# The behaviour of stock returns under price limits, a truncated time series approach

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**Abstract:** The Tunisian Stock Exchange is subject to some authorities' regulations, constraints and limitations such as price limits. Hence, both of the following distortions will occur as a consequence to price limitations: unconditional equilibrium prices, 'shadowed prices', are unobservable by agents due to the fact that the asset valuation will be guided by the limited future prices assumption and the conditional equilibrium prices, 'shadowing prices', which are not observed by agents because they can exercise only a price that is within a limited range. Thus, the estimation of the 'fair' value of the asset will be complex and the existing trading strategies that focus only on observed prices will be inefficient. In this paper we will discuss the impact of price limitations on the stock returns behaviour and develop an inference methodology in order to extract and collect useful information about the shadowing prices based on truncated population. Finally, we develop a heuristic truncated normality test based on the JB test.

**Keywords:** price limits; truncated time series; truncated normal distribution; JB test; maximum likelihood estimator.

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## 1 Introduction

In order to limit the volatility of asset prices, many stock exchanges lay down daily price limits, i.e., prices of today cannot be out of a certain range that depends on yesterday closing price. The debate about the usefulness of price limits is a long run debate that dragged the financial community for decades. On one hand, price limit advocates claim that price limits decrease stock price volatility, counter overreaction, and do not interfere with trading activity. On the other hand, price limit critics advance that price limits cause several negative effects: higher volatility levels on subsequent days (volatility spill-over hypothesis), prevent prices from efficiently reaching their equilibrium level (delayed price discovery hypothesis), and interfere with trading due to limitations imposed by price limits (trading interference hypothesis).

Many academics showed interest to study several phenomena related to price limits. Kim and Rhee (1997) stated that price limits prevent stock prices from falling below or rising above predetermined boundaries. This allows them to control for volatility by establishing constraints and providing time for rational reassessment during panic times. They check whether price limits increase volatility levels on the upcoming days, if they do not allow prices to reach their equilibrium levels and whether they interfere with trading because of the limitations they impose. Their study is conducted on the Tokyo Stock Exchange price limits system. To do so, they use daily stock price data between 1989 and 1992 and compare stocks reaching price limits to those that almost attained their limits on volatility, price continuation, reversal and trading activities levels. Their findings suggest that for stocks that reached the price limits, volatility does not return to normal level as fast as those that did not attain the price limits. Besides, price continuation occurs more frequently when the price limits are reached. Finally, it is documented that price limits cause trading activities to increase. These results lead the authors to question the effectiveness of price limits in reducing volatility.

Chen et al. (2005) checked the effectiveness of the price limits system in the Shanghai and the Shenzhen Stock Exchanges, more specifically the A shares, by testing the volatility spillover hypothesis of Fama (1989), the delayed price discovery hypothesis of Fama (1989) and the trading interference hypothesis of Lauterbach and Ben-Zion (1993). Moreover, they test if stocks that hit the price limits have certain attributes such as being volatile, being actively traded, having small capitalisation, etc. to do so, they used Chinese A share individual stock prices and volume between December 1996 to December 2003. The findings show that the effect of the price limits is asymmetric for the upwards and downwards movement and different for the bullish and bearish sample periods. During the bullish periods, price limits effectively reduce the stock volatility for the downwards movement but not for the upwards movement, while it is the opposite case for the bearish sample period. In addition, price limits delay efficient price discovery for only the upward price movements. Also, actively traded stock hit their price limits more frequently, specifically the lower boundary when the market is bearish while stock with high book to market hit the upper limit more often. Finally, no evidence showing that price limits badly interfere with the trading process was documented.

Wang et al. (2014) studied the effects of price limits on the Chinese stock market during periods of global instability. The purpose is to examine the characteristics of stocks that hit the price limits more frequently during periods of turmoil. Specifically, these periods are the Asian financial crisis of 1997 and the global financial crisis. To do so, they tackled volatility spillovers, delayed price discovery and trading interference with such periods using daily A-share stock prices and trading volume from the Shanghai Stock Exchange and the Shenzhen Stock Exchange. The findings are quite interesting. First, the price limits mechanism increases the volatility significantly, especially in the downward movement, during the global financial crisis. Second, they found that the efficient price discovery is delayed by price limits. Moreover, price limits interfere in the trading activity one day after the stock hitting that limit, in the upwards movement. Finally, actively traded stocks in the property and industrial sectors which are highly and positively correlated with the market reach the price limits frequently while those with large book to market value or large size hardly reach the upper price limit during periods of turmoil.

Errais and Bahri (2016) claim that for investors trading across assets and countries with different price limits, volatility estimates (measured by standard deviation) are biased. This biasness is due to the fact that price limits are imposed. When this happens, equilibrium prices are unobservable and observed prices are truncated, which is the cause of the bias. Their methodology consists in using censored stochastic volatility (CSV) model and options' pricing. The findings show that the prices of stocks traded on markets with price limits exhibit an option's lookalike payoff. Consequently, when options are inexistent in a market, traders gain options' payoff as well as the regular linear payoff observed in stocks.

Mathematically, if we denote today's price by $\{p_t^0\}_{t\in T}$ we can write:

$$p_t^0 \in [d_t p_{t-1}^0, u_t p_{t-1}^0] \quad d_t < u_t \tag{1}$$

By simply deviding $p_t^0$ by $p_{t-1}^0$ from equation (1) we can notice that the returns are limited as well:

$$R_t^0 := \frac{p_t^0}{p_{t-1}^0} \in [d_t, u_t] \tag{2}$$

if we suppose that these limits are symmetric and time independant, there exists a limit $0 \leqslant l \leqslant 1$ such as:

$$d_t = 1 - l \quad u_t = 1 + l$$

With this configuration, one can think about censored time series to model $\{R_t^0\}_{t\in T'}$ as $R_t^0$ being a stochastic process with a finite support. We will see in this paper that the observed returns $\{R_t^0\}_{t\in T'}$ have similar but not identical features to truncated time series when we establish its link with the shadowing returns. Thus the price limits do not boil down to truncating or censoring a time series.

The purpose of this paper is to discuss in more details the impacts of price limits requirements on the behaviour of stock returns and on decision parameters estimation (mean, variance, covariance, etc.), then, propose an estimation methodology to reconstruct the unobserved returns density function. Hence, we will try to answer these questions: how does limited price rules influence both traders' decision making and valuation of assets? How can we model the observed returns to link them with equilibrium prices?

The paper is organised as follows. In Section 2, we discuss the impact of price limitations on the behaviour of stock returns. Section 3 sets up the theoretical background of censored and truncated time series. Section 4 presents an estimation methodology of shadowing returns density function based on maximum entropy methods and kernel estimation. Section 5 analyses the relevant normality tests for truncated times series. Section 6 runs empirical examples from the Tunisian Stock Exchange (TSE). Section 7 concludes.

## 2 Impact of price limits on the behaviour of stock returns

The impacts of price limits on the behaviour of stock returns can be devided into two major effects:

- Valuation effect: Imposing price limits on future returns influences the value of asset today. If we suppose that there is a stochastic discount factors series $\{m_t\}_{t\in T}$, the unconditional equilibrium prices $\{\pi_t\}_{t\in T}$ and the conditional equilibrium prices $\{p_t\}_{t\in T}$ can be defined as:

$$\pi_t := \mathbb{E}_t\left[\sum_{i\in\{x|t+x\in T\}} m_{t+i}X_{t+i}\right] \tag{3}$$

$$p_t := \mathbb{E}_t\left[\sum_{i\in\{x|t+x\in T\}} m_{t+i}\bar{X}_{t+i}\right] \tag{4}$$

  with

  1 $\{X_t\}_{t\in T}$: Cashflow series generated by the asset.

  2 $\{\bar{X}_t\}_{t\in T}$: Cashflow series generated by the asset when price limits are imposed.

- Value effect: From equation (4), we do not have any guarantee that $p_t$ is convergent and thus authorised under price limit imposition. If we suppose that $p_t$ at time $t$ is known, we can define the observed price $\{p_t^0\}_{t\in T}$ as follows

$$p_t^0 := \arg\min_{p\in\mathcal{A}_t}|p_t - p| \quad \mathcal{A}_t := [d_t p_{t-1}^0, u_t p_{t-1}^0] \tag{5}$$

Equation (5) means that the observed price is the most accepted price close to the shadowing price. This is true if we suppose that the investor will exercice the $p_t$. With minor manipulations we can reformulate (5) to get a more explicit relationship between $p_t^0$ and $p_t$:

$$p_t^0 = \begin{cases} d_t p_{t-1}^0 & \text{if } \frac{p_t}{p_{t-1}^0} < d_t \\ p_t & \text{if } d_t \leqslant \frac{p_t}{p_{t-1}^0} \leqslant u_t \\ u_t p_{t-1}^0 & \text{if } \frac{p_t}{p_{t-1}^0} > u_t \end{cases} \tag{6}$$

from equation (6), we can define the observed return as

$$R_t^0 := \frac{p_t^0}{p_{t-1}^0} = \begin{cases} d_t & \text{if } \frac{p_t}{p_{t-1}^0} < d_t \\ \frac{p_t}{p_{t-1}^0} & \text{if } d_t \leqslant \frac{p_t}{p_{t-1}^0} \leqslant u_t \\ u_t & \text{if } \frac{p_t}{p_{t-1}^0} > u_t \end{cases} \tag{7}$$

if we have $d_t \leqslant \frac{p_t}{p_{t-1}^0} \leqslant u_t$, we cannot guarantee that $R_t^0 = R_t$ as $p_{t-1}^0$ can be different from $p_{t-1}$.

To make the treatment simpler, we will supose that price limitiations are time independant and symmetric, thus, we will have $d_t = 1 - l$ and $u_t = 1 + l$. The following example will help illustrating the difference between a simple censored series and the process described by equation (7).

*Example 1:* Suppose the following configuration

$$p_{t-1} = (1 + l + d)p_{t-2} \text{ with } d \geqslant 0$$

$$p_t = (1 + r_t)p_{t-1} \text{ with } 0 \leqslant r_t < l$$

$$p_{t-2}^0 = p_{t-2}$$

we will have according to equation (6):

$$\frac{p_{t-1}}{p_{t-2}^0} = (1 + l + d) \Rightarrow R_{t-1}^0 = 1 + l$$

$$\frac{p_t}{p_{t-1}^0} = \frac{(1 + r_t)(1 + l + d)}{(1 + l)} = (1 + r_t) + \frac{d(1 + r_t)}{(1 + l)}$$

for $d \geqslant (l - r_t)\frac{1+l}{1+r_t}$, $R_t^0 = 1 + l$.

For $d < (l - r_t)\frac{1+l}{1+r_t}$, $p_t^0 = p_t$, and $R_t^0 = (1 + r_t) + \frac{d(1+r_t)}{(1+l)}$.

From the example above we can conclude that

$$\mathbb{P}[R_t^0 = 1 + l] \neq \mathbb{P}[R_t \geqslant 1 + l]$$

$$\mathbb{P}[a < R_t^0 < b] \neq \mathbb{P}[a < R_t < b]; \quad a, b \in [1 - l; 1 + l]$$

*Theorem 1:* If $R_{t-1}^0 \neq (1 \pm l)$ then

$$R_t^0 = \begin{cases} 1 - l & \text{if } R_t < 1 - l \\ R_t & \text{if } 1 - l \leqslant R_t \leqslant 1 + l \\ 1 + l & \text{if } R_t > 1 + l \end{cases}$$

*Proof:* If $R^0_{t-1} \neq (1 \pm l)$ then $R^0_{t-1} = \frac{p_{t-1}}{p^0_{t-2}} \Leftrightarrow p_{t-1} = R^0_{t-1}.p^0_{t-2} = p^0_{t-1}$. Thus, using equation (7) we have:

$$R^0_t = \begin{cases} 1-l & \text{if}\, R_t < 1-l \\ \frac{p_t}{p^0_{t-1}} = \frac{p_t}{p_{t-1}} = R_t & \text{if}\, 1-l \leqslant R_t \leqslant 1+l \\ 1+l & \text{if}\, R_t > 1+l \end{cases}$$

□

Theorem 1 allows us to make a link between the process of observed returns $\{R^0_t\}$ and censored and truncated variables.

## 3 Truncation

Truncation and censoring are sampling-related phenomena; truncation occurs when the sample is drown from a non-fully representative subpopulation, while censoring occurs when a group of values is replaced by a unique value. Greene (2003) used the example of "studies of income based on incomes above or below some poverty line" to explain the difference between the two phenomena: truncation occurs when out we totally neglect observations that are out of the subpopulation studied, i.e., observations with income higher or lower than the chosen poverty line while censoring occurs when we replace these observations by 'higher than'/'lower than' the chosen poverty line.

### *3.1 A brief history*

The early statistical treatment of truncation dates back to 1897, the date by which Galton (1897) published his "an examination into the registered speeds of American trotting horses, with remarks on their value as hereditary data." In this article, the author studied the speed of trotting horses from samples published by The American Trotting Association. However the association did not reported the speed of unsuccessful trollers, that is, the horses with a speed below a certain threshold were not reported. This is what we can call in modern econometrics language a left truncation. Galton (1897) considered that his sample is normal $\mathcal{N}(\mu, \sigma)$ and used the sample mode as an estimator for $\mu$, then he used sample inter-quartile to estimate the standard deviation. This procedure was juged satisfactory of the author's needs. Pearson (1902) criticised this procedure and proposed the use of fitting parabolas to the logarithms of the sample frequencies, however the results were slightly different from those found by Galton (1897). Later on, Pearson continued his investigation on the truncated normal samples, Pearson and Lee (1908) used the method of moments to derive estimators of the mean and standard deviation of left truncated normal distribution. Fisher (1931) used the maximum likelihood method to estimate the normal distribution parameters.

The treatment of censored samples needed to wait until 1937. Bliss and Stevens (1937) derived maximum liklihood equations to estimate normal parameters for singly and doubly truncated.

Further development of statistical methods to deal with truncated and cosored samples continued with many other scholars such as Cohen, Saw, Whitten, etc.

### 3.2 Truncation

We will focus in this subsection on parallel truncation, that is when we truncate the distribution from above and below:

$$X|d \leqslant X \leqslant u \quad d, u \in \mathbb{R}$$

**Figure 1** (–5, 5) – truncated Cauchy ($x_0$, 1) PDF (see online version for colours)

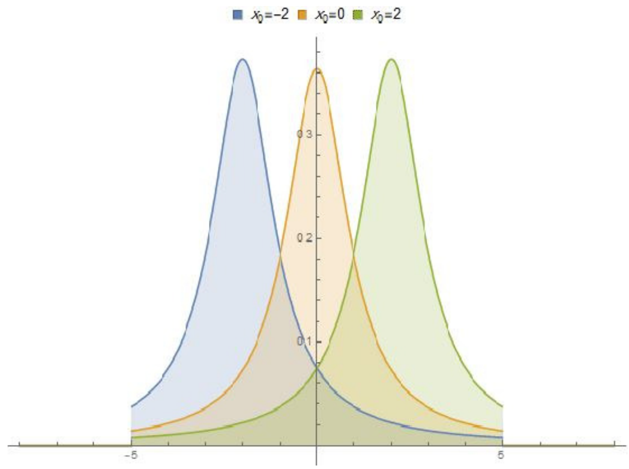
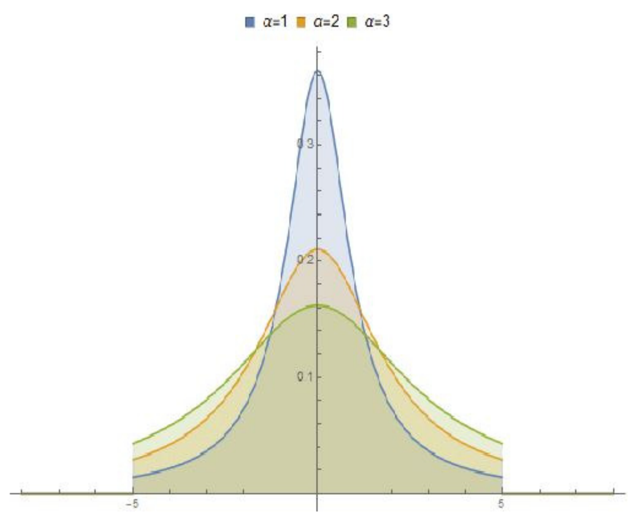



**Figure 2** (–5, 5) – truncated Cauchy (0, $\alpha$) PDF (see online version for colours)




#### 3.2.1 *Effects of truncation on distribution moments and characteristics*

The study of truncated moments may be of negligeable importance in infering the full population moments if we have no or limited information on the population distribution.

In fact, the mean of any continuous random variable on a finite support is finite even if this random variable is the truncation of a random variable that has an infinite or even undefined mean. To see this we can use for example the Cauchy distribution that has an undefined mean. There are two main reasons for exploring some properties of the truncated cauchy distribution: the first concerns the fact that the Cauchy distribution is amongst the few stable distribution that have a density function. The second concerns the presence of fat tails in returns distribution (Fama, 1963; Lux, 1998; Tsay, 2005; etc.) which is captured with the Cauchy distribution.

*Example 2:* Let $X \sim \text{Cauchy}(x_0, \alpha)$.

We have

$$f_{X|d\leqslant X\leqslant u}(x) := \frac{f_X(x)}{\mathbb{P}(d \leqslant X \leqslant u)} = c.\frac{1}{\left[1+\left(\frac{x-x_0}{\alpha}\right)^2\right]} \quad d \leqslant x \leqslant u$$

where $c := \left[\alpha\left(\arctan\left(\frac{u-x_0}{\alpha}\right) - \arctan\left(\frac{d-x_0}{\alpha}\right)\right)\right]^{-1}$.

Now we can calcualte for example the mean of the truncated variable:

$$\begin{aligned}\mathbb{E}(X|d \leqslant X \leqslant u) &= \int_d^u x.f_{X|d\leqslant X\leqslant u}(x)dx \\ &= \frac{c\alpha^2}{2}\left[\ln\left((x-x_0)^2+\alpha^2+x_0\alpha\arctan\left(\frac{x-x_0}{\alpha}\right)\right)\right]_d^u \in \mathbb{R}\end{aligned}$$

However, having a prior knowledge of the full population distribution may lead to a complete knowledge of its moments from the truncated ones. This is the case for example of normal random variables.

*Example 3:* Let $X \sim \mathcal{N}(\mu, \sigma)$.

We have:

$$f_{X|d\leqslant X\leqslant u}(x) := \frac{f_X(x)}{\mathbb{P}(d \leqslant X \leqslant u)} = \frac{1}{\sigma}\frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi(\bar{u})-\Phi(\bar{d})} \quad d \leqslant x \leqslant u \tag{8}$$

where $\bar{u} := \frac{u-\mu}{\sigma}, \bar{d} := \frac{d-\mu}{\sigma}$, $\phi(\bullet)$: the standard normal density function and $\Phi(\bullet)$ The standard normal cumulative density function.

Similarly to the untruncated normal distribution, we can relate the truncated moments with a recursive formula (Orjebin, 2014):

$$m_k = (k-1)\sigma^2 m_{k-2} + \mu m_{k-1} - \sigma\frac{b^{k-1}\phi(\upsilon) - a^{k-1}\phi(\delta)}{\Phi(\upsilon)-\Phi(\delta)} \tag{9}$$

with $m_k := \mathbb{E}(X|d \leqslant X \leqslant u)$ for $k \geqslant 1, m_0 = 1$ and $m_{-1} = 0$.

**Figure 3** (–5, 5) – truncated normal ($\mu$, 1) PDF (see online version for colours)

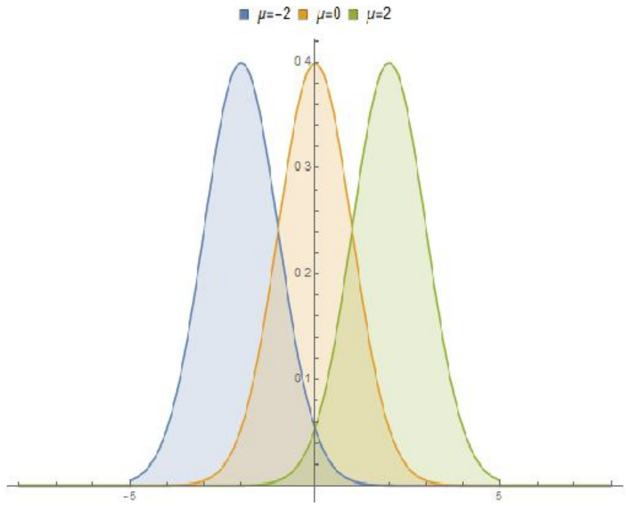
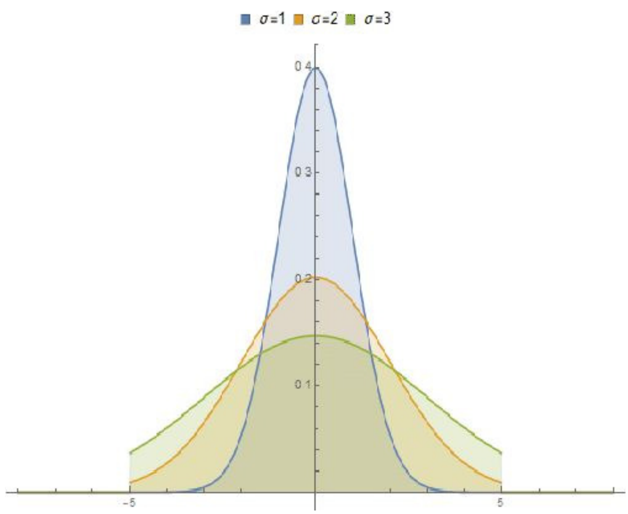



**Figure 4** (–5, 5) – truncated normal (0, $\sigma$) PDF (see online version for colours)




## 4 Estimation approach

In this section, we will develop parametric method to estimate a reconstruction of the parent distribution of returns. We will limit our focus on the special case of log normal returns.

From the results obtained in Section 2, we should proceed to some restrictions of the data in order to obtain a truncated or censored sample. However, in this article we will focus only on using truncation method and leave the case of censoring for further invistigations. In this section we will restrict the full sample $T$ to $\tau$ difined as follows:

$$\tau := \{t \in T | R_t \neq (1 \pm l) \,\text{and}\, R_{t-1} \neq (1 \pm l)\} \,\text{with card}\, (\tau) = \theta$$

We can easilly see that in the restricted sample $\tau$ the observed returns coincide with shadowing returns. It thus a truncated sample of the shadowing price.

The general hypothesis retained here is that $\{R_t^0\}_{t\in\tau}$ is an iid sample of the variable $R|u < R < d$. Statistically speaking, the object of this section is to estimate the distribution of $R$.

The log-normality of returns is an assumption that is commonly used in financial litterature (Tsay, 2005). It combines two important features:

- It has a bijective link with normal distribution: this makes the switch between the two distributions quite easy and direct.
- It can model the boundedness from below of gross returns.

The first step is to log-transform the observed return to obtain $\{r_t\}_{t\in\tau}$ which is now an iid sample of $r|\delta < r < \upsilon$ where $r := \log(R)$, $\upsilon := \log(u)$ and $\delta := \log(d)$.

As $R$ is considered here to follow a log normal distribution, $r$ will follow a normal distribution with mean $\mu$ and standard deviation $\sigma$. In order to estimate the population parameters, we will only need the first and second order moments estimators. However, we will need the third and forth moments estimators in order to test the hypothesis of log-normality.

From equation (8), we have

$$r \sim \mathcal{N}(\mu, \sigma)$$

$$f_{r|\delta<r<\upsilon}(x) = \frac{1}{\sigma} \frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi(\bar{\upsilon}) - \Phi(\bar{\delta})} \quad d \leqslant x \leqslant u$$

Cohen (1950, 1959) used two ways for estimating the moments of normal distribution from truncated samples using the method of maximum likelihood then the method of moments.

## 4.1 The MLE estimator

By using the results found in equation (8) we can deduce directly the liklihood function:

$$\mathcal{L}(\{r_t^0\}_{t\in\tau}; \mu; \sigma) := \prod_{t\in\tau} \frac{1}{\sigma} \frac{\phi\left(\frac{r_t-\mu}{\sigma}\right)}{\Phi(\bar{\upsilon}) - \Phi(\bar{\delta})}$$

by applying the log we can find:

$$\begin{aligned}
\log \mathcal{L} &= \sum_{t\in\tau} -(\log\sigma + \log(\Phi(\bar{\upsilon}) - \Phi(\bar{\delta}))) + \log\left(\phi\left(\frac{r_t - \mu}{\sigma}\right)\right) \\
&= \sum_{t\in\tau} -(\log\sigma + \log(\Phi(\bar{\upsilon}) - \Phi(\bar{\delta}))) - \frac{1}{2}\left(\frac{r_t - \mu}{\sigma}\right)^2 + \log\left(\frac{1}{\sqrt{2\pi}}\right)
\end{aligned}$$

As we can see, log-liklihood is more complex that its equivalent in the case of no truncation. This is due to the presence of the $\log(\Phi(\bar{\upsilon}) - \Phi(\bar{\delta}))$ term which, we should remind, incorporates $\mu$ and $\sigma$. By taking the derivative we obtain:

$$\nabla_{\mu,\sigma} \log \mathcal{L} = \begin{bmatrix} \frac{\theta}{\sigma} \frac{\varphi(\bar{\upsilon})-\varphi(\bar{\delta})}{(\Phi(\bar{\upsilon})-\Phi(\bar{\delta}))} + \frac{1}{\sigma^2} \sum_{t\in\tau}[r_t - \mu] \\ -\frac{\theta}{\sigma^2} \frac{\varphi(\bar{\upsilon})-\varphi(\bar{\delta})}{(\Phi(\bar{\upsilon})-\Phi(\bar{\delta}))} - \frac{\theta}{\sigma} + \frac{1}{\sigma^3} \sum_{t\in\tau}[r_t - \mu] \end{bmatrix}'$$

The maximum likelihood estimator of $\mu$ and $\sigma$, denoted $\hat{\mu}_{\mathrm{ML}}$ and $\hat{\sigma}_{\mathrm{ML}}$, can be obtained by equating the gradient to 0.

$$\begin{bmatrix} \frac{\theta}{\sigma} \frac{\varphi(\bar{\upsilon})-\varphi(\bar{\delta})}{(\Phi(\bar{\upsilon})-\Phi(\bar{\delta}))} + \frac{1}{\sigma^2} \sum_{t\in\tau}[r_t - \mu] \\ -\frac{\theta}{\sigma^2} \frac{\varphi(\bar{\upsilon})-\varphi(\bar{\delta})}{(\Phi(\bar{\upsilon})-\Phi(\bar{\delta}))} - \frac{\theta}{\sigma} + \frac{1}{\sigma^3} \sum_{t\in\tau}[r_t - \mu] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

However the nonlinearity of the system of equations obtained due to the presence of $\frac{\varphi(\bar{\upsilon})-\varphi(\bar{\delta})}{(\Phi(\bar{\upsilon})-\Phi(\bar{\delta}))}$ makes the solution quite complex and needs a numeric computational assistance.

Cohen and Whitten (1988) gave an approach to find a solution and provided tables for this task. However we will use a nonlinear optimisation algorithm directly in the empirical part.

### 4.2 *Impacts of using the (untruncated) normal distribution MLE estimators*

*Remark 1:* If $X \sim \mathcal{N}_d^u(\mu,\sigma)$, with $u = \mu + l$ and $d = \mu - l$ for $l \in \mathbb{R}_+$, we can prove using the truncated normal distribution expectation formula (Greene, 2003) and the symmetry of $\varphi$

$$\mathbb{E}(X) = \mu + \frac{\varphi\left(-\frac{l}{\sigma}\right) - \varphi\left(\frac{l}{\sigma}\right)}{\Phi\left(\frac{l}{\sigma}\right) - \Phi\left(-\frac{l}{\sigma}\right)} = \mu = \frac{u+d}{2}$$

This result can be easilly to be proven to work in the other direction.

Let $X \sim \mathcal{N}_d^u(\mu,\sigma)$ with $\mathbb{E}(X) = \mu$, we have then:

$$\frac{\varphi\left(\frac{d-\mu}{\sigma}\right) - \varphi\left(\frac{u-\mu}{\sigma}\right)}{\Phi\left(\frac{u-\mu}{\sigma}\right) - \Phi\left(\frac{d-\mu}{\sigma}\right)} = 0 \Leftrightarrow \begin{cases} d-\mu = u-\mu \Leftrightarrow d = u \\ u-\mu = \mu - d = l \end{cases}$$

If $X$ is non-degenerated, we have $X \sim \mathcal{N}_{d-l}^{\mu+l}(\mu,\sigma)$.

We have (by applying the strong law of large numbers):

$$\bar{r}_\tau := \frac{\sum_{t\in\tau}}{\theta} \overset{a.s}{\to} \mathbb{E}(r) = \mu + \frac{\varphi\left(\frac{\delta-\mu}{\sigma}\right) - \varphi\left(\frac{\upsilon-\mu}{\sigma}\right)}{\Phi\left(\frac{\upsilon-\mu}{\sigma}\right) - \Phi\left(\frac{\delta-\mu}{\sigma}\right)}$$

For a sufficiently large $\theta$, if we have, that the sample mean $\bar{\rho}_\tau = \delta - l = \upsilon + l$, for $l > 0$ (i.e., $\overline{\rho}_\tau = \frac{\upsilon+d}{2}$). We can assume (for a well behaving $\bar{r}_\tau$) $\bar{r}_\tau \overset{a.s}{\to} \frac{\upsilon+d}{2}$ which means (as both $\mathbb{E}(r)$ and $\frac{\upsilon+d}{2}$ are real scalars) that

$$\mathbb{E}(r) = \frac{\upsilon+d}{2} \Leftrightarrow \mathbb{E}(r) = \mu \Leftrightarrow \bar{r}_\tau \overset{a.s}{\to} \mu$$

Thus, by using the sample mean simply we can find a consistant estimator for $\mu$ without the complicated calculation needed for the MLE estimator. For the estimation of $\sigma$, we can use either the method of moments or the MLE.

### 4.2.1 Assymptotic bias of the sample mean

Let $\varepsilon \in ]-l, +\infty[$, we will focus here on truncated normal distribution of the $r \sim \mathcal{N}_{\mu-l-\varepsilon}^{\mu+l}(\mu, \sigma)$, we have now:

$$\mathbb{E}(r) = \mu + \frac{\varphi\left(-\frac{l+\varepsilon}{\sigma}\right) - \varphi\left(\frac{l}{\sigma}\right)}{\Phi\left(\frac{l}{\sigma}\right) - \Phi\left(-\frac{l+\varepsilon}{\sigma}\right)} = \mu + \frac{\varphi\left(\frac{l+\varepsilon}{\sigma}\right) - \varphi\left(\frac{l}{\sigma}\right)}{\Phi\left(\frac{l}{\sigma}\right) + \Phi\left(\frac{l+\varepsilon}{\sigma}\right) - 1}$$

It follows that:

$$\mathbb{E}(\bar{r}_\tau) - \mu = \mathbb{E}(r) - \mu = \frac{\varphi\left(\frac{l+\varepsilon}{\sigma}\right) - \varphi\left(\frac{l}{\sigma}\right)}{\Phi\left(\frac{l}{\sigma}\right) + \Phi\left(\frac{l+\varepsilon}{\sigma}\right) - 1} =: \beta(\lambda, \epsilon)$$

with $\lambda = \frac{l}{\sigma}$ and $\epsilon = \frac{\epsilon}{\sigma}$.

*Note 1:* We should note here that $\mu, l, \varepsilon$ are related by the following system:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \mu \\ l \\ \varepsilon \end{bmatrix} = \begin{bmatrix} v \\ \delta \end{bmatrix}$$

Which happends to be underspecified. We will carry here the analysis in terms of these unknowns to have an understanding of the factors that impact the bias and its seriousness.

The two equations above shows that the bias of the estimator depends on the relative (to $\sigma$) importance of truncation from above and from below and not the width of truncation in general. In other words, both the width of truncation and its position around the mean are what impacts the bias of the sample mean. For example, $\mathcal{N}_{-0.8}^{1.2}$ (0, 1), we have $\beta$ (1, 0.2) = 0.066, for $\mathcal{N}_{-0.6}^{1.4}$ (0, 1), we have $\beta$ (1.2, 0.4) = –0.1 and for $\mathcal{N}_{-1}^{1.2}$ (0, 1), we have $\beta$ (1.2, 0.2) = –0.05 $>$ while the first two distributions are restricted to the same intevrval width of 1.8. and that the first and the last share the same truncation position ($\epsilon$), all three distribution induced different biases.

The assymetry of $\beta(\lambda, \epsilon)$ over $\epsilon$ around 0, is thus explained by the fact that negative $\epsilon$ induces a wider truncation range, while a positive $\epsilon$ induces a tighter truncation range.

However, the most practicle remark about $\beta$ is that for large $\lambda$ (even for $\lambda$ = 10), $\beta(\lambda, \epsilon)$ becomes negligeable.

### 4.2.2 Efficiency of the sample mean

We have

$$\sigma_{\bar{r}} = \frac{1}{\sqrt{n}} \sigma_r = \frac{1}{\sqrt{n}}$$

If we have a sufficiently large sample, the sample mean will have an acceptable level for efficiency.

As it was mentioned several times here, the maximum liklihood estimator needs elaborated calculations. We will look in the next subsection to the quality of this estimator using various simulations.

*Note 2:* The statistical litterature produced other estimators for the truncated normal distribution parameters based on the method of moments. However many studies confirmed its higher bias and lower eficiency compared to the MLE method.

## 5 Testing for truncated normality

### 5.1 The use of the ordinary JB test

As discussed earlier in the previous section, the effects of truncation on the first and second moments of normal variable gets lower as the truncation range gots wider and in case of symmetric truncation around the mean. We can intuitivly extend this result to higher order moments and in particular skewness and curtosis. This being said, a truncated normal sample, on a large truncation range (compared to its standard deviation) can pass the JB test.

**Table 1** Acceptance rate of $H_0$ of JB test for a sample following $\mathcal{N}_{-l}^{l}(0,1)$ at 5%

| $l$ | *Size* | | |
|---|---|---|---|
| | *10* | *100* | *1,000* |
| 1 | 0.9983 | 0.8293 | 0.0000 |
| 2 | 0.9964 | 0.9991 | 0.0001 |
| 3 | 0.9935 | 0.9928 | 0.9547 |
| 4 | 0.9892 | 0.9610 | 0.9640 |
| 5 | 0.9902 | 0.9563 | 0.9519 |
| 6 | 0.9901 | 0.9550 | 0.9512 |
| 7 | 0.9886 | 0.9555 | 0.9497 |
| 8 | 0.9903 | 0.9604 | 0.9498 |
| 9 | 0.9905 | 0.9629 | 0.9494 |
| 10 | 0.9898 | 0.9599 | 0.9501 |

**Table 2** Acceptance rate of $H_0$ of JB test for a sample following $\mathcal{N}_{-10+e}^{10}(0,1)$ at 5%

| $e$ | *Size* | | |
|---|---|---|---|
| | *10* | *100* | *1,000* |
| 1 | 0.985 | 0.530 | 0.000 |
| 2 | 0.994 | 0.952 | 0.038 |
| 3 | 0.994 | 0.974 | 0.954 |
| 4 | 0.993 | 0.960 | 0.957 |
| 5 | 0.99 | 0.96 | 0.952 |
| 6 | 0.991 | 0.963 | 0.959 |
| 7 | 0.992 | 0.974 | 0.956 |
| 8 | 0.992 | 0.949 | 0.038 |
| 9 | 0.986 | 0.538 | 0.000 |
| 10 | 0.956 | 0.04 | 0.000 |

We have assessed the efficiency of the JB test through a series of simulations.

We can conclude that a symmetrically truncated normal sample have a great chance to pass the JB test of normality if the sample size is small or medium or if the truncation range exceeds six time the standard deviation. Unlike range, assymetry do not have a monotone impact on the chance of the truncated normal sample to pass the JB test. In the next subsections we will try to modify the ordinary JB test to enhance its efficiency to detect truncated normal samples.

## 5.2 *The modified JB test*

In this subsection, we will develop a Jarque and Bera (1980) normality test that takes into account the truncation effects on moments

### 5.2.1 *Linear transformation of a truncated normal variable*

$$X \sim \mathcal{N}_d^u(\mu, \sigma) \Leftrightarrow f_X(x) = \frac{1}{\sigma} \frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi(\bar{\upsilon}) - \Phi(\bar{\delta})}$$

Let $Z = aX + b, a \in R_+^*, b \in \mathbb{R}$

For $x \in [\delta, \upsilon]$, $z = ax + b \in [a\delta + b, a\upsilon + b]$ we have:

$$F_Z(z) = \mathbb{P}(Z \leqslant z) = \mathbb{P}\left(X \leqslant \frac{z-b}{a}\right) = F_X\left(\frac{z-b}{a}\right)$$

$$\begin{aligned} f_Z(z) &= \frac{d}{dz}\left(F_x\left(\frac{z-b}{a}\right)\right) = \frac{1}{a} f_X(z) \\ &= \frac{1}{a\sigma} \frac{\phi\left(\frac{z-(a\mu+b)}{a\sigma}\right)}{\Phi(\bar{\upsilon}) - \Phi(\bar{\delta})} \\ &\propto \phi\left(\frac{z-\mu'}{\sigma'}\right); \mu' = a\mu + b, \sigma' = a\sigma \end{aligned} \tag{10}$$

The result found in equation (10) yields that:

$$Z = aX + b \sim \mathcal{N}_{a\delta+b}^{a\upsilon+b}(\mu', \sigma')$$

by taking $a = \frac{1}{\sigma}$ and $b = -\frac{\mu}{\sigma}$ we get,

$$Z \sim \mathcal{N}_{\frac{\delta-\mu}{\sigma}}^{\frac{\upsilon-\mu}{\sigma}}(0, 1)$$

## 5.3 *Kurtosis and skewness of a truncated standard normal variable*

After understanding the effects of linear transformation of a truncated normal variable, we can focus now on deriving the kurtosis and skeweness of a truncated standard normal variable.

Let $Z \sim \mathcal{N}_d^u(0, 1)$, we have

$$
\begin{aligned}
\mathbb{E}(Z^3) &= \int_d^u x^3 f_Z(x)dx = \frac{1}{(\Phi(d)-\Phi(u))}\int_d^u x^3\varphi(x)dx \\
&= \frac{1}{(\Phi(d)-\Phi(u))}[(x^2+2)\varphi(x)]_d^u = \varsigma(d,u)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}(Z^4) &= \int_d^u x^4 f_Z(x)dx = \frac{1}{\Phi(u)-\Phi(d)}\int_d^u x^4\varphi(x)dx \\
&= \frac{1}{(\Phi(d)-\Phi(u))}\left[\frac{1}{8\sqrt{2}}\,\mathrm{erf}(x) - \left(\frac{1}{2}x^3+\frac{3}{4}x\right)\varphi(x)\right]_d^u \\
&= \kappa(d,u)
\end{aligned}
$$

*Remark 2:*

$$
\lim_{+\infty}\varsigma(-x,x) = \lim_{+\infty}\frac{1}{(\Phi(d)-\Phi(u))}[(x^2+2)\varphi(x)]_d^u = 0
$$

$$
\lim_{+\infty}\kappa(-x,x) = \lim_{+\infty}\frac{1}{(\Phi(d)-\Phi(u))}\left[\frac{1}{8\sqrt{2}}\,\mathrm{erf}(x) - \left(\frac{1}{2}x^3+\frac{3}{4}x\right)\varphi(x)\right]^u = 3
$$

### *5.4 Test hypothesis and statistic*

Let $\{X_i\}$ be an iid sample of $X \backsim \mathcal{N}_a^b$.

This test is used to check whether the data have a normalised skewness and kurtosis of a truncated normal variable:

$$
\begin{cases}
H_0 : \begin{pmatrix}\tilde{S}\\ \tilde{K}\end{pmatrix} = \begin{pmatrix}\varsigma(d,u)\\ \kappa(d,u)\end{pmatrix}\\
H_a : \begin{pmatrix}\tilde{S}\\ \tilde{K}\end{pmatrix} \neq \begin{pmatrix}\varsigma(d,u)\\ \kappa(d,u)\end{pmatrix}
\end{cases}
$$

with

$$
\begin{cases}
\tilde{S} = \sum\left(\frac{X_i-\mu}{\sigma}\right)^3\\
\tilde{K} = \sum\left(\frac{X_i-\mu}{\sigma}\right)^4\\
d = \sigma a - \frac{\mu}{\sigma}\\
u = \sigma b - \frac{\mu}{\sigma}
\end{cases}
$$

The test statistic can be written as follows:

$$
\widetilde{\mathrm{JB}} = \frac{N}{6}\left((\hat{S}-\hat{\varsigma}(\hat{d},\hat{u}))^2 + \left(\frac{\tilde{K}-\hat{\kappa}(\hat{d},\hat{u})}{3}\right)^2\right)
$$

with

$$
\begin{cases}
\hat{S} = \sum\left(\frac{X_i-\hat{\mu}}{\hat{\sigma}}\right)^3\\
\hat{K} = \sum\left(\frac{X_i-\hat{\mu}}{\hat{\sigma}}\right)^4\\
\hat{d} = \frac{a-\hat{\mu}}{\hat{\sigma}}\\
\hat{u} = \frac{b-\hat{\mu}}{\hat{\sigma}}\\
N = \text{Samplesize}
\end{cases}
$$

The smaller the $\widetilde{\mathrm{JB}}$ statistic, the more likely the sample to be drawn from a truncated normal population. We will use simulation techniques to derive the distribution of $\widetilde{\mathrm{JB}}$ for different population sizes and truncation limits.

## 6 Application to the TSE

In this part we have used the log-return data from the most liquid 17 different stocks of the TSE.

TSE was created in February 1969 as a state owned institution. Afterwards, in November 1995, it was established as a private company equally owned by the 23 brokers. The TSE is positioned in the heart of the Tunisian financial system – which contains the brokers that represent the trading monopoles, the financial market council as the legal authority supervising the financial system of the country and the guarantee funds which job is to protect investors from various risks. The TSE is made of the principal market which contains the listed big companies, the alternative market in which small and medium size firms are listed, the bond market, the mutual funds market and the hors cote market (designed for unlisted firms that desiring financing).

The normal trading hours start at 9 am and end at 2 pm, with a pre-opening session from 9 to 10 am and a pre-closing session from 2 to 2:05 pm. Trading has to be within a ±3% window of the previous closing price. Once the price of a stock hits this limit, its trading is stopped for 15 minutes. As a result, the ceiling and floor are increased by an extra 1.5% until the limit of 6.09% of yesterday's closing price is reached

### *6.1 A first look to the data*

#### *6.1.1 Log-returns distribution*

**Figure 5** TSE stock exchange log-return distribution (see online version for colours)

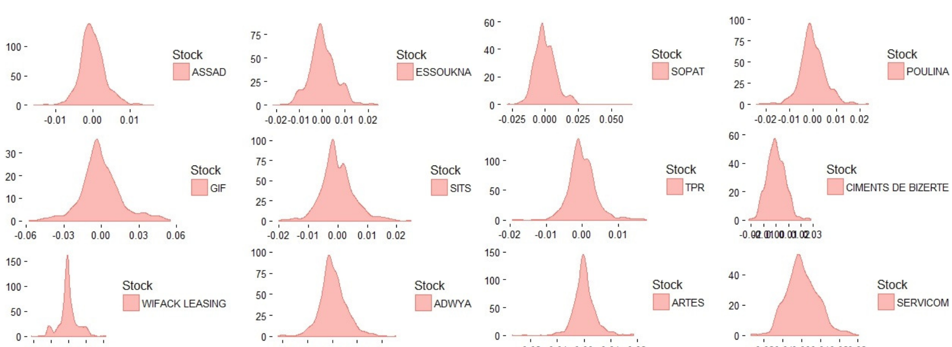
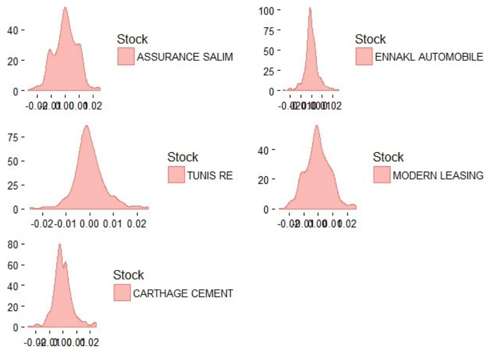


**Figure 5** TSE stock exchange log-return distribution (continued) (see online version for colours)




### *6.1.2 Log-returns main statistics*

We display in Table 3 the main statistics of the selected stocks along with the JB statistic.

**Table 3** Main statistics of daily log-returns

| | *Mean* | *SD* | *Skewness* | *Kurtosis* |
|---|---|---|---|---|
| ASSAD | 0 | 0.003 | 0.290 | 5.331 |
| GIF | 0 | 0.016 | 0.381 | 4.361 |
| WIFACK LEASING | 0 | 0.005 | 0.141 | 4.473 |
| ESSOUKNA | 0 | 0.006 | 0.334 | 4.102 |
| SITS | 0 | 0.006 | 0.393 | 4.592 |
| ADWYA | 0 | 0.006 | 0.185 | 4.989 |
| SOPAT | 0 | 0.008 | 0.922 | 8.122 |
| TPR | 0 | 0.004 | 0.413 | 6.730 |
| ARTES | 0 | 0.004 | –0.075 | 8.377 |
| POULINA | 0 | 0.006 | 0.142 | 5.339 |
| CIMENTS DE BIZERTE | 0 | 0.007 | 0.317 | 3.420 |
| SERVICOM | 0 | 0.009 | 0.283 | 3.496 |
| ASSURANCE SALIM | 0 | 0.008 | –0.078 | 3.036 |
| TUNIS RE | 0 | 0.007 | 0.467 | 5.274 |
| CARTHAGE CEMENT | 0 | 0.007 | 0.561 | 5.155 |
| ENNAKL AUTOMOBILE | 0 | 0.006 | 0.196 | 5.955 |
| MODERN LEASING | 0 | 0.008 | 0.187 | 3.216 |

### *6.2 Use of truncation method*

We remark here that $\hat{\sigma}$ for all the 17 daily log-returns are significantly lower than theire empiric standard deviation which is not a plausible result. As the truncation range is wide, it is better to use the empiric standard deviation to calculate the $\widetilde{\text{JB}}$ test statistic. The result of the calculation are in Table 4.

**Table 4** MLE estimation results for daily log-returns

| | $\hat{\mu}$ | $\hat{\sigma}$ |
|---|---|---|
| ASSAD | 0 | 0.000 |
| GIF | 0 | 0.000 |
| WIFACK LEASING | 0 | 0.001 |
| ESSOUKNA | 0 | 0.001 |
| SITS | 0 | 0.000 |
| ADWYA | 0 | 0.000 |
| SOPAT | 0 | 0.000 |
| TPR | 0 | 0.000 |
| ARTES | 0 | 0.000 |
| POULINA | 0 | 0.000 |
| CIMENTS DE BIZERTE | 0 | 0.000 |
| SERVICOM | 0 | 0.001 |
| ASSURANCE SALIM | 0 | 0.001 |
| TUNIS RE | 0 | 0.001 |
| CARTHAGE CEMENT | 0 | 0.000 |
| ENNAKL AUTOMOBILE | 0 | 0.001 |
| MODERN LEASING | 0 | 0.001 |

**Table 5** $\widetilde{\mathrm{JB}}$ test statistics for daily log-returns

| | *JB* | $\widetilde{\mathrm{JB}}$ |
|---|---|---|
| ASSAD | 198.029 | 57.589 |
| GIF | 90.109 | 39.919 |
| WIFACK LEASING | 69.154 | 18.841 |
| ESSOUKNA | 54.181 | 24.235 |
| SITS | 106.459 | 41.856 |
| ADWYA | 141.890 | 38.613 |
| SOPAT | 1,017.540 | 10.103 |
| TPR | 490.258 | 138.615 |
| ARTES | 972.870 | 241.908 |
| POULINA | 186.480 | 48.122 |
| CIMENTS DE BIZERTE | 17.487 | 13.385 |
| SERVICOM | 19.164 | 12.791 |
| ASSURANCE SALIM | 0.667 | 0.635 |
| TUNIS RE | 216.365 | 76.904 |
| CARTHAGE CEMENT | 203.430 | 82.757 |
| ENNAKL AUTOMOBILE | 292.572 | 76.189 |
| MODERN LEASING | 5.971 | 4.803 |

The $\widetilde{\mathrm{JB}}$ was smaller for all the 17 daily log-returns. This mean that the modified JB test statistic succeded in capturing the impact of truncation on the skewness and kurtosis. However no other daily log-return succeded the $\widetilde{\mathrm{JB}}$ test than the ones that already pqssed the JB test.

## 7 Conclusions and recommendations

### *7.1 Conclusions*

- In the case of log-normal daily returns (and other thin tailed distributions) taking into account the effect of truncation is of significant interest if the truncation range is wide enough and symmetric around the mean.
- The majority of daily log return of stocks studied in this dissertation fail to pass the JB and $\widetilde{\text{JB}}$ tests. It is not probable that they follow a log-normal distribution.
- The observation of daily returns that hit the price limit in the TSE are negligeable, in fact, for 5 years and 17 stocks, the price limit was hitted 3 times.

The impact of price limits (if it exists) cannot related to the exercice of prices (shadowing prices) that exeed the limits. However this impact maybe more important in the estimation of shadowing price in itself. The price limits sets a psychological barrier for stock exchange agents for what they believe is a 'fair price'.

### *7.2 Recommendations*

It is highly important for the development of the stock exchange activities and the implication of its stakeholder to understand the psychological effects of price limits on the behaviour of stock returns. In fact, if this psychological effect is making the stock exchange agents underevluating the stock prices (and thus the attractiveness of the investment in stocks), authorities can boost the activity of stock exchange by reducing or eleminating the price limits. In the other hand, if price limits are giving stock exchange agents more secured in the market, authorities should preserve the price limits or tighten them. Thus, it is a question of finding the optimal price limits that will give enough security to the stock exchange agents without pushing them to underevaluate the stock prices

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## Appendix

### *Assessmment of MLE estimator of truncated normal distribution*

### *A1 Simulation script (R)*

```
install.packages ("tmvtnorm")
library ("tmvtnorm")
#set the distribution parameters
lower = d
upper = u
mu = m
sigma = s
#inicite the simulation parameters and variables
repetion = 100,000
mu_mle = sigma_mle = matrix (nrow = 1, ncol = 3)

for (i in seq (1, repetion)) {
```

```
 for (size in seq (1:3)) {
  #Create a sample of (lower, upper) – truncated normal (mu, sigma)
  X <- rtmvnorm (n = 10^size, mu, sigma, lower, upper)
  method <- 'BFGS'
  #Estimate the parameters of the created sample using MLE
  mle.fit1 <- mle.tmvnorm (X, lower = lower, upper = upper)
  #Extract the estimated variables for each sample size
  mu_mle[size] <- mle.fit1@coef[1]
  sigma_mle[size] <- mle.fit1@coef[2]
 }
#Append the estimated variables to the corresponding csv file for further treatments
write.table (mu_mle, file = "mu_mle.csv", append = TRUE, col.names = FALSE, sep = ",")
write.table (sigma_mle, file = "sigma_mle.csv", append = TRUE, col.names = FALSE, sep = ",")

}
```

*A2 Simulation results*

The simulation object is to assess the MLE estimator (distribution, bias, etc.). The algorithm detailed below will make 100,000 iterations of estimating the normal paramaters of 10, 100 and 1,000 (–1, 1) truncated standard normal variables. As there is no computational formula to calculate the MLE estimators directly, we are in fact assessing the practicality of the use of the MLE method and not its theoritical qualities.

The baseline is the case of $\mathcal{N}_{-1}^{1}(0,1)$, the we change each time the truncation range in order to study its impact on the MLE performance. Finally we used the log TUNINDEX return data to assess the performance of the estimator in a real world case.

*Notation 1:* We will note here $\hat{\mu}_d^u(\mu,\sigma)$ and $\hat{\sigma}_d^u(\mu,\sigma)$ the MLE estimators for a sample of $\mathcal{N}_d^u(\mu,\sigma)$.

*A.2.1 Simulation output*

*The case of $\mathcal{N}_{-1}^{1}$ (0, 1)*

**Table 6** Quantiles of $\hat{\mu}_{-1}^{1}(0,1)$

| | *Min* | *25%* | *50%* | *75%* | *Max* |
|---|---|---|---|---|---|
| 10 | –57.880 | –0.352 | –0.001 | 0.346 | 48.380 |
| 100 | –25.605 | –0.134 | –0.001 | 0.132 | 17.533 |
| 1,000 | –0.830 | –0.040 | 0.000 | 0.040 | 2.096 |

**Table 7** Moments and JB test results of $\hat{\mu}_{-1}^{1}(0,1)$

| | *Bias* | *SD* | *Skewness* | *Kurtosis* | *JB* | *p-value* |
|---|---|---|---|---|---|---|
| 10 | –0.005 | 3.950 | –0.190 | 20.535 | 883,790 | 0.000 |
| 100 | –0.011 | 1.078 | –0.800 | 52.823 | 9,175,300 | 0.000 |
| 1,000 | 0.000 | 0.063 | 0.335 | 19.449 | 757,970 | 0.000 |

**Figure 6** $\hat{\mu}^1_{-1}(0,1)$ simulated distribution (see online version for colours)

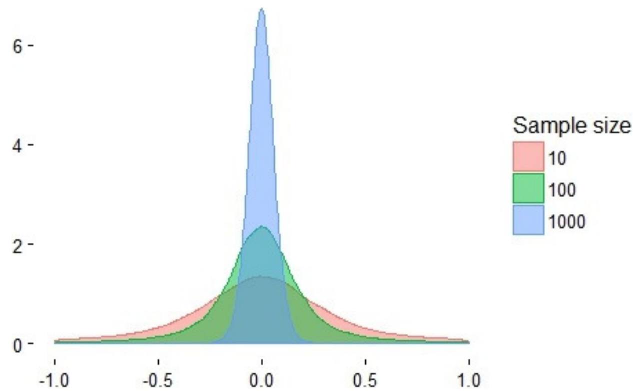


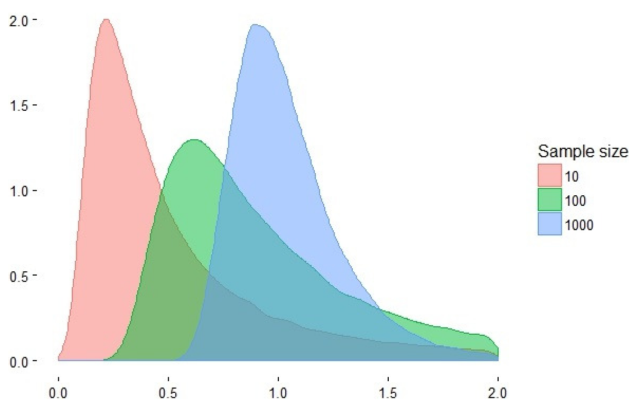

**Table 8** Quantiles of $\hat{\sigma}^1_{-1}(0,1)$

| | *Min* | *25%* | *50%* | *75%* | *Max* |
|---|---|---|---|---|---|
| 10 | 0.018 | 0.294 | 0.663 | 8.719 | 81.822 |
| 100 | 0.203 | 0.652 | 0.960 | 1.753 | 67.687 |
| 1,000 | 0.494 | 0.866 | 0.995 | 1.170 | 23.325 |

**Table 9** Moments and JB test results of $\hat{\sigma}^1_{-1}(0,1)$

| | *Mean* | *SD* | *Skewness* | *Kurtosis* | *JB* | *p-value* |
|---|---|---|---|---|---|---|
| 10 | 4.074 | 5.807 | 1.533 | 5.503 | 65,474 | 0.000 |
| 100 | 2.907 | 5.760 | 3.394 | 14.318 | 728,120 | 0.000 |
| 1,000 | 1.053 | 0.303 | 9.038 | 453.029 | $<$1,000,000 | 0.000 |

**Figure 7** $\hat{\sigma}^1_{-1}(0,1)$ simulated distribution (see online version for colours)



*The case of $\mathcal{N}^2_{-2}$ (0, 1)*

**Table 10** Quantiles of $\hat{\mu}^2_{-2}(0,1)$

| *Sample* | *Min* | *25%* | *50%* | *75%* | *Max* |
|---|---|---|---|---|---|
| 10 | –95.959 | –0.255 | 0 | 0.254 | 27.262 |
| 100 | –0.672 | –0.077 | 0 | 0.078 | 0.894 |
| 1,000 | –0.159 | –0.0.024 | 0 | 0.024 | 0.15 |

**Table 11** Moments and JB test results of $\hat{\mu}^2_{-2}(0,1)$

| *Sample* | *Bias* | *SD* | *Skewness* | *Kurtosis* | *JB* | *p-value* |
|---|---|---|---|---|---|---|
| 10 | –0.002 | 1.111 | –2.093 | 166.571 | >100,000 | 0.000 |
| 100 | 0 | 0.117 | –0.008 | 3.264 | 291 | 0.000 |
| 1,000 | 0 | 0.036 | –0.007 | 3.006 | 1 | 0.605 |

**Figure 8** $\hat{\mu}^2_{-2}(0,1)$ simulated distribution (see online version for colours)

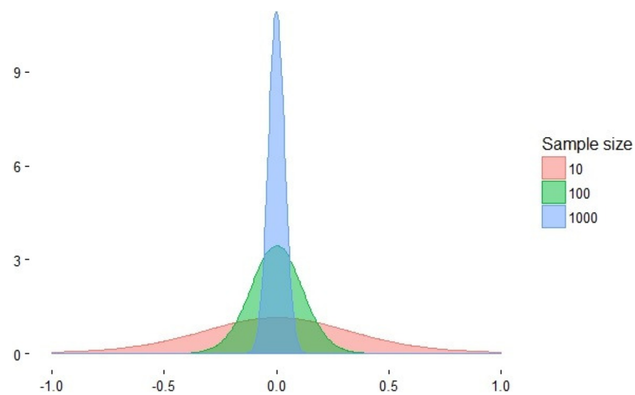


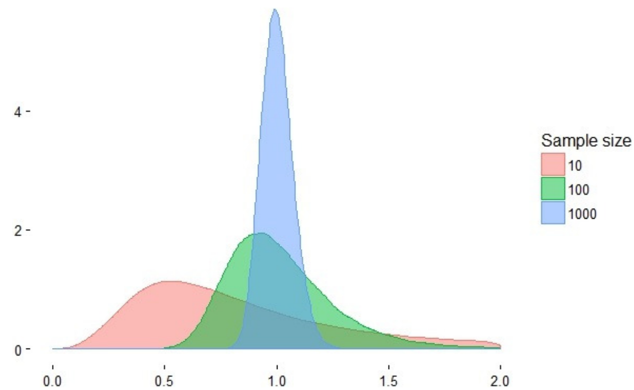

**Table 12** Quantiles of $\hat{\sigma}^2_{-2}(0,1)$

| *Sample* | *Min* | *25%* | *50%* | *75%* | *Max* |
|---|---|---|---|---|---|
| 10 | 0.043 | 0.532 | 0.822 | 1.391 | 113.035 |
| 100 | 0.372 | 0.851 | 0.982 | 1.147 | 18.573 |
| 1000 | 0.762 | 0.953 | 0.999 | 1.047 | 1.373 |

**Table 13** Moments and JB test results of $\hat{\sigma}^2_{-2}(0,1)$

| *Sample* | *Mean* | *SD* | *Skewness* | *Kurtosis* | *JB* | *p-value* |
|---|---|---|---|---|---|---|
| 10 | 1.830 | 3.685 | 4.820 | 34.125 | >100,000 | 0.000 |
| 100 | 1.024 | 0.277 | 10.762 | 543.802 | >100,000 | 0.000 |
| 1,000 | 1.002 | 0.071 | 0.0314 | 3.188 | 1791 | 0.000 |

**Figure 9** $\hat{\sigma}^2_{-2}(0,1)$ simulated distribution (see online version for colours)




*The case of $\mathcal{N}^1_{-3}$ (0, 1)*

**Figure 10** $\hat{\mu}^1_{-3}(0,1)$ simulated distribution (see online version for colours)

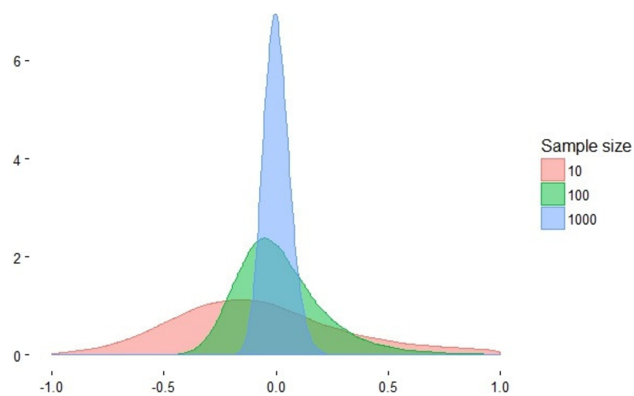



**Table 14** Quantiles of $\hat{\mu}^1_{-3}(0,1)$

| *Sample* | *Min* | *25%* | *50%* | *75%* | *Max* |
|---|---|---|---|---|---|
| 10 | –4.863 | –0.306 | –0.041 | 0.364 | 90.163 |
| 100 | –0.565 | –0.115 | –0.007 | 0.127 | 9.905 |
| 1,000 | –0.213 | –0.038 | 0 | 0.04 | 0.399 |

**Table 15** Moments and JB test results of $\hat{\mu}^1_{-3}(0,1)$

| *Sample* | *Bias* | *SD* | *Skewness* | *Kurtosis* | *JB* | *p-value* |
|---|---|---|---|---|---|---|
| 10 | –0.787 | 3.211 | 5.229 | 41.582 | 883,790 | 0.000 |
| 100 | –0.026 | 0.216 | 2.627 | 52.823 | 9,175,300 | 0.000 |
| 1,000 | 0.002 | 0.058 | 0.306 | 19.449 | 757,970 | 0.000 |

**Figure 11** $\hat{\sigma}^{1}_{-3}(0,1)$ simulated distribution (see online version for colours)

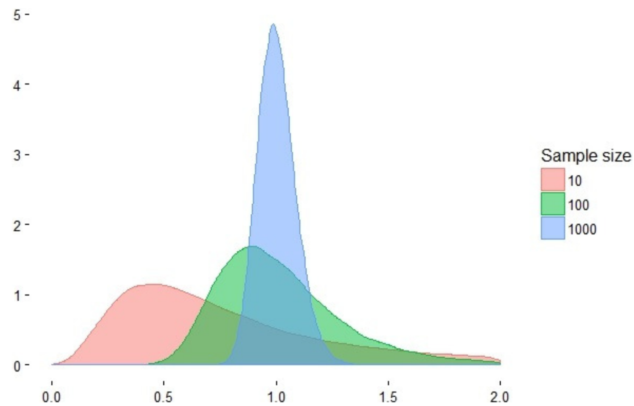



**Table 16** Quantiles of $\hat{\sigma}^{1}_{-3}(0,1)$

| *Sample* | *Min* | *25%* | *50%* | *75%* | *Max* |
|---|---|---|---|---|---|
| 10 | 0.027 | 0.465 | 0.765 | 1.412 | 99.659 |
| 100 | 0.337 | 0.822 | 0.976 | 1.174 | 20.995 |
| 1,000 | 0.711 | 0.944 | 0.998 | 1.056 | 1.506 |

**Table 17** Moments and JB test results of $\hat{\sigma}^{1}_{-3}(0,1)$

| *Sample* | *Mean* | *SD* | *Skewness* | *Kurtosis* | *JB* | *p-value* |
|---|---|---|---|---|---|---|
| 10 | 2.108 | 4.505 | 4.317 | 26.142 | >100,000 | 0 |
| 100 | 1.034 | 0.322 | 3.247 | 160.286 | >100,000 | 0 |
| 1,000 | 0.998 | 0.084 | 0.384 | 3.296 | 2,828.7 | 0 |

*The case of* $\mathcal{N}_2^4$ *(0, 1)*

**Figure 12** $\hat{\mu}_2^4(0,1)$ simulated distribution (see online version for colours)

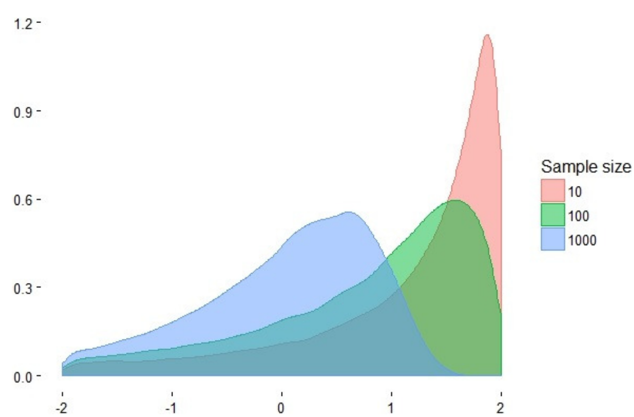


**Table 18** Quantiles of $\hat{\mu}_2^4(0,1)$

| *Sample* | *Min* | *25%* | *50%* | *75%* | *Max* |
|---|---|---|---|---|---|
| 10 | –61.414 | –4.590 | 1.766 | 2.188 | 2.790 |
| 100 | –44.155 | –4.865 | 0.364 | 1.364 | 2.29 |
| 1,000 | –29.59 | –0.857 | 0.042 | 0.589 | 1.746 |

**Table 19** Moments and JB test results of $\hat{\mu}_2^4(0,1)$

| *Sample* | *Bias* | *SD* | *Skewness* | *Kurtosis* | *JB* | *p-value* |
|---|---|---|---|---|---|---|
| 10 | –1.722 | 6.205 | –1.605 | 4.942 | 58,734 | 0 |
| 100 | –2.696 | 5.915 | –1.351 | 3.683 | 32,411 | 0 |
| 1,000 | –0.545 | 2.131 | –3.885 | 24.741 | >100,000 | 0 |

**Figure 13** $\hat{\sigma}_2^4(0,1)$ simulated distribution (see online version for colours)

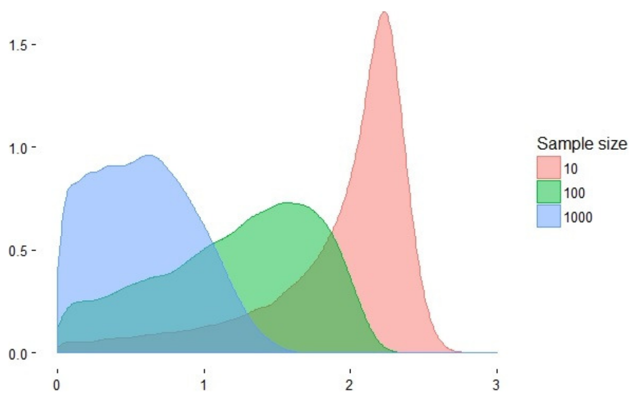



**Table 20** Quantiles of $\hat{\sigma}_2^4(0,1)$

| *Sample* | *Min* | *25%* | *50%* | *75%* | *Max* |
|---|---|---|---|---|---|
| 10 | 0.003 | 0.117 | 0.312 | 2.302 | 66.904 |
| 100 | 0.089 | 0.473 | 0.862 | 2.841 | 31.986 |
| 1,000 | 0.317 | 0.774 | 0.985 | 1.327 | 14.984 |

**Table 21** Moments and JB test results of $\hat{\sigma}_2^4(0,1)$

| *Sample* | *Mean* | *SD* | *Skewness* | *Kurtosis* | *JB* | *p-value* |
|---|---|---|---|---|---|---|
| 10 | 1.79 | 3.121 | 3.209 | 20.858 | >100,000 | 0 |
| 100 | 2.023 | 2.308 | 1.613 | 5.951 | 79,669 | 0 |
| 1,000 | 1.206 | 0.809 | 3.862 | 24.847 | >100,000 | 0 |

*The case of a real stock $\mathcal{N}_{-0.063}^{0.059}$ (0, 0.003) (TUNINDEX)*

**Figure 14** $\hat{\mu}_{-0.063}^{0.059}(0, 0.003)$ simulated distribution (see online version for colours)

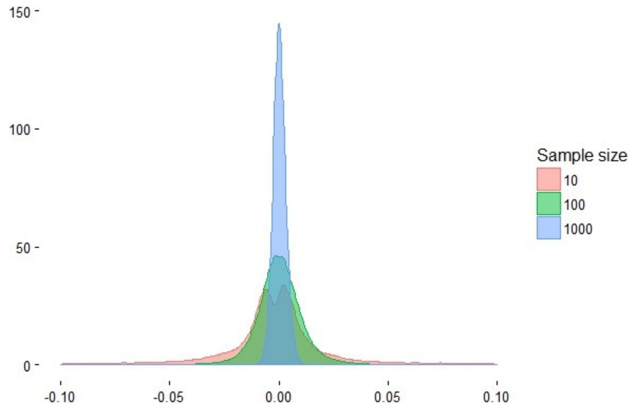



**Table 22** Quantiles of $\hat{\mu}_{-0.063}^{0.059}(0, 0.003)$

| *Sample* | *Min* | *25%* | *50%* | *75%* | *Max* |
|---|---|---|---|---|---|
| 10 | –7.864 | –2.133 | 0 | 2.774 | 8.933 |
| 100 | –4.916 | –0.002 | 0 | 0.007 | 4.851 |
| 1,000 | –22.04 | –0.002 | 0 | 0.002 | 9.996 |

**Table 23** Moments and JB test results of $\hat{\mu}_{-0.063}^{0.059}(0, 0.003)$

| *Sample* | *Bias* | *SD* | *Skewness* | *Kurtosis* | *JB* | *p-value* |
|---|---|---|---|---|---|---|
| 10 | 0.229 | 3.662 | 0.151 | 2.501 | 1,417 | 0 |
| 100 | 0.039 | 0.564 | 2.037 | 39.011 | $>$100,000 | 0 |
| 1,000 | –0.021 | 0.637 | –27.252 | 797.227 | $>$100,000 | 0 |

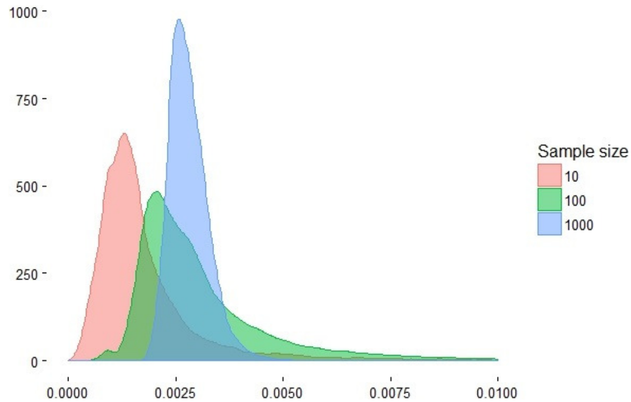
**Table 24** Quantiles of $\hat{\sigma}_{-0.063}^{0.059}(0, 0.003)$

| *Sample* | *Min* | *25%* | *50%* | *75%* | *Max* |
|---|---|---|---|---|---|
| 10 | 0 | 0.002 | 0.533 | 0.697 | 87.888 |
| 100 | 0 | 0.002 | 0.003 | 0.004 | 472.144 |
| 1,000 | 0 | 0.003 | 0.003 | 0.003 | 19,891.762 |

**Table 25** Moments and JB test results of $\hat{\sigma}_{-0.063}^{0.059}(0, 0.003)$

| *Sample* | *Mean* | *SD* | *Skewness* | *Kurtosis* | *JB* | *p-value* |
|---|---|---|---|---|---|---|
| 10 | 0.476 | 0.627 | 43.298 | 4721.817 | $>$100,000 | 0 |
| 100 | 0.144 | 3.888 | 67.262 | 5717.042 | $>$100,000 | 0 |
| 1,000 | 1.894 | 25.047 | 25.036 | 1032.692 | $>$100,000 | 0 |

**Figure 15** $\hat{\sigma}^{0.059}_{-0.063}(0, 0.003)$ simulated distribution (see online version for colours)




## *A3 Interpretation of results*

The simulated distribution of $\hat{\mu}$ for different sample sizes shows a bell-shaped which indicates a possible normality of the estimator. We will check for normality using JB test of the 100,000 simulations. As an MLE estimator, $\hat{\mu}$ is guarenteed to be assymptotically normal, the object of performing the normality tests is to understand the speed of convergence of the estimator.

The quantiles of $\hat{\mu}$ shows two remarks about MLE estimator, first the symmetry and this fact is most likely to be related to the fact that the truncation is symmetric aorund the mean (as we will check later), the second which is more important (and most likeley more general) is the presence of significant outliers especially for small samples (10 and 100). in a similar sumulation for non-truncated normal distribution, the outliers are less significant (for the case of 10 samples, the range for mu was [–1.42, 1.42]). The problem of large outliers can be caused by three possible reasons, either the quality of the semi-random generator used by the library (in this case, using the MLE estimation algorithm will be likely to produce good estimation for $\mu$ for real truncated normal data). The second possible reason is that the estimation algorithm, in case of existence of multiple solutions, may choose non-reasonable admissible solution (in this case, one needs to play with the algorithm parameters until finding a 'reasonable' estimation).

According to Table 7, the bias of of the MLE estimator is reasonable even for a sample of small size and the standard deviation of the estimator is decreasing. The skewness for all the sample sizes is similar to a normal distribution (near 0), however, the kurtosis observed is significantly higher then normal distribution, this made the simulated distribution of $\hat{\mu}$, fail to pass the JB test. As it is easy to detect if $\hat{\mu}$ is an outlier, it will be easy to eliminate them. For example, if we remove all $\hat{\mu} \notin [-2\sigma_{\hat{\mu}}, 2\sigma_{\hat{\mu}}]$, we will find that the kurtosis droped down significantly to attain 2.479 or the 1,000 sample size, with loosing less than 5% of observations. Even with removing the exuberant observations, the remaining sample fail the JB test, however the JB statistic dropped very significantly.

For the estimation of standard deviation, the first thing we remark is the importance of the bias for small and even for medium samples. However the median of $\hat{\sigma}$ stood close to the right value of $\sigma$ for all sample sizes. Thus, it is almost equiprobable to over or under estimate for all sample sizes.

The shape of $\hat{\sigma}$ distribution shows a slow convergence to symmetry and normality. However, the evolution of skewness shows the inverse. The skewness inflation for the 1,000 sample can explained by the influence of exuberant values on the right, the presence of 23.325, for example (the maximum value for the 1,000 sample size) added, on its own $\frac{1}{100{,}000}\left(\frac{23.325-1.035}{0.303}\right)^3 = 3.98$ to the skewness, while the value of 0.494 only reduced the skewness by $-\frac{1}{100{,}000}\left(\frac{0.494-1.035}{0.303}\right)^3 \backsimeq 0.000$. It is better though to compare the symmetry by more robust statistics such as quartiles for example, for our case, quartiles distribution confirms the graphical intuition as the median is approaching the centre of the first and third quartiles as we increase the sample size.

By doubling the truncation range, and conserving its symmetry around the mean as in the case of $\mathcal{N}^2_{-2}(0, 1)$, the bias of $\hat{\mu}$ was reduced, but more importantly, the range of the estimator got significantly limited for 100 and 1,000 sample sizes (from [–25.605, 17.533] to [–0.672, 0.894] and from [–0.83, 2.096] respectively) (the range for the small sample size of ten got wider for the same reasons explained above, however the distance between the second and third quartile got reduced). The standard deviation of $\hat{\mu}$ also got significantly reduced. Furthermore, the shape of the distribution got closer to normal with the estimator of the sample of size 1,000 succeeding in passing the JB normality test. bu doubling the truncation range, the quality of $\mu$ estimator got significantly ameliorated specially for medium sized samples.

For $\hat{\sigma}$, the same remarks about $\hat{\mu}$ apply: the bias and standard deviation got significantly reduced, the range got tighter and the distribution shape converged more rapidly to the normal shape.

When we conserved the range length of 4 and distorted the symmetry as in the case of $\mathcal{N}^1_{-3}(0, 1)$, the observed bias increased compared to the previous cases especially for the small sample case, the standard deviation however got slightly reduced compared to the baseline case. The distribution shape shows a slight assymetry especially for the small and medium size samples. This is confirmed by the difference between the mean and median of the estimator. If compared with the case of $\mathcal{N}^2_{-2}(0, 1)$, we can see the performance of $\hat{\mu}$ has dropped.

The impact of a slight assymetry was lower on $\hat{\sigma}$ (compared to $\bar{\mu}$), in fact, the quartiles of $\hat{\sigma}$ for the medium and large samples are close to the quartiles found for the symmetric case ($\mathcal{N}^2_{-2}(0, 1)$), idem for the bias and standard deviation of the estimator.

In case the real $\mu$ is out of the truncation range (we can consider it a case of severe assymetry), as in the case of $\mathcal{N}^4_2(0, 1)$, we remark that the performance of the MLE estimators dropped down significantly in terms of bias, standard deviavtion, interquartile distance and distribution shape (very slow convergence to normality). However, similarly to the previous case, $\hat{\mu}$ is were more sensitive to assymetry than $\hat{\sigma}$.

In the last case [we use the sample mean and standard deviation for TUNINDEX log-return for five years (2010–2014)] to assess the performance of the MLE estimator for a case with parameters in the same scale of magnitude as the cases studied in the empirical part.