete-time systems with delays (Gao and Chen, 2007; Zhang et al., 2008; Kwon et al., 2013; Xu et al., 2014; Feng et al., 2015; Nam et al., 2015). Several techniques are available for the estimation of forward difference namely, free weighting matrix (FWM)-based method, inequality based method and many more to tackle the sum terms. (FWM)-based method is computationally demanding due to the involvement of free weighting matrices. Inequality based methods such as the Jensen inequality (Gu, 2003), improved Jensen inequalities (Moon et al., 2001; Zhang et al., 2005; Park et al., 2011), reciprocally convex combination inequality, Wirtinger's inequality (Liu and Fridman, 2012) and Wirtinger-based integral inequality (Seuret and Gouaisbaut, 2013a, 2013b) have been reported. Criteria obtained by Wirtinger inequality method provide less conservative results than those developed by Jensen inequality method. However, all these methods convey conservatism to some extent.

Although an anti-windup compensator has been designed to tackle the saturation nonlinearty subjected to linear time varying delay systems by using Jensen's inequality (Negi et al., 2012) or Wirtinger inequality (Pal et al., 2020, Singh et al., 2021). To the best of authors' knowledge none of the work has been accomplished with the consideration of saturation nonlinearity, external disturbances by using the Wirtinger inequality to derive the less conservative results for discrete time delayed systems. To bridge this gap, the present work is motivated by aforesaid papers to take up the problem of a discretised delayed system with anti-windup compensator designed for this stability criterion using Wirtinger inequality and exponential decaying disturbance for stability analysis of the system. The novelty of the paper is that a H_∞ controller is designed for a discrete time-delayed system with saturating actuator, disturbance and anti-windup strategy using Wirtinger inequality to stabilise the system considered.

The contribution of the paper is as follows:

1 A novel output feedback H_∞ controller with anti-windup strategies is designed for discrete time-delayed system subjected to input saturation and disturbance

employing Wirtinger inequality and reciprocal convex inequality to reduce/diminish the influence of wind up phenomenon and disturbance.

2 Optimisation technique is used to calculate basin of attraction for different delay range.

3 Application of Wirtinger inequality and reciprocal convex approach to the system establishes less conservative results than the existing ones.

4 Numerical example is specified to prove advantage of the results obtained.

The paper is organised as follows. Section 2 introduces the system considered. By exploiting discrete time-delayed system in context to LMI based conditions with saturation and H_∞ controller, the global asymptotic stability is established in Section 3. In Section 4, numerical instances are specified to prove the advantage of the obtained results.

Notations: $\Re^{q\times n}$ is the set of $q \times n$ real matrices, $\Re^q$ denotes set of $q \times 1$ real matrices, $\boldsymbol{P} > \mathbf{0}$ ($\geq \mathbf{0}$) denotes that $\boldsymbol{P}$ is real symmetric and positive definite (positive semidefinite) matrix, $\mathbf{0}$ is a null matrix or null vector, $\boldsymbol{I}$ is an identity matrix with appropriate dimension, $\lambda_{\max}(\bar{\boldsymbol{A}})$ denotes maximum eigen value of any given matrix $\bar{\boldsymbol{A}}$, symbol '*' represents symmetric terms in symmetric matrix, $\|.\|$ represents norm of matrix or vector, $He(\boldsymbol{A}) = \boldsymbol{A} + \boldsymbol{A}^T$, $diag(A, B)$ denotes block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $\|\mathbf{w}\|_2 = \sqrt{\sum_{i=0}^{\infty} \|\mathbf{w}(i)\|^2}$ is l_2 norm of signal $\boldsymbol{w}(i) \in l_2\{(0, \infty)\}$, if $\|\boldsymbol{w}\|_2 < \infty$.

2 System description

Consider the following discrete-time system consisting of actuator saturation, time-varying delay, and disturbance:

$$\boldsymbol{x}(\vartheta+1) = \boldsymbol{A}\boldsymbol{x}(\vartheta) + \boldsymbol{A}_d\boldsymbol{x}\left(\vartheta - g(\vartheta)\right) + \boldsymbol{B}_w\boldsymbol{w}(\vartheta) + \boldsymbol{B}\left(\boldsymbol{u}(\vartheta)\right) \tag{1a}$$

$$\boldsymbol{y}(\vartheta) = \boldsymbol{C}\boldsymbol{x}(\vartheta) \tag{1b}$$

$$\boldsymbol{z}(\vartheta) = \boldsymbol{C}_z\boldsymbol{x}(\vartheta) + \boldsymbol{D}_z\boldsymbol{w}(\vartheta) \tag{1c}$$

$$\boldsymbol{x}(\vartheta) = \Upsilon(\vartheta),\ \vartheta = -g_u, -g_u + 1, \ldots, 0 \tag{1d}$$

where $\boldsymbol{x}(\vartheta) \in \Re^n$, $\boldsymbol{u}(\vartheta) \in \Re^m$, $\boldsymbol{y}(\vartheta) \in \Re^p$ and $\boldsymbol{z}(k) \in \Re^m$ are state, input, measured and controlled output vectors respectively. The external interference in system is denoted as $\boldsymbol{w}(\vartheta) \in \Re^q$. Matrices $\boldsymbol{A}$, $\boldsymbol{A}_d$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}_z$, $\boldsymbol{C}_z$ are constant matrices of appropriate dimensions, and $g(\vartheta)$ is the time-varying delay satisfying

$$g_u \leq g(\vartheta) \leq g_l \tag{2}$$

where g_u and g_l are constant non-negative integers representing the upper and lower bounds respectively.

For system (1), the dynamic output stabilising controller is given as

$$\boldsymbol{x}_c(\vartheta+1) = \boldsymbol{A}_c\boldsymbol{x}_c(\vartheta) + \boldsymbol{B}_c\boldsymbol{y}(\vartheta) \tag{3a}$$

$$\boldsymbol{v}_c(\vartheta) = \boldsymbol{C}_c\mathbf{x}_c(\vartheta) + \boldsymbol{D}_c\boldsymbol{C}\boldsymbol{x}(\vartheta) \tag{3b}$$

where $\boldsymbol{x}_c(\vartheta) \in \Re^{n_c}$ denotes controller state and $\boldsymbol{v}_c(\vartheta)$ is the controller output. $\boldsymbol{A}_c$, $\boldsymbol{B}_c$, $\boldsymbol{C}_c$, $\boldsymbol{D}_c$ are constant controller matrices.

The input $\boldsymbol{u}$ is subjected to the amplitude constraint defined as

$$-\boldsymbol{u}_{0(i)} \le \boldsymbol{u}_{(i)} \le \boldsymbol{u}_{0(i)} \tag{4}$$

where $\boldsymbol{u}_{0(i)} > 0$, $i = 1, \ldots, m$, represents the control amplitude bounds. Thus, the actual control signal given into the plant is

$$\boldsymbol{u}(\vartheta) = sat\left(\boldsymbol{v}_c(\vartheta)\right) = sat\left(\boldsymbol{C}_c\boldsymbol{x}_c(\vartheta) + \boldsymbol{D}_c\boldsymbol{C}\boldsymbol{x}(\vartheta)\right) \tag{5}$$

The saturation nonlinearities are given by

$$sat\left(\boldsymbol{v}_c(\vartheta)\right)_{(i)} = \begin{cases} -\boldsymbol{u}_{0(i)} & \text{if} \quad \boldsymbol{v}_{c(i)} < -\boldsymbol{u}_{0(i)} \\ \boldsymbol{v}_{c(i)} & \text{if} \quad -\boldsymbol{u}_{0(i)} \le \boldsymbol{v}_{c(i)} \le \boldsymbol{u}_{0(i)}, \; i = 1, \ldots, m, \\ \boldsymbol{u}_{0(i)} & \text{if} \quad \boldsymbol{v}_{c(i)} > \boldsymbol{u}_{0(i)} \end{cases} \tag{6}$$

Substituting (5) in (1), we obtain

$$\begin{aligned} \boldsymbol{x}(\vartheta+1) = \boldsymbol{A}\boldsymbol{x}(\vartheta) + \boldsymbol{A}_d\boldsymbol{x}\left(\vartheta - g(\vartheta)\right) + \boldsymbol{B}_w\boldsymbol{w}(\vartheta) + \boldsymbol{B}\left(\boldsymbol{C}_c\boldsymbol{x}_c(\vartheta) + \boldsymbol{D}_c\boldsymbol{C}\boldsymbol{x}(\vartheta)\right) \\ -\boldsymbol{B}\boldsymbol{\Psi}\left(\boldsymbol{C}_c\boldsymbol{x}_c(\vartheta) + \boldsymbol{D}_c\boldsymbol{C}\boldsymbol{x}(\vartheta)\right) \end{aligned} \tag{7}$$

where

$$\boldsymbol{\Psi}(v) = \boldsymbol{v} - sat(\boldsymbol{v}) \tag{8}$$

An anti-windup term given as $\boldsymbol{E}_c(sat(\boldsymbol{v}_c(\vartheta)) - \boldsymbol{v}_c(\vartheta))$ can be injected to controller as follows

$$\boldsymbol{x}_c(\vartheta+1) = \boldsymbol{A}_c\boldsymbol{x}_c(\vartheta) + \boldsymbol{B}_c\boldsymbol{C}\boldsymbol{x}(\vartheta) - \boldsymbol{E}_c\boldsymbol{\Psi}\left(\boldsymbol{C}_c\boldsymbol{x}_c(\vartheta) + \boldsymbol{D}_c\boldsymbol{C}\boldsymbol{x}(\vartheta)\right) \tag{9}$$

Now extended state vector can be defined as

$$\xi(\vartheta+1) = \begin{bmatrix} \boldsymbol{x}(\vartheta+1) \\ \boldsymbol{x}_c(\vartheta+1) \end{bmatrix} \in \Re^{n+n_c}, \boldsymbol{W}(\vartheta) = \begin{bmatrix} \boldsymbol{w}(\vartheta) \\ \boldsymbol{0} \end{bmatrix} \in \Re^{q+1}. \tag{10}$$

and the matrices

$$\begin{aligned} &\bar{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} + \boldsymbol{B}\boldsymbol{D}_c\boldsymbol{C} & \boldsymbol{B}\boldsymbol{C}_c \\ \boldsymbol{B}_c\boldsymbol{C} & \boldsymbol{A}_c \end{bmatrix}, \bar{\boldsymbol{A}}_d = \begin{bmatrix} \boldsymbol{A}_d & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \bar{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix}, \boldsymbol{R} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_{n_c} \end{bmatrix}, \\ &\bar{\boldsymbol{K}} = \begin{bmatrix} \boldsymbol{D}_c\boldsymbol{C} & \boldsymbol{C}_c \end{bmatrix}, \bar{\boldsymbol{B}}_w = \begin{bmatrix} \boldsymbol{B}_w & \boldsymbol{0} \end{bmatrix}, \end{aligned} \tag{11}$$

Utilising (1)–(11), closed-loop system is represented as

$$\xi(\vartheta+1) = \bar{\boldsymbol{A}}\xi(\vartheta) + \bar{\boldsymbol{A}}_d\xi\left(\vartheta - g(\vartheta)\right) - \left(\bar{\boldsymbol{B}} + \boldsymbol{R}\boldsymbol{E}_c\right)\boldsymbol{\Psi}\left(\bar{\boldsymbol{K}}\xi(\vartheta)\right) + \bar{\boldsymbol{B}}_w\boldsymbol{W}(\vartheta) \tag{12}$$

The initial condition as in (Negi et al., 2012) is

$$\xi_0 = \Phi_{\xi}(\vartheta), \vartheta = -g_u, -g_u+1, \ldots., 0 \text{ be } \varphi\left(\vartheta, \xi_0\right) \tag{13}$$

The basin of attraction of the origin of (12) is defined as

$$\mathbf{\Gamma} \triangleq \left\{ \phi_\xi(\vartheta), \vartheta = -g_u, -g_u + 1, \ldots, 0 : \lim_{\vartheta \to \infty} \varphi\left(\vartheta, \xi_0\right) = 0 \right\} \tag{14}$$

Definition 1 (Negi et al., 2012): Consider a matrix $\mathcal{G} \in \Re^{m \times (n+n_c)}$ and define the polyhedral set as (15)

$$\ell \triangleq \left\{ \begin{array}{c} \xi \in \Re^{(\eta+\eta_c)}; -u_{0(i)} \leq \left(\bar{\boldsymbol{K}}_{(i)} - \mathcal{G}_{(i)} \xi(\vartheta) \leq u_{0(i)}, \\ i = 1, 2, \ldots, m \end{array} \right\} \tag{15}$$

Definition 2 (Silva and Tarbouriech, 2005):

$$\partial = \mathbf{\Psi}^T\left(\bar{\boldsymbol{K}}\xi(\vartheta)\right)\boldsymbol{D}\left\{\mathbf{\Psi}\left(\bar{\boldsymbol{K}}\xi(\vartheta)\right) - \mathcal{G}\xi(\vartheta)\right\} \leq 0 \tag{16}$$

where $\boldsymbol{\xi}(\vartheta) \in \ell$ and $\boldsymbol{D} \in \Re^{m \times m}$ is a positive definite diagonal matrix.

- *Lemma 1 (Seuret et al., 2015):* "In a given symmetric positive definite matrix $\boldsymbol{U} = \Re^{n \times n}$, the sequence of a discrete-time variable $\boldsymbol{x}(\vartheta)$ in $[-g, 0] \cap \mathbb{Z} \to \Re^n$, where $g \geq 1$, the inequality is as follows:

  $$\sum_{p=-g+1}^{0} \mathbf{\Omega}^T(p)\boldsymbol{U}\mathbf{\Omega}(p) \geq \frac{1}{g} \begin{bmatrix} \mathbf{\Xi}_0 \\ \mathbf{\Xi}_1 \end{bmatrix}^T \begin{bmatrix} \boldsymbol{U} & \mathbf{0} \\ \mathbf{0} & 3\left(\frac{g+1}{g-1}\right)U \end{bmatrix} \begin{bmatrix} \mathbf{\Xi}_0 \\ \mathbf{\Xi}_1 \end{bmatrix} \tag{17}$$

  where $\mathbf{\Omega}(p) = \boldsymbol{x}(p) - \boldsymbol{x}(p-1)$,

  $\mathbf{\Xi}_0 = \boldsymbol{x}(0) - \boldsymbol{x}(-g)$,

  $\mathbf{\Xi}_1 = \boldsymbol{x}(0) + \boldsymbol{x}(-g) - \dfrac{2}{g+1}\sum_{p=-g}^{0} \boldsymbol{x}(p).$

  In some practical systems having time-varying delay, the factor $\left(\dfrac{g+1}{g-1}\right)$ is difficult to tackle. Hence, it is removed using Lemma 2."

- *Lemma 2 (Seuret et al., 2015):* "In a given symmetric positive definite matrix $\boldsymbol{U} = \Re^{n \times n}$, the sequence of a discrete-time variable $\boldsymbol{x}(\vartheta)$ in $[-\vartheta, 0] \cap \mathbb{Z} \to \Re^n$, where $g \geq 1$, the following inequality holds:

  $$\sum_{p=-g+1}^{0} \mathbf{\Omega}^T(p)\boldsymbol{U}\mathbf{\Omega}(p) \geq \frac{1}{g} \begin{bmatrix} \mathbf{\Xi}_0 \\ \mathbf{\Xi}_1 \end{bmatrix}^T \begin{bmatrix} \boldsymbol{U} & 0 \\ 0 & 3\boldsymbol{U} \end{bmatrix} \begin{bmatrix} \mathbf{\Xi}_0 \\ \mathbf{\Xi}_1 \end{bmatrix}, \tag{18}$$

  where

  $\mathbf{\Omega}(p) = \boldsymbol{x}(p) - \boldsymbol{x}(p-1)$,

  $\mathbf{\Xi}_0 = \boldsymbol{x}(0) - \boldsymbol{x}(-g)$,

$$\Xi_1 = \boldsymbol{x}(0) + \boldsymbol{x}(-g) - \frac{2}{g+1}\sum_{p=-g}^{0} \boldsymbol{x}(p).\text{''}$$

- *Lemma 3 (Park et al., 2011):* "For any vectors $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ matrices $\boldsymbol{T}$, $\boldsymbol{S}$ and real numbers $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ satisfying

$$\begin{bmatrix} \boldsymbol{T} & \boldsymbol{S} \\ * & \boldsymbol{T} \end{bmatrix} \geq \boldsymbol{0},\ \alpha_1 + \alpha_2 = 1, \tag{19a}$$

$$\boldsymbol{\sigma}_k = \boldsymbol{0} \text{ if } \alpha_k = 0\ (k = 1, 2), \tag{19b}$$

then

$$-\frac{1}{\alpha_1}\boldsymbol{\sigma}_1^T \boldsymbol{T}\boldsymbol{\sigma}_1 - \frac{1}{\alpha_2}\boldsymbol{\sigma}_2^T \boldsymbol{T}\boldsymbol{\sigma}_2 \leq \begin{bmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 \end{bmatrix}^T \begin{bmatrix} \boldsymbol{T} & \boldsymbol{S} \\ * & \boldsymbol{T} \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 \end{bmatrix}. \tag{19c}$$

In this section, the main results are discussed as follows:

3 Main results

The objective of this article is to obtain:

a anti-windup gain $\boldsymbol{E}_c$

b a largest possible scalar δ, such that asymptotic stability of closed-loop system (12) is achieved for all time varying delays fulfilling (2)

c an estimate of basin of attraction $X_\delta \subset \boldsymbol{\Gamma}$ where

$$X_\delta \triangleq \left\{ \phi_\xi(\vartheta),\ \vartheta = -g_u, -g_u + 1, \ldots., 0 : \max_{\vartheta \to \infty} |\phi_\xi(\vartheta)| \leq \delta \right\} \tag{20}$$

Now, using above system and Lemmas, Theorem 1 is stated as follows.

Theorem 1: For the scalars $\wp$, $\boldsymbol{\sigma}$, g_l, g_u, satisfying $0 \leq g_l \leq g_u$ consider system (12) if there exists symmetric matrices $\boldsymbol{0} < \boldsymbol{P} = \text{diag}(\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3) \in \Re^{(n+n_c)\times(n+n_c)}$, $\boldsymbol{0} < \boldsymbol{Q}_\vartheta\ (\vartheta = 1, 2) \in \Re^{(n+n_c)\times(n+n_c)}$, $\boldsymbol{0} < \boldsymbol{W}_\vartheta\ (\vartheta = 1, 2) \in \Re^{(n+n_c)\times(n+n_c)}$, $\boldsymbol{0} < \boldsymbol{R}_\vartheta\ (\vartheta = 1, 2) \in \Re^{(n+n_c)\times(n+n_c)}$, a diagonal positive definite matrix $\boldsymbol{\mathcal{L}} \in \Re^{m\times m}$, $\boldsymbol{\mathcal{G}} \in \Re^{m\times(n+n_c)}$, $\boldsymbol{\mathcal{H}} \in \Re^{n_c\times m}$, matrices $S_\vartheta(\vartheta = 1, 2, 3, 4)$ with suitable dimensions fulfilling inequalities (21)–(23)

$$\boldsymbol{\varphi} = \begin{bmatrix} \tilde{\boldsymbol{R}}_1 & \boldsymbol{0} & \boldsymbol{0} \\ * & \tilde{\boldsymbol{R}}_2 & \boldsymbol{S} \\ * & * & \tilde{\boldsymbol{R}}_2 \end{bmatrix} > \boldsymbol{0} \tag{21}$$

$$\begin{bmatrix} P & K_i^T - \boldsymbol{\mathcal{G}}_i^T \\ K_i - \boldsymbol{\mathcal{G}}_i & hu_{0(i)}^2 \end{bmatrix} \geq \boldsymbol{0},\ i = 1, 2, \ldots, m \tag{22}$$

where $h = \dfrac{1}{\left(1 + \wp^2 \sigma^2\right)}$

yields

$$\boldsymbol{\Delta}_{11} =$$

$$\begin{bmatrix}
\yen_{11} & -2\boldsymbol{R}_1 & \mathbf{0} & \mathbf{0} & 6\boldsymbol{R}_1 & \mathbf{0} & \mathbf{0} & \boldsymbol{\mathcal{G}}^T & \mathbf{0} & \boldsymbol{A}^T & g_l \hat{\boldsymbol{A}}^T & g_{lh} \hat{\boldsymbol{A}}^T & \mathbf{C}_z \\
* & \yen_{22} & \yen_{23} & \yen_{24} & 6\boldsymbol{R}_1 & 6\boldsymbol{R}_2 & 2\boldsymbol{S}_2 + 2\boldsymbol{S}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & \yen_{33} & \yen_{34} & \mathbf{0} & \yen_{36} & \yen_{37} & \mathbf{0} & \mathbf{0} & \boldsymbol{A}_d^T & g_l \boldsymbol{A}_d^T & g_{lh} \boldsymbol{A}_d^T & \mathbf{0} \\
* & * & * & \yen_{44} & \mathbf{0} & -2\boldsymbol{S}_3^T + 2\boldsymbol{S}_4^T & 6\boldsymbol{R}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & -12\boldsymbol{R}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & -12\boldsymbol{R}_2 & -4\boldsymbol{S}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & -12\boldsymbol{R}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & -2\boldsymbol{\mathcal{L}} & \mathbf{0} & -\hat{\boldsymbol{B}}^T & -g_l \hat{\boldsymbol{B}}^T & -g_{lh} \hat{\boldsymbol{B}}^T & \mathbf{0} \\
* & * & * & * & * & * & * & * & -\wp^2 \mathbf{I} & \boldsymbol{B}_w^T & -g_l \boldsymbol{B}_w^T & -g_l \boldsymbol{B}_w^T & \boldsymbol{D}_z \\
* & * & * & * & * & * & * & * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & * & * & * & -2\boldsymbol{X}_2 + \boldsymbol{X}_2 \boldsymbol{R}_1 \boldsymbol{X}_2 & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & * & * & * & * & -2\boldsymbol{X}_3 + \boldsymbol{X}_3 \boldsymbol{R}_2 \boldsymbol{X}_3 & \mathbf{0} \\
* & * & * & * & * & * & * & * & * & * & * & * & -\boldsymbol{I}
\end{bmatrix} < \mathbf{0} \tag{23}$$

$$\begin{aligned}
&\yen_{22} = -\boldsymbol{P}_2 + \boldsymbol{P}_3 - \boldsymbol{Q}_1 + \boldsymbol{Q}_2 - 4\boldsymbol{R}_1 - 4\boldsymbol{R}_2 \yen_{11} = -\boldsymbol{P}_1 + \boldsymbol{P}_2 + \boldsymbol{Q}_1 - 4\boldsymbol{R}_1, \\
&\yen_{23} = -2\boldsymbol{R}_2 - \boldsymbol{S}_1 - \boldsymbol{S}_2 - \boldsymbol{S}_3 - \boldsymbol{S}_4, \yen_{24} = \boldsymbol{S}_1 - \boldsymbol{S}_2 + \boldsymbol{S}_3 - \boldsymbol{S}_4, \\
&\yen_{33} = -8\boldsymbol{R}_2 + \boldsymbol{S}_1^T + \boldsymbol{S}_1 + \boldsymbol{S}_2^T + \boldsymbol{S}_2 - \boldsymbol{S}_3^T - \boldsymbol{S}_3 - \boldsymbol{S}_4^T - \boldsymbol{S}_4, \\
&\yen_{34} = -2\boldsymbol{R}_2 - \boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{S}_3 - \boldsymbol{S}_4, \yen_{36} = 6\boldsymbol{R}_2 + 2\boldsymbol{S}_3^T + 2\boldsymbol{S}_4^T, \\
&\yen_{37} = 6\boldsymbol{R}_2 - 2\boldsymbol{S}_2 + 2\boldsymbol{S}_4, \yen_{44} = -4\boldsymbol{R}_2 - \boldsymbol{P}_3 - \boldsymbol{Q}_2, \\
&\tilde{\boldsymbol{R}}_1 = \operatorname{diag}\left(\boldsymbol{R}_1, 3\boldsymbol{R}_1\right), \tilde{\boldsymbol{R}}_2 = \operatorname{diag}\left(\boldsymbol{R}_2, 3\boldsymbol{R}_2\right), g_{lh} = g_u - g_l, \\
&\hat{\boldsymbol{B}} = \overline{\boldsymbol{B}} + \boldsymbol{R}\boldsymbol{E}_c, \hat{\boldsymbol{A}} = (\boldsymbol{A} - \boldsymbol{I}), \\
&\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_1 & \boldsymbol{S}_2 \\ \boldsymbol{S}_3 & \boldsymbol{S}_4 \end{bmatrix} \in \Re^{2t \times 2t}
\end{aligned}$$

then for the gain matrix $\boldsymbol{E}_c = \boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{L}}^{-1}$, the closed-loop system (12) has a stipulated H_∞ interference attenuation level $\wp$ for all primary conditions satisfying $\Gamma_\delta \leq 1$ and the region of asymptotic stability is defined by an ellipsoid

$$\varepsilon\left(\boldsymbol{P}, 1 + \wp^2 \boldsymbol{\sigma}^2\right) = \left\{\xi \in \Re^{n+n_c}; \xi^T \boldsymbol{P} \xi \leq 1 + \wp^2 \boldsymbol{\sigma}^2\right\}. \tag{24}$$

and an estimate of basin of attraction is given by

$$\begin{aligned}
\Gamma_\delta = \delta^2 \big[&\lambda_{\max}\left(\boldsymbol{P}_1\right) + g_l \lambda_{\max}\left(\boldsymbol{P}_2\right) + g_{lh} \lambda_{\max}\left(\boldsymbol{P}_3\right) + g_l \lambda_{\max}\left(\boldsymbol{Q}_1\right) + \left(g_u - g_l\right) \lambda_{\max}\left(\boldsymbol{Q}_2\right) \\
&+ 2g_l\left(g_l + 1\right) \lambda_{\max}\left(\boldsymbol{R}_1\right) + 2g_{lh}\left(g_u + g_l + 1\right) \lambda_{\max}\left(\boldsymbol{R}_2\right) \big] \leq 1.
\end{aligned} \tag{25}$$

Proof of Theorem 1 is given in Appendix A.

Remark 1: Primary guess for a positive definite symmetric matrix $\boldsymbol{X}_1$ could be $\boldsymbol{X}_1 = \alpha_c(\boldsymbol{A}^T\boldsymbol{A} + \boldsymbol{I})^{-1}$.

Remark 2: Although the complexities of space and time have increased but the conservativeness of stability is reduced in terms of discrete delay system, as seen in previous papers. There is a trade-off between space and time complexities and conservativeness. The dimension/size of the undertaken system is 13 × 13, as seen in equation (23). It is a large order system; it occupies sufficient RAM/system memories. Since the simulation is performed on 2.26 GHz, exploring the optimum solution of the defined problem takes significant time. To optimise space and time complexities for the defined problem (12), a high-end computation platform is required.

Corollary 1: For the scalars $\wp$, $\boldsymbol{\sigma}$, g_l, g_u, satisfying $0 \leq g_l \leq g_u$ consider system (12) if there exists symmetric matrices $\mathbf{0} < \boldsymbol{P} = \mathrm{diag}(\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3) \in \Re^{(n+n_c)\times(n+n_c)}$, $\mathbf{0} < \boldsymbol{R}_\vartheta (\vartheta = 1, 2) \in \Re^{(n+n_c)\times(n+n_c)}$, $\mathbf{0} < \boldsymbol{Q}_\vartheta (\vartheta = 1, 2) \in \Re^{(n+n_c)\times(n+n_c)}$, $\mathbf{0} < \boldsymbol{W}_\vartheta (\vartheta = 1, 2) \in \Re^{(n+n_c)\times(n+n_c)}$, a diagonal positive definite matrix $\boldsymbol{\mathcal{L}} \in \Re^{m\times m}$, $\boldsymbol{\mathcal{G}} \in \Re^{m\times(n+n_c)}$, $\boldsymbol{\mathcal{H}} \in \Re^{n_c\times m}$, matrices S_ϑ (ϑ = 1, 2, 3, 4) such that (21)–(22) hold, for the gain matrix $\boldsymbol{E}_c = \boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{L}}^{-1}$ the closed-loop system (12) is globally asymptotically stable for all initial conditions satisfying $\Gamma_\delta \leq 1$ with prescribed H_∞ disturbance attenuation level .

Proof: Consider $\boldsymbol{\mathcal{G}} = \bar{\boldsymbol{K}}$. It follows that (14) is verified for all $\xi(\vartheta) \in \Re^{n+n_c}$, then (23) corresponds to (26) yields

$$
\boldsymbol{\Delta}_{22} = \begin{bmatrix}
¥_{11} & -2\boldsymbol{R}_1 & \mathbf{0} & \mathbf{0} & 6\boldsymbol{R}_1 & \mathbf{0} & \mathbf{0} & \bar{\boldsymbol{K}}^T & \mathbf{0} & \boldsymbol{A}^T & g_l\hat{\boldsymbol{A}}^T & g_{lh}\hat{\boldsymbol{A}}^T & \boldsymbol{C}_z \\
* & ¥_{22} & ¥_{23} & ¥_{24} & 6\boldsymbol{R}_1 & 6\boldsymbol{R}_2 & 2\boldsymbol{S}_2 + 2\boldsymbol{S}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & ¥_{33} & ¥_{34} & \mathbf{0} & ¥_{36} & ¥_{37} & \mathbf{0} & \mathbf{0} & \boldsymbol{A}_d^T & g_l\boldsymbol{A}_d^T & g_{lh}\boldsymbol{A}_d^T & \mathbf{0} \\
* & * & * & ¥_{44} & \mathbf{0} & -2\mathbf{S}_3^T + 2\boldsymbol{S}_4^T & 6\boldsymbol{R}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & -12\boldsymbol{R}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & -12\boldsymbol{R}_2 & -4\boldsymbol{S}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & -12\boldsymbol{R}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & -2\boldsymbol{\mathcal{L}} & \mathbf{0} & -\hat{\boldsymbol{B}}^T & -g_l\hat{\boldsymbol{B}}^T & -g_{lh}\hat{\boldsymbol{B}}^T & \mathbf{0} \\
* & * & * & * & * & * & * & * & -\wp^2\boldsymbol{I} & \boldsymbol{B}_w^T & -g_l\boldsymbol{B}_w^T & -g_l\boldsymbol{B}_w^T & \boldsymbol{D}_z \\
* & * & * & * & * & * & * & * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & * & * & * & -2\boldsymbol{X}_2 + \boldsymbol{X}_2\boldsymbol{R}_1\boldsymbol{X}_2 & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & * & * & * & * & -2\boldsymbol{X}_3 + \boldsymbol{X}_3\boldsymbol{R}_2\boldsymbol{X}_3 & \mathbf{0} \\
* & * & * & * & * & * & * & * & * & * & * & * & -\boldsymbol{I}
\end{bmatrix} < \mathbf{0} \tag{26}
$$

Basin of attraction

Theorem 2: Consider the closed-loop system (12) with the initial conditions (13), the maximum attraction basin can be estimated if r is minimised where

$$
\begin{aligned}
r = r_1 + g_l r_2 + (g_u - g_l) r_3 + g_l r_4 + (g_u - g_l) r_5 + 2g_l (g_l + 1) r_6 \\
+2(g_u - g_l)(g_u + g_l + 1) r_7
\end{aligned} \tag{27}
$$

subjected to (21)–(23) and

$$
\begin{aligned}
&r_1\boldsymbol{I} - \boldsymbol{P}_1 \geq \mathbf{0}, r_2\boldsymbol{I} - \boldsymbol{P}_2 \geq \mathbf{0}, r_3\boldsymbol{I} - \boldsymbol{P}_3 \geq \mathbf{0}, r_4\boldsymbol{I} - \boldsymbol{Q}_1 \geq \mathbf{0}, r_5\boldsymbol{I} - \boldsymbol{Q}_2 \geq \mathbf{0}, \\
&r_6\boldsymbol{I} - \boldsymbol{R}_1 \geq \mathbf{0}, r_7\boldsymbol{I} - \boldsymbol{R}_2 \geq \mathbf{0}
\end{aligned} \tag{28}
$$

has a feasible solution for the weighting parameters $r_i > 0$, $i = 1, 2, \ldots, 11$, positive definite symmetric matrices $\boldsymbol{P}_k (k = 1, 2, 3) \in \Re^{(n+n_c)\times(n+n_c)}$, $\mathbf{Q}_k (k = 1, 2) \in \Re^{(n+n_c)\times(n+n_c)}$, $\boldsymbol{W}_k (k = 1, 2) \in \Re^{(n+n_c)\times(n+n_c)}$, $\mathbf{0} <_k \boldsymbol{R}(k = 1, 2) \in \Re^{(n+n\ c)\times(n+n_c)}$, $\mathbf{0} < \boldsymbol{X}_k (k = 1, \ldots, 4) \in \Re^{(n+n_c)\times(n+n_c)}$, a diagonal positive definite matrix $\mathcal{L} \in \Re^{m\times m}, \mathcal{G} \in \Re^{m\times(n+n_c)}, \mathcal{H} \in \Re^{n_c\times m}$, matrices $\boldsymbol{S}_k$ $(k = 1, 2, 3, 4)$.

The gain $\boldsymbol{E}_c = \mathcal{H}\mathcal{L}^{-1}$ provides attraction basin by $\delta_{\max} = 1/\sqrt{\boldsymbol{E}2}$, where

$$\begin{aligned}\boldsymbol{E2} = {} & \lambda_{\max}(\boldsymbol{P}_1) + g_l\lambda_{\max}(\boldsymbol{P}_2) + (g_u - g_l)\lambda_{\max}(\boldsymbol{P}_3) + g_l\lambda_{\max}(\boldsymbol{Q}_1) \\ & + (g_u - g_l)\lambda_{\max}(\boldsymbol{Q}_2) + 2g_l(g_l+1)\times\lambda_{\max}(\boldsymbol{R}_1) + 2g_{lh}(g_u + g_l + 1)\lambda_{\max}(\boldsymbol{R}_2)\end{aligned} \tag{29}$$

Proof of Theorem 2 is given in Appendix B.

4 Numerical examples

This section provides examples to depict essence of the main results.

Example 1: Consider the discrete time system (1) with controller (3)

$$\boldsymbol{A} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.97 \end{bmatrix}, \boldsymbol{A}_d = \begin{bmatrix} -0.1 & -0.1 \\ 0 & -0.01 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} -0.04 & 0.18 \\ 0 & 0 \end{bmatrix},$$

$$\boldsymbol{A}_c = \begin{bmatrix} 0.2 & 0.0 \\ -0.16 & 0.084 \end{bmatrix}, \boldsymbol{B}_c = \begin{bmatrix} 1.0 & -20.0 \\ 0.0 & 0.0 \end{bmatrix}, \boldsymbol{C}_c = \begin{bmatrix} -0.1 & -0.1 \end{bmatrix},$$

$$\boldsymbol{D}_c = \begin{bmatrix} 0.0 & 0.0 \end{bmatrix}, \boldsymbol{B}_w = \begin{bmatrix} 0.01 & 0 \\ 0.007 & 0.008 \end{bmatrix}, \boldsymbol{C}_z = \begin{bmatrix} 0.01 & 0 \\ 0.006 & 0.002 \end{bmatrix},$$

$$\boldsymbol{D}_z = \begin{bmatrix} 0.006 & 0.005 \\ 0.002 & 0.006 \end{bmatrix}, \alpha = 0.01, \boldsymbol{w}(\vartheta) = \begin{bmatrix} 0.05e^{-0.05\vartheta} & 0.05e^{-0.05\vartheta} \end{bmatrix}^T.$$

The saturated control signal (6) is given into the plant where $u_{0(i)} = [1 \quad 1]^T$. By utilising the LMI workbench (Gahinet et al., 1995), the LMI constraints (21)–(23) asserted in Theorem 1 are encountered feasible for $1 \le g(\vartheta) \le 45, \wp = 1.003$ and $\boldsymbol{E}_c = \mathcal{H}\mathcal{L}^{-1} = \begin{bmatrix} -0.1738 & -0.1309 \end{bmatrix}^T$.

The trajectories of plant states and controller states are revealed in Figures 1 and 2 for initial plant states $x_0 = [6 \quad -6]^T$. The states of the plant given by $x_1(\vartheta)$ and $x_2(\vartheta)$ converge to zero (see Figure 1). As shown in Figure 2, the controller states given by $x_{c1}(\vartheta)$ and $x_{c2}(\vartheta)$ also converge to zero. The plot of plant input $u(\vartheta)$ and unconstrained controller output $v_c(\vartheta)$ is displayed in Figure 3. Thus, anti-windup gain stabilises the system in presence of delay. The maximised estimate of basin of attraction provided by anti-windup gain is given by $\delta_{\max} = 9.5054 \times 10^{-6}$.

A comparison of maximum allowed g_u obtained using Theorem 1 and criteria used in (Negi et al., 2012; Xu et al., 2012; Qian et al., 2015) for global asymptotic stability of the system is depicted in Table 1. It is clear from Table 1 that Theorem 1 produces superior results than in Negi et al. (2012), Xu et al. (2012) and Qian et al. (2015).

Figure 1 Plant states trajectory (see online version for colours)

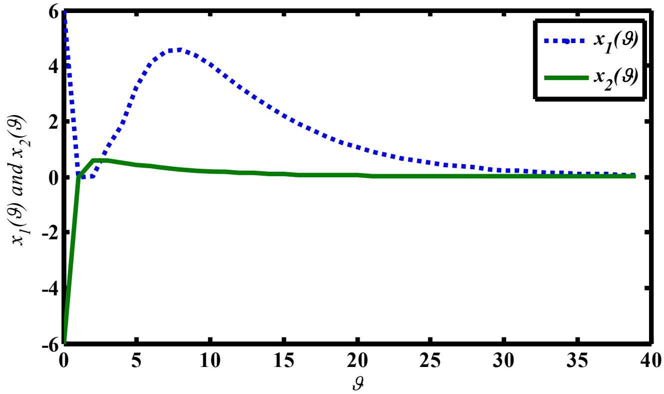

Figure 2 Controller states trajectory (see online version for colours)

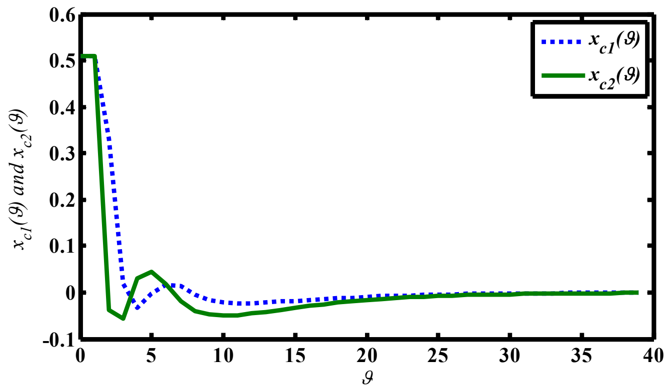

Figure 3 Plot of $v_c(\vartheta)$ and $u(\vartheta)$ (see online version for colours)

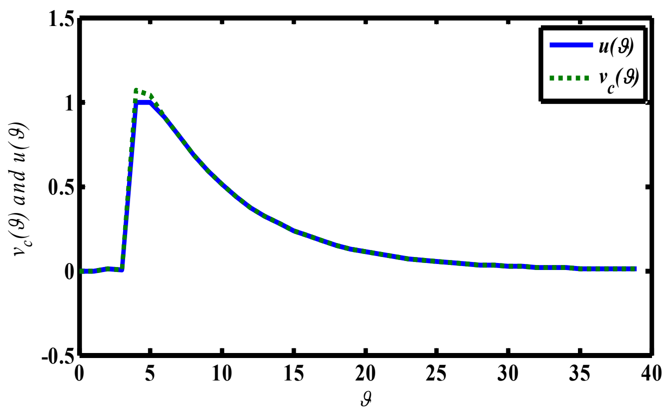

Table 1 Comparison of delay ranges for stability analysis of system using different techniques

Method	*Delay range* ($g_l \leq g(\vartheta) \leq g_u$)	*Nonlinearities*
Theorem 2 (Negi et al., 2012)	$1 \leq g(\vartheta) \leq 5$	System with saturation and delay
Theorem 1 (Xu et al., 2012)	$1.4 \leq g(\vartheta) \leq 3.8$	System with saturation, delay and interference
Theorem 1 (Qian et al., 2015)	$1 \leq g(\vartheta) \leq 4$	System with saturation, delay and interference
Theorem 2 (Qian et al., 2015)	$1 \leq g(\vartheta) \leq 3$	System with saturation, delay and interference
Corollary 2 (Pal and Negi, 2018)	$1 \leq g(\vartheta) \leq 9$	With saturation, time varying delay, external interference and uncertainties with H_∞ level $\lambda = 1$
Theorem 1, Corollary 1 (de Souza et al., 2019)	$1 \leq g(\vartheta) \leq 11$	With saturation, time varying delay
Theorem 1 (Chen et al., 2019)	$1 \leq g(\vartheta) \leq 5$	With saturation, time varying delay and uncertainty
Theorem 6 (Singh et al., 2022)	$1 \leq g(\vartheta) \leq 11$	With saturation, time varying delay, external interference and uncertainties with H_∞ level $\lambda = 0.8$
Theorem 3.1 [proposed work]	$1 \leq g(\vartheta) \leq 45$	System with saturation, stabilising controller, delay and interference

Remark 3: This strategy utilises all information correctly. Hence, the system reaches the error band correctly and settles down soon. It suffers from disadvantage that all the system parameters are responsible for the stability of the system but we cannot put stability criterion on some parameters. In the considered study, the design of override controller may be seen as a main limitation.

5 Conclusions

The problem for discrete time delay systems with input saturation and disturbance using anti-windup strategies is analysed in this paper. Time varying delay is considered here. A delay-dependent approach is utilised along with Wirtinger inequality and reciprocal convex inequality. It is shown in example that the delay range $1 \leq g(\vartheta) \leq 45$ has increased as compared to the previous existing results. An analogous LMI based anti-windup gain is evaluated. Estimate of basin of attraction of the origin is obtained for the system considered with various time delay ranges.

As demonstrated in Meng et al. (2010), the idea of delay partition may result in less conservative stability criteria. Thus, the same problem can be done using delay partitioning, for 1D or 2D systems or by introducing various nonlinearities such as uncertainty and finite wordlength. The extension of the proposed work may be seen in the field of telecommunication along with probability density distribution communication delay by event-triggered mechanism (Gu et al., 2022a) and for network fault detection in interval type-2 adaptive memory-event-triggered mechanism (Gu et al., 2022b).

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Appendix A

Proof: When disturbance is present in a system (12), disturbance attenuation can be intended in sense of H_∞ by considering the following equation (A1)

$$\boldsymbol{\beta}(\vartheta) = \Delta v\left(\boldsymbol{x}(\vartheta)\right) + \boldsymbol{z}^T(\vartheta)z(\vartheta) - \wp^2 \boldsymbol{w}^T(\vartheta)\boldsymbol{w}(\vartheta). \tag{A1}$$

Consider a Lyapunov function as

$$V\left(\xi(\vartheta)\right) = V_1\left(\xi(\vartheta)\right) + V_2\left(\xi(\vartheta)\right) + V_3\left(\xi(\vartheta)\right), \tag{A2a}$$

with

$$V_1\left(\boldsymbol{\xi}(k)\right) = \prod\nolimits_2^T(\vartheta)\boldsymbol{P}\prod\nolimits_2(\vartheta), \tag{A2b}$$

$$V_2\left(\boldsymbol{\xi}(\vartheta)\right) = \sum_{i=\vartheta-g_l}^{\vartheta-1} \xi^T(i)\boldsymbol{Q}_1\xi(i) + \sum_{i=\vartheta-g_u}^{\vartheta-g_l-1} \xi^T(i)\boldsymbol{Q}_2\boldsymbol{\xi}(i), \tag{A2c}$$

$$V_3\xi(\vartheta) = \sum_{i=-g_l+1}^{0} \sum_{j=\vartheta+i}^{\vartheta} g_l \boldsymbol{\Lambda}^T(j)\boldsymbol{R}_1\boldsymbol{\Lambda}(j) + g_{12} \sum_{i=g_2+1}^{-g_1} \sum_{j=\vartheta+i}^{\vartheta} \boldsymbol{\Lambda}^T(j)\boldsymbol{R}_2\boldsymbol{\Lambda}(j), \tag{A2d}$$

where

$$\prod\nolimits_2^T(\vartheta) = \begin{bmatrix} \boldsymbol{\xi}^T(\vartheta) & \sum_{i=\vartheta-g_l}^{\vartheta-1} \boldsymbol{\xi}^T(i) & \sum_{i=\vartheta-g_u}^{\vartheta-g_l-1} \boldsymbol{\xi}^T(i) \end{bmatrix}, \tag{A3}$$

$$\boldsymbol{\Lambda}(\vartheta) = \boldsymbol{\xi}(\vartheta+1) - \boldsymbol{\xi}(\vartheta).$$

The augmented vector is defined as

$$\left[\boldsymbol{\xi}^T(\vartheta) \quad \boldsymbol{\xi}^T\left(\vartheta-g_l\right) \quad \boldsymbol{\xi}^T\left(\vartheta-g(\vartheta)\right) \quad \boldsymbol{\xi}^T\left(\vartheta-g_u\right) \quad v_1^T(\vartheta) \quad v_2^T(\vartheta) \quad v_3^T(\vartheta) \quad \boldsymbol{\Psi}^T\left(\bar{\boldsymbol{K}}\boldsymbol{\xi}(\vartheta)\right) \quad w(\vartheta)\right], \tag{A4}$$

where

$$v_1(\vartheta) = \frac{1}{g_l + 1} \sum_{i=\vartheta-g_l}^{\vartheta} \boldsymbol{\xi}(i), \tag{A5}$$

$$v_2(\vartheta) = \frac{1}{g(\vartheta) - g_l + 1} \sum_{i=\vartheta-g(\vartheta)}^{\vartheta-g_l} \boldsymbol{\xi}(i), \tag{A6}$$

$$v_3(\vartheta) = \frac{1}{g_2 - g(\vartheta) + 1} \sum_{i=\vartheta-g_2}^{\vartheta-g(\vartheta)} \boldsymbol{\xi}(i). \tag{A7}$$

The forward difference of Lyapunov function (A1) along the trajectories of the system (12) is

$$\begin{aligned} \Delta V\left(\boldsymbol{\xi}(\vartheta)\right) &= V\left(\boldsymbol{\xi}(\vartheta+1)\right) - V\left(\boldsymbol{\xi}(\vartheta)\right) \\ &= \Delta V_1\left(\boldsymbol{\xi}(\vartheta)\right) + \Delta V_2\left(\boldsymbol{\xi}(\vartheta)\right) + \Delta V_3\left(\boldsymbol{\xi}(\vartheta)\right) \end{aligned} \tag{A8}$$

where

$$\Delta V_1\left(\boldsymbol{\xi}(\vartheta)\right) = \boldsymbol{\xi}^T(\vartheta)\left[\boldsymbol{\omega}_1^T \boldsymbol{P}\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2^T \boldsymbol{P}\boldsymbol{\omega}_2\right] + He\left[\boldsymbol{\beta}^T\left(g(\vartheta)\right)\boldsymbol{P}\left(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\right)\right]\xi(\vartheta) \tag{A9}$$

$\Delta V_1(\boldsymbol{\xi}(\vartheta))$ is obtained by the following relation

$$\begin{aligned} \begin{bmatrix} \boldsymbol{\xi}(\vartheta+1) \\ \sum\limits_{i=\vartheta-g_l+1}^{\vartheta} \xi(i) \\ \sum\limits_{i=\vartheta-g_u+1}^{\vartheta-g_l} \xi(i) \end{bmatrix} &= \begin{bmatrix} \hat{\boldsymbol{A}}\xi(\vartheta) + \bar{\boldsymbol{A}}_d\xi\left(\vartheta - g(\vartheta)\right) - \hat{\boldsymbol{B}}\boldsymbol{\Psi}\left(\bar{\boldsymbol{K}}\boldsymbol{\xi}(\vartheta)\right) + w(\vartheta) \\ -\boldsymbol{\xi}\left(\vartheta - g_l\right) + v_1(\vartheta) \\ -\xi\left(\vartheta - g(\vartheta)\right) - \boldsymbol{\xi}\left(\vartheta - g_u\right) + v_2(\vartheta) + v_3(\vartheta) \end{bmatrix} \\ &\quad + \begin{bmatrix} \boldsymbol{\xi}(\vartheta) \\ g_l v_1(\vartheta) \\ \left(g(\vartheta) - g_l\right)v_2(\vartheta) + \left(g_u - g(\vartheta)\right)v_3(\vartheta) \end{bmatrix} \\ &= \boldsymbol{\omega}_1 + \boldsymbol{\beta}\left(g(\vartheta)\right)\xi(\vartheta) \end{aligned} \tag{A10}$$

and

$$\begin{bmatrix} \xi(\vartheta) \\ \sum_{s=\vartheta-g_l}^{\vartheta-1} \xi(s) \\ \sum_{s=\vartheta-gh_1}^{\vartheta-g_l-1} \xi(s) \\ \sum_{s=\vartheta-g_u}^{\vartheta-gh_1-1} \xi(s) \end{bmatrix} = \begin{bmatrix} 0 \\ -\xi(\vartheta)+v_1(\vartheta) \\ -\xi(\vartheta-g_l)-\xi(\vartheta-g(\vartheta))+v_2(\vartheta)+v_3(\vartheta) \\ -\xi(\vartheta-gh_1)-\xi(\vartheta-g(\vartheta))+v_4(\vartheta)+v_5(\vartheta) \end{bmatrix}$$

$$+\begin{bmatrix} \xi(\vartheta) \\ g_l v_1(\vartheta) \\ (g(\vartheta)-g_l)v_2(\vartheta)+(gh_1-g(\vartheta))v_3(\vartheta) \\ (g(\vartheta)-gh_1)v_4(\vartheta)+(g_u-g(\vartheta))v_5(\vartheta) \end{bmatrix} \tag{A11}$$

$$= \boldsymbol{\omega}_2+\boldsymbol{\beta}(g(\vartheta))\boldsymbol{\xi}(\vartheta)$$

where

$$\boldsymbol{\omega}_1 = \begin{bmatrix} \hat{\boldsymbol{A}} & \mathbf{0} & \bar{\boldsymbol{A}}_d & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\hat{\boldsymbol{B}} & \boldsymbol{B}_w \\ \mathbf{0} & -\boldsymbol{I} & \mathbf{0} & \mathbf{0} & \boldsymbol{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\boldsymbol{I} & -\boldsymbol{I} & \mathbf{0} & \boldsymbol{I} & \boldsymbol{I} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\boldsymbol{\omega}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\boldsymbol{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{I} & -\boldsymbol{I} & \mathbf{0} & \mathbf{0} & \boldsymbol{I} & \boldsymbol{I} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{A12}$$

$$\boldsymbol{\beta}(g(\vartheta)) = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & g_l\boldsymbol{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (g(\vartheta)-g_l)\boldsymbol{I} & (g_h-g(\vartheta))\boldsymbol{I} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\begin{aligned} \Delta V_2(\xi(\vartheta)) &= V_2(\xi(\vartheta+1))-V_2(\xi(\vartheta)) \\ &= \boldsymbol{\xi}^T(\vartheta)\boldsymbol{Q}_1\xi(\vartheta)-\boldsymbol{\xi}^T(\vartheta-g_l)(\boldsymbol{Q}_1-\boldsymbol{Q}_2)\boldsymbol{\xi}(\vartheta-g_l) \\ &\quad -\boldsymbol{\xi}^T(\vartheta-g_u)\mathbf{Q}_2\boldsymbol{\xi}(\vartheta-g_u) \\ &= \boldsymbol{\xi}^T(\vartheta)\tilde{\boldsymbol{Q}}\xi(\vartheta), \end{aligned} \tag{A13}$$

where

$$\tilde{\boldsymbol{Q}} = diag(\boldsymbol{Q}_1, -\boldsymbol{Q}_1+\boldsymbol{Q}_2, \mathbf{0}, -\boldsymbol{Q}_2, \mathbf{0}, \mathbf{0}, \mathbf{0}) \tag{A14}$$

$$\begin{aligned} \Delta \boldsymbol{V}_3 &= \left[\boldsymbol{\xi}^T(\vartheta+1)-\boldsymbol{\xi}^T(\vartheta)\right]\left(g_l^2\boldsymbol{R}_1+g_{lh}^2\right)\boldsymbol{R}_2\left[\xi(\vartheta+1)-\xi(\vartheta)\right] \\ &\quad -g_l\sum_{j=\vartheta-g_l+1}^{\vartheta}\boldsymbol{\Lambda}^T(i)\boldsymbol{R}_1\boldsymbol{\Lambda}(i)-\frac{g_{lh}}{g(\vartheta)-g_l}(g(\vartheta)-g_l)\sum_{j=\vartheta-g(\vartheta)+1}^{\vartheta-g_l}\boldsymbol{\Lambda}^T(j)\boldsymbol{R}_2\boldsymbol{\Lambda}(j) \\ &\quad -\frac{g_{lh}}{g_u-g(\vartheta)}(g_u-g(\vartheta))\sum_{j=\vartheta-g_u+1}^{\vartheta-g(\vartheta)}\boldsymbol{\Lambda}^T(j)\boldsymbol{R}_2\boldsymbol{\Lambda}(j). \end{aligned} \tag{A15}$$

On applying Lemma 1 and 2 on (A15) we get

$$\Delta V_3 \le \xi^T(\vartheta)\boldsymbol{\omega}_1^T\tilde{\boldsymbol{R}}_{12}\boldsymbol{\omega}_1\xi(\vartheta) - \begin{bmatrix} \xi(\vartheta)-\xi(\vartheta-g_l) \\ \xi(\vartheta)+\xi(\vartheta-g_l)-\dfrac{2}{g_l+1}\sum\limits_{i=\vartheta-g_l}^{\vartheta}\xi(i) \end{bmatrix}^T \begin{bmatrix} \boldsymbol{R}_1 & \boldsymbol{0} \\ \boldsymbol{0} & 3\boldsymbol{R}_1 \end{bmatrix}$$

$$\begin{bmatrix} \xi(\vartheta)-\xi(\vartheta-g_l) \\ \xi(\vartheta)+\xi(\vartheta-g_l)-\dfrac{2}{g_l+1}\sum\limits_{i=\vartheta-g_l}^{\vartheta}\xi(i) \end{bmatrix}$$

$$-\begin{bmatrix} \xi(\vartheta-g_l)-\xi(\vartheta-g(\vartheta)) \\ \xi(\vartheta-g_l)-\xi(\vartheta-g(\vartheta))-\dfrac{2}{g(\vartheta)-g_l+1}\sum\limits_{i=\vartheta-g(\vartheta)}^{\vartheta-g_1}\xi(i) \end{bmatrix}^T$$

$$\begin{bmatrix} \dfrac{g_{lh}}{g(\vartheta)-g_l}\boldsymbol{R}_2 & \boldsymbol{0} \\ \boldsymbol{0} & 3\dfrac{g_{lh}}{g(\vartheta)-g_l}\boldsymbol{R}_2 \end{bmatrix} \times \begin{bmatrix} \xi(\vartheta-g_l)-\xi(\vartheta-g(\vartheta)) \\ \xi(\vartheta-g_l)-\xi(\vartheta-g(\vartheta))-\dfrac{2}{g(\vartheta)-g_l+1}\sum\limits_{i=\vartheta-g(\vartheta)}^{\vartheta-g_1}\xi(i) \end{bmatrix}$$

$$-\begin{bmatrix} \xi(\vartheta-g(\vartheta))-\xi(\vartheta-g_u) \\ \xi(\vartheta-g(\vartheta))-\xi(\vartheta-g_u)-\dfrac{2}{g_u-g(\vartheta)+1}\sum\limits_{i=\vartheta-g_u}^{\vartheta-g(\vartheta)}\xi(i) \end{bmatrix}^T \quad \text{(A16)}$$

$$\begin{bmatrix} \dfrac{g_{12}}{g(\vartheta)-g_l}\boldsymbol{R}_2 & \boldsymbol{0} \\ \boldsymbol{0} & 3\dfrac{g_{12}}{g(\vartheta)-g_l}\boldsymbol{R}_2 \end{bmatrix} \times \begin{bmatrix} \xi(\vartheta-g(\vartheta))-\xi(\vartheta-g_u) \\ \xi(\vartheta-g(\vartheta))-\xi(\vartheta-g_u)-\dfrac{2}{g_u-g(\vartheta)+1}\sum\limits_{i=\vartheta-g_u}^{\vartheta-g(\vartheta)}\xi(i) \end{bmatrix}$$

$$-\begin{bmatrix} \xi(\vartheta-g(\vartheta))-\xi(\vartheta-g_u) \\ \xi(k-g(\vartheta))+\xi(\vartheta-g_u)-\dfrac{2}{g_u-g(\vartheta)+1}\sum\limits_{i=\vartheta-g_u}^{\vartheta-g(\vartheta)}\xi(i) \end{bmatrix}^T$$

$$\begin{bmatrix} \dfrac{g_{12}}{g_u-g(\vartheta)}\boldsymbol{R}_2 & \boldsymbol{0} \\ \boldsymbol{0} & 3\dfrac{g_{12}}{g_u-g(\vartheta)}\boldsymbol{R}_2 \end{bmatrix} \times \begin{bmatrix} \xi(\vartheta-g(\vartheta))-\xi(\vartheta-g_u) \\ \xi(\vartheta-g(\vartheta))+\xi(\vartheta-g_u)-\dfrac{2}{g_u-g(\vartheta)+1}\sum\limits_{i=\vartheta-g_u}^{\vartheta-g(\vartheta)}\xi(i) \end{bmatrix}$$

where $\boldsymbol{R}_1 = diag\left[\boldsymbol{R}_1,\quad 3\boldsymbol{R}_1\right]$, $\boldsymbol{R}_2 = diag\left[\boldsymbol{R}_2,\quad 3\boldsymbol{R}_2\right]$, $\boldsymbol{R}_{12} = diag\left[g_l^2\boldsymbol{R}_1 + g_{lh}^2\boldsymbol{R}_2,\quad \boldsymbol{0},\quad \boldsymbol{0}\right]$.

$$\begin{aligned} \Delta V\left(\xi(\vartheta)\right) &= \xi^T(\vartheta)\tilde{\boldsymbol{Q}}\xi(\vartheta) + \xi^T(\vartheta)\left[\boldsymbol{\omega}_1^T\boldsymbol{P}\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2^T\boldsymbol{P}\boldsymbol{\omega}_2\right] \\ &\quad + He\left[\boldsymbol{\beta}^T\left(g(\vartheta)\right)\boldsymbol{P}(\boldsymbol{\omega}_1-\boldsymbol{\omega}_2)\right]\xi(\vartheta) \\ &\quad + \xi^T(\vartheta)\boldsymbol{\omega}_1^T\tilde{\boldsymbol{R}}_{12}\boldsymbol{\omega}_1\xi(\vartheta) - \xi^T(\vartheta)\boldsymbol{\theta}^T(\vartheta)\boldsymbol{\varphi}\boldsymbol{\theta}(\vartheta)\xi(\vartheta). \end{aligned} \quad \text{(A17)}$$

Using (A9)–(A17), we have following inequality

$$
\begin{aligned}
\Delta V\left(\xi(\vartheta)\right) \le\ & \boldsymbol{\xi}^T(\vartheta)\tilde{\boldsymbol{Q}}\boldsymbol{\xi}(\vartheta) + \boldsymbol{\xi}^T(\vartheta)\left[\boldsymbol{\omega}_1^T \boldsymbol{P}\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2^T \boldsymbol{P}\boldsymbol{\omega}_2\right] \\
& + He\left[\boldsymbol{\beta}^T\left(g(\vartheta)\right)\boldsymbol{P}\left(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\right)\right]\xi(\vartheta) \\
& + \boldsymbol{\xi}^T(\vartheta)\boldsymbol{\omega}_1^T\tilde{\boldsymbol{R}}_{12}\boldsymbol{\omega}_1\boldsymbol{\xi}(\vartheta) - \boldsymbol{\xi}^T(\vartheta)\boldsymbol{\theta}^T(\vartheta)\boldsymbol{\varphi}\boldsymbol{\theta}(\vartheta)\boldsymbol{\xi}(\vartheta) - 2\partial.
\end{aligned} \tag{A18}
$$

where

$$
\boldsymbol{\varphi} = \begin{bmatrix} \tilde{\boldsymbol{R}}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\boldsymbol{R}}_2 & \boldsymbol{S} \\ \mathbf{0} & \boldsymbol{S}^T & \tilde{\boldsymbol{R}}_2 \end{bmatrix}, \boldsymbol{\theta}(\vartheta) = \begin{bmatrix} \boldsymbol{M} & \mathbf{0}_{2n\times n} & \\ \mathbf{0}_{2n\times n} & \boldsymbol{M} & \mathbf{0}_{2n\times 3n} \\ \mathbf{0}_{2n\times 2n} & \boldsymbol{M} & \mathbf{0}_{2n\times 2n} \\ & \mathbf{0}_{2n\times 9n} & \end{bmatrix}, \boldsymbol{M} = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{I} & \boldsymbol{I} & \mathbf{0} & \mathbf{0} & -2\boldsymbol{I} \end{bmatrix},
$$

∂ is given by (16),

$$
¥_2 =
$$

$$
\begin{bmatrix}
¥_{11}+\nabla_{11} & -2\boldsymbol{R}_1 & \nabla_{13} & \mathbf{0} & 6\boldsymbol{R}_1 & \mathbf{0} & \mathbf{0} & \mathcal{G}^T\boldsymbol{D}+\nabla_{18} & \nabla_{19} \\
* & ¥_{22} & ¥_{23} & ¥_{24} & 6\boldsymbol{R}_1 & 6\boldsymbol{R}_2 & 2\boldsymbol{S}_2+2\boldsymbol{S}_4 & \mathbf{0} & \mathbf{0} \\
* & * & ¥_{33}+\nabla_{33} & ¥_{34} & \mathbf{0} & ¥_{36} & ¥_{37} & \nabla_{38} & \nabla_{39} \\
* & * & * & ¥_{44} & \mathbf{0} & -2\boldsymbol{S}_3^T+2\boldsymbol{S}_4^T & 6\boldsymbol{R}_2 & \mathbf{0} & \mathbf{0} \\
* & * & * & * & -12\boldsymbol{R}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & -12\boldsymbol{R}_2 & -4\boldsymbol{S}_4 & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & -12\boldsymbol{R}_2 & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & -2\boldsymbol{D}+\nabla_{88} & \nabla_{89} \\
* & * & * & * & * & * & * & * & \nabla_{99}
\end{bmatrix} < \mathbf{0}, \tag{A19}
$$

$$\nabla_{11} = (\boldsymbol{A}-\boldsymbol{I})^T g_l^2 \boldsymbol{R}_1\hat{\boldsymbol{A}} + \hat{\boldsymbol{A}}^T g_{lh}^2 \boldsymbol{R}_2\hat{\boldsymbol{A}} + \hat{\boldsymbol{A}}^T \boldsymbol{P}_1\hat{\boldsymbol{A}}$$

$$\nabla_{13} = \hat{\boldsymbol{A}}^T g_l^2 \boldsymbol{R}_1\boldsymbol{A}_d + \hat{\boldsymbol{A}}^T g_{lh}^2 \boldsymbol{R}_2\boldsymbol{A}_d + \hat{\boldsymbol{A}}^T \boldsymbol{P}\boldsymbol{A}_d$$

$$\nabla_{18} = -\hat{\boldsymbol{B}}^T g_l^2 \boldsymbol{R}_1\hat{\boldsymbol{B}} - \hat{\boldsymbol{A}}^T g_{lh}^2 \boldsymbol{R}_2\hat{\boldsymbol{B}} - \hat{\boldsymbol{A}}^T \boldsymbol{P}_1\hat{\boldsymbol{A}}$$

$$\nabla_{19} = \hat{\boldsymbol{A}}^T \boldsymbol{P}_1\boldsymbol{B}_w + \boldsymbol{P}_1\boldsymbol{B}_w + \hat{\boldsymbol{A}}^T g_l^2 \boldsymbol{R}_1\boldsymbol{B}_w + \hat{\boldsymbol{A}}^T g_{lh}^2 \boldsymbol{R}_2\boldsymbol{B}_w + \boldsymbol{C}_z^T\boldsymbol{D}_z$$

$$\nabla_{33} = \boldsymbol{A}_d^T \boldsymbol{P}_1\boldsymbol{A}_d + \boldsymbol{A}_d^T g_l^2 \boldsymbol{R}_1\boldsymbol{A}_d + \boldsymbol{A}_d^T g_{lh}^2 \boldsymbol{R}_2\boldsymbol{A}_d$$

$$\nabla_{38} = -\boldsymbol{A}_d^T \boldsymbol{P}_1\hat{\boldsymbol{B}}$$

$$\nabla_{39} = \boldsymbol{A}_d^T \boldsymbol{P}_1\boldsymbol{B}_w + \boldsymbol{A}_d^T g_l^2 \boldsymbol{R}_1\boldsymbol{B}_w + \boldsymbol{A}_d^T g_{lh}^2 \boldsymbol{R}_2\boldsymbol{B}_w$$

$$\nabla_{88} = \hat{\boldsymbol{B}}^T g_l^2 \boldsymbol{R}_1\hat{\boldsymbol{B}} + \hat{\boldsymbol{B}}^T g_{lh}^2 \boldsymbol{R}_2\hat{\boldsymbol{B}} + \hat{\boldsymbol{B}}^T \boldsymbol{P}_1\hat{\boldsymbol{B}}$$

$$\nabla_{89} = -\hat{\boldsymbol{B}}^T \boldsymbol{P}_1\boldsymbol{B}_w - \hat{\boldsymbol{B}}^T g_l^2 \boldsymbol{R}_1\boldsymbol{B}_w - \hat{\boldsymbol{B}}^T g_{lh}^2 \boldsymbol{R}_2\boldsymbol{B}_w$$

$$\nabla_{99} = \boldsymbol{B}_w^T \boldsymbol{P}_1\boldsymbol{B}_w + \boldsymbol{B}_w^T g_l^2 \boldsymbol{R}_1\boldsymbol{B}_w - \boldsymbol{B}_w^T g_{lh}^2 \boldsymbol{R}_2\boldsymbol{B}_w - \delta^2\boldsymbol{I} + \boldsymbol{D}_z^T\boldsymbol{D}_z$$

On applying Schur's complement on (A19) we get

$$
\yen_3 = \begin{bmatrix}
\yen_{11} & -2\boldsymbol{R}_1 & \mathbf{0} & \mathbf{0} & 6\boldsymbol{R}_1 & \mathbf{0} & \mathbf{0} & \mathcal{G}^T\boldsymbol{D} & \mathbf{0} & \boldsymbol{A}^T & g_l\hat{\boldsymbol{A}}^T & g_{lh}\hat{\boldsymbol{A}}^T & \boldsymbol{C}_z \\
* & \yen_{22} & \yen_{23} & \yen_{24} & 6\boldsymbol{R}_1 & 6\boldsymbol{R}_2 & 2\boldsymbol{S}_2+2\boldsymbol{S}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & \yen_{33} & \yen_{34} & \mathbf{0} & \yen_{36} & \yen_{37} & \mathbf{0} & \mathbf{0} & \mathbf{A}_d^T & g_l\boldsymbol{A}_d^T & g_{lh}\mathbf{A}_d^T & \mathbf{0} \\
* & * & * & \yen_{44} & \mathbf{0} & -2\boldsymbol{S}_3^T+2\boldsymbol{S}_4^T & 6\boldsymbol{R}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & -12\boldsymbol{R}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & -12\boldsymbol{R}_2 & -4\boldsymbol{S}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & -12\boldsymbol{R}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & -2\boldsymbol{D} & \mathbf{0} & -\hat{\boldsymbol{B}}^T & -g_l\hat{\mathbf{B}}^T & -g_{lh}\hat{\boldsymbol{B}}^T & \mathbf{0} \\
* & * & * & * & * & * & * & * & -\delta^2\boldsymbol{I} & \boldsymbol{B}_w^T & -g_l\mathbf{B}_w^T & -g_l\boldsymbol{B}_w^T & \mathbf{D}_z \\
* & * & * & * & * & * & * & * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & * & * & * & -2\boldsymbol{X}_2+\boldsymbol{X}_2\boldsymbol{R}_1\boldsymbol{X}_2 & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & * & * & * & * & -2\boldsymbol{X}_3+\boldsymbol{X}_3\boldsymbol{R}_2\boldsymbol{X}_3 & \mathbf{0} \\
* & * & * & * & * & * & * & * & * & * & * & * & -\boldsymbol{I}
\end{bmatrix} < \mathbf{0} \tag{A20}
$$

Pre and post multiplying (A20) by $diag\left(\boldsymbol{I}, \ \boldsymbol{I}, \ \boldsymbol{I}, \ \boldsymbol{I}, \ \boldsymbol{I}, \ \boldsymbol{I}, \ \boldsymbol{I}, \ \boldsymbol{D}^{-1} \ \boldsymbol{I}, \ \boldsymbol{I}, \ \boldsymbol{I}\right)$, yields

$$
\yen_4 = \begin{bmatrix}
\yen_{11} & -2\boldsymbol{R}_1 & \mathbf{0} & \mathbf{0} & 6\boldsymbol{R}_1 & \mathbf{0} & \mathbf{0} & \mathcal{G}^T & \mathbf{0} & \boldsymbol{A}^T & g_l\hat{\boldsymbol{A}}^T & g_{lh}\hat{\boldsymbol{A}}^T & \boldsymbol{C}_z \\
* & \yen_{22} & \yen_{23} & \yen_{24} & 6\boldsymbol{R}_1 & 6\boldsymbol{R}_2 & 2\boldsymbol{S}_2+2\boldsymbol{S}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & \yen_{33} & \yen_{34} & \mathbf{0} & \yen_{36} & \yen_{37} & \mathbf{0} & \mathbf{0} & \boldsymbol{A}_d^T & g_l\boldsymbol{A}_d^T & g_{lh}\boldsymbol{A}_d^T & \mathbf{0} \\
* & * & * & \yen_{44} & \mathbf{0} & -2\boldsymbol{S}_3^T+2\boldsymbol{S}_4^T & 6\boldsymbol{R}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & -12\boldsymbol{R}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & -12\boldsymbol{R}_2 & -4\mathrm{S}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & -12\boldsymbol{R}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & -2\mathcal{L} & \mathbf{0} & -\hat{\boldsymbol{B}}^T & -g_l\hat{\boldsymbol{B}}^T & -g_{lh}\hat{\boldsymbol{B}}^T & \mathbf{0} \\
* & * & * & * & * & * & * & * & -\delta^2\boldsymbol{I} & \boldsymbol{A}_w^T & -g_l\boldsymbol{B}_w^T & -g_l\boldsymbol{B}_w^T & \boldsymbol{D}_z \\
* & * & * & * & * & * & * & * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & * & * & * & -2\boldsymbol{X}_2+\boldsymbol{X}_2\boldsymbol{R}_1\boldsymbol{X}_2 & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & * & * & * & * & -2\boldsymbol{X}_3+\boldsymbol{X}_3\boldsymbol{R}_2\boldsymbol{X}_3 & \mathbf{0} \\
* & * & * & * & * & * & * & * & * & * & * & * & -\boldsymbol{I}
\end{bmatrix} < \mathbf{0} \tag{A21}
$$

The LMI conditions (21)–(22) together with $\boldsymbol{\beta}(\vartheta) < \mathbf{0}$ are sufficient conditions for asymptotic stability of the system (12). For zero initial condition, it can be shown that $\| \boldsymbol{z}(\vartheta) \|_2^2 < \wp^2 \| w(\vartheta) \|_2^2$. The satisfaction of LMI (22) for all initial states $\Gamma_\delta < 1$, follows that $\xi^T(\vartheta)\boldsymbol{P}\boldsymbol{\xi}(\vartheta) < 1 + \wp^2\boldsymbol{\sigma}^2$. So the trajectories of the system starting from $\Gamma_\delta < 1$ will remain within ellipsoid given by $\varepsilon(\boldsymbol{P}, 1 + \wp^2\boldsymbol{\sigma}^2)$.

The steps involved to solve the inequalities (21)–(23) are explained in Pseudo code 1 and Flow chart 1.

Pseudo code 1

Step 1 We will set the upper delay bound g_u satisfying $1 \le g_l \le g(\vartheta) \le g_u$ and check the LMIs (21)–(22) of Theorem 1. If they hold, go to Step 2.

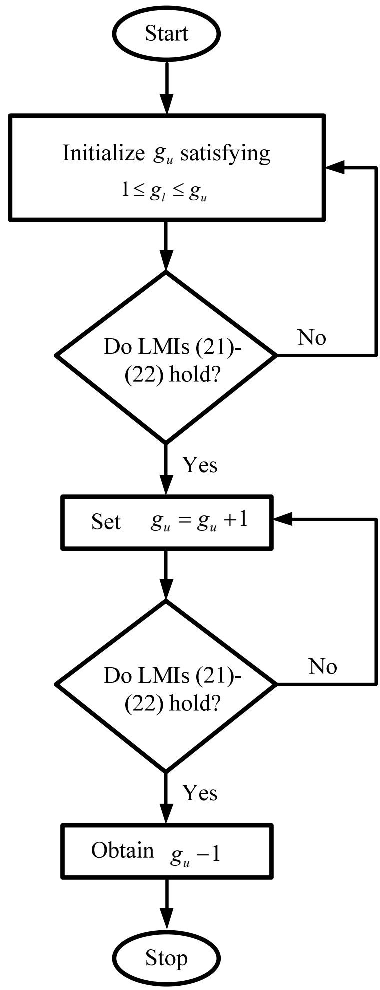
Step 2 Now we will set $g_u = g_u + 1$ and check LMIs (21)–(22) of Theorem 1.

Step 3 If there exist integer g_u such that (21)–(22) hold, then we will repeat Step 2.

If not, we will stop the process and obtain the maximum upper delay bound as $g_u - 1$.

Step by step visibility is shown in Figure A1.

Figure A1 Flowchart to solve inequalities (21)–(23)



Appendix B

Proof: The satisfaction of relation (28) implies that

$$\lambda_{\max}(\boldsymbol{P}_1) \le r_1\boldsymbol{I},\ \lambda_{\max}(\boldsymbol{P}_2) \le r_2\boldsymbol{I},\ \lambda_{\max}(\boldsymbol{P}_3) \le r_3\boldsymbol{I},\ \lambda_{\max}(\boldsymbol{Q}_1) \le r_4\boldsymbol{I},\ \lambda_{\max}(\boldsymbol{Q}_2) \le r_5\boldsymbol{I},$$
$$\lambda_{\max}(\boldsymbol{R}_1) \le r_6\boldsymbol{I},\ \lambda_{\max}(\boldsymbol{R}_2) \le r_7\boldsymbol{I}$$

From (25), one has $\delta = \Gamma_\delta / \sqrt{\boldsymbol{E2}}$. Thus, if we minimise (27), δ is being maximised. In other words, the optimisation problem given in Theorem 2 orients the solution of (21)–(23) in order to obtain domain of attraction as large as possible.