Construction of a fuzzy probability space with Gumbel function, Gaussian function, derivative of Gaussian function and Weibull function

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Abstract: Random variable basically addresses a probability space and fuzzy random variable (FRV) will address the fuzzy probability space. Concepts of FRV valued functions such as exponential function, logarithmic function and power function have been already researched. But applications in the field of failure analysis of structures very often are dealt with extreme value probability distribution functions such as Gumbel, Frechet (Type-I and Type II) and Weibull function. However such functions are well defined in presence of a large number of data. But the failure analysis of structures with insufficient information in the similar footing needs corresponding FRV valued functions. Therefore the basic thrust of this paper is to propose a concept of formulating FRV valued such type of extreme value distribution functions viz., Gumbel, Frechet and the Weibull. In this paper we have proposed the FRV valued Gumbel and Weibull function. In addition to this we have also proposed the similar concept for FRV valued Gaussian and its derivative function. Fundamental properties of these functions in the fuzzy probability space are also discussed in this paper.

Keywords: failure analysis; fuzzy probability space; Gumbel function; Gaussian function; Weibull function.

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1 Introduction

Fuzzy random variables (FRVs) are random variables whose values are represented as fuzzy numbers. In particular a special type of fuzzy set is addressed as fuzzy number. The concept of FRV has been investigated by many researchers (Kwakernaak, 1978, 1979; Puri and Ralescu, 1986). Considering an example of expert's opinion poll about the velocity of wind on any structures such as 'very high', 'high' and 'medium', randomness occurs because it is not known which response may be expected from any given attributes. Once the response is available, there exists an uncertainty about the precise meaning of the response. The latter uncertainty (subjective or knowledge uncertainty) is characterised by fuzziness, in the sense that each of the linguistic responses 'very high', 'high' and 'medium' is represented by a fuzzy set. In this context, utility of the FRV valued function is essential and appropriate to model the situation. From the utility point FRV valued exponential function, logarithmic function and power function has been studied by many researchers (Zhang and Wang, 1993; Zhong and Wang, 1966; Zhong et al., 1994). The outcome of the research on the FRV valued exponential, logarithmic and power function has been further extended on the Gumbel, Weibull, Gaussian and the derivative of Gaussian function to propose that these functions can also behave as FRV valued function and hence a fuzzy probability space can easily be constructed with these FRVs. Fuzzy set theory being a tool to quantify the uncertainty analysis with variables having insufficient information FRV valued such type of functions will help the reliability study of the structures in presence of the epistemic uncertainty environment. If randomness is the characteristic of a classical random variable then fuzziness can be interpreted as the characteristics of a fuzzy variable. Joint ventures of randomness and fuzziness can model the uncertainty in a better way (Moller Bernd and Beer, 2004). Triangular and trapezoidal membership function of a fuzzy number only has been explored by many researchers to address the uncertainty modelling for many science and engineering problems. However, FRV plays a major role for failure analysis of structures in any nuclear installation in presence of uncertainty during any rare events such as rainfall, earthquake, and flood. In order to deal with uncertainty modelling in the fuzzy probability space, we have proposed the similar concept of FRV for Gumbel function, Weibull function, Gaussian function and the derivative of Gaussian function. In order to proof their uniqueness and identity as FRV we have discussed their fundamental mathematical properties.

Let us first highlight the fundamental concepts of fuzzy numbers, fuzzy random variable and fuzzy probability space.

Let $\mathbb{F}_0(\mathbb{R})$ denote the set of all bounded closed fuzzy numbers, i.e., if $\tilde{r} \in \mathbb{F}_0(\mathbb{R})$, then satisfies

1 $\{x \mid x \in \mathbb{R}, \; \tilde{r}(x) = 1\} \neq \phi$

2 for every $\alpha \in (0, 1]$, we can write a finite closed interval on $\mathbb{R}$ as $\tilde{r} = \{x \mid x \in \mathbb{R}, \tilde{r}(x) \geq \alpha\} = [r_\alpha^-, r_\alpha^+]$, where $\mathbb{R} = (-\infty,\infty)$.

Let us have the following definitions:

Definition 1.1: FRV on the probability space (Ω, A, P) is defined as a mapping $\tilde{a} : \Omega \to \mathbb{F}_0(\mathbb{R})$, if for every $b \in B$, the equation, $\{w \mid w \in \Omega, \; \tilde{a}_\alpha(w) \cup B \neq \phi\} \in A$, is satisfied for every $\alpha \in [0, 1]$, where B represents the Borel subsets on $\mathbb{R}$ and $\tilde{a}_\alpha(w) = \{x \mid x \in \mathbb{R}, \tilde{a}_\alpha(w)(x) \geq \alpha\}$ is the level set of $\tilde{a}_\alpha(w)$. From now onward we shall denote the set of FRV in the probability space (Ω, A, P) as $\mathbb{FR}(\Omega)$.

Theorem 1.1: If $\tilde{a}_\alpha^-(w)$ and $\tilde{a}_\alpha^+(w)$ are two random variables such that $\tilde{a}_\alpha^-(w) \leq \tilde{a}_\alpha^+(w)$ everywhere on Ω then there exists a closed interval random number $\tilde{a}_\alpha(w) = [\tilde{a}_\alpha^-(w), \tilde{a}_\alpha^+(w)]$ on the probability space (Ω, A, P).

Definition 1.2: If $\tilde{a}, \tilde{b} \in \mathbb{FR}(\Omega)$, we can define

1 $\tilde{a} \leq \tilde{b}$ if and only if $\tilde{a}_\alpha(w) \leq \tilde{b}_\alpha(w)$ for every $w \in \Omega$ and for any $\alpha \in [0, 1]$.

2 $\tilde{a} < \tilde{b}$ if and only if $\tilde{a}_\alpha(w) \leq \tilde{b}_\alpha(w)$ for every $w \in \Omega$ and for any $\alpha \in [0, 1]$.

Definition 1.3: Let $o \in \{+, -, *, /\}$ be an algebraic operation on $\mathbb{F}_0(\mathbb{R})$. Then the corresponding operation o on $\mathbb{FR}(\Omega)$ can be defined as $(\tilde{a}o\tilde{b})(w) \triangleq \tilde{a}(w)o\tilde{b}(w)$ for any $w \in \Omega$, where $\tilde{a}, \tilde{b} \in \mathbb{FR}(\Omega)$.

Standard arithmetic operation of closed interval numbers can be found elsewhere (Dubois and Prade, 1980; Kaufman and Gupta, 1985; Moore, 1979). Therefore, it is obvious that $\mathbb{R} \subset \mathbb{F}_0(\mathbb{R}) \subset \mathbb{FR}(\Omega)$, and $\mathbb{R} \subset \mathbb{R}(\omega) \subset \mathbb{FR}(\Omega)$. where $\mathbb{R}(\Omega)$ is the set of all random variables on (Ω, A, P).

2 FRV valued Gumbel function

Safety while designing any structures under extreme wind load is basically carried out using extreme value distribution function (IAEA, 2003). Gumbel distribution function being one of the members of the extreme value distribution plays a major role in that context. General tendency is to fit the wind data using this distribution. However lack of wind data either due to the failure of the instrument (sensors) or the high cost to fix up the sensors at the target locations results in an uncertainty in the data. As a result Gumbel distribution function through such data does not fit appropriately. Presence of fuzzy randomness within this data is the main cause of this bad fit. Remedy of this is only possible if we can satisfy that Gumbel distribution function can be re-shaped into a FRV valued Gumbel distribution function. In this context effort has been made in this paper to do the same. On the basis of FRV valued power function (Nather, 2001; Nguyen and Wu, 2006) we can write the following polynomial function:

$$f(\tilde{x}) = \tilde{a}_0 + \tilde{a}_1\tilde{x} + \tilde{a}_2\tilde{x}_2 + \cdots + \tilde{a}_n\tilde{x}^n, \tag{1}$$

where $\tilde{a}_j \in \mathbb{FR}(\Omega)$, $\forall j$ represents a given FRV and $\tilde{x} \in \mathbb{FR}(\Omega)$ represents an independent variable. Therefore, $f(\tilde{x})$ is signified as a FRV valued polynomial function. As safety always targets the maximum value we can define now the FRV valued Gumbel distribution function for maxima using the $f(\tilde{x})$.

Definition 2.1: Gumbel distribution function for maxima (Castillo, 1988) is defined as

$$G(x) = \exp\left[-\exp\left(-\frac{x-\lambda}{\delta}\right)\right] \tag{2}$$

where λ and δ are constants known as the location and scale parameters respectively. Let us denote, $y = \frac{x-\lambda}{\delta}$. We can rewrite the Gumbel distribution function as

$$G(x) = \exp\left[-\exp(-y)\right]. \tag{3}$$

Let us write, $\Psi(\tilde{y}) = \log(G(\tilde{y})) = -\exp(-\tilde{y})$, where $\tilde{y} = \frac{\tilde{x}-\lambda}{\delta}$. $\Psi(\tilde{y})$ is signified here as FRV valued Gumbel distribution function. According to fuzzy random variable valued exponential function (Wang and Zhong, 1994; Wang and Zhang, 1993) we can construct the following theorem.

Theorem 2.1: This theorem tells about the following properties of the fuzzy random variable valued Gumbel distribution function $\Psi(\tilde{y})$.

1 $\Psi(\tilde{y})(\omega) \in \mathbb{F}_0(\mathbb{R})$ for every $\omega \in \Omega$

2 for any $\alpha \in (0, 1]$,

$$\begin{aligned}(\Psi(\tilde{y}))_\alpha &= \log\left(G\left(\tilde{y}_\alpha\right)\right) = -\exp\left(-\tilde{y}_\alpha\right),\\ &= -\exp\left(-\left[y_\alpha^-, y_\alpha^+\right]\right),\\ &= -\left[\exp\left(-y_\alpha^-\right), \exp\left(-y_\alpha^+\right)\right].\end{aligned}$$

3 $y_\alpha^- = \frac{1}{\delta}[x_\alpha^- - \lambda]$ and $y_\alpha^+ = \frac{1}{\delta}[x_\alpha^+ - \lambda]$.

3 FRV valued Weibull function

In the standard probability space, Weibull function in the form of a Weibull probability density function plays a major role to estimate the failure probability and fatigue of any structure (Modarres et al., 2010). Very often limitation or non-availability of sufficient experimental data restricts the application of Weibull probability distribution. Hence, it is essential to tackle those situations using the concept of FRV or fuzzy probability space. In this context, it is essential to investigate the fundamental properties of fuzzy random variable valued Weibull function. Therefore, this section presents the fuzzy random variable valued Weibull function and its fundamental properties. On the basis of the fuzzy random variable defined in Section 2 we can define the FRV valued Weibull function.

Definition 3.1: Let $f(x) = (\frac{\beta}{\gamma})(\frac{x}{\gamma})^{\beta-1}\exp[(-\frac{x}{\gamma})^{\beta}]$ be the Weibull function, where β represents the shape parameter and γ is the scale parameter and all the parameters are positive. Therefore, $\psi(x) = \log[f(x)] = \log(\frac{\beta}{\gamma}) + (\beta-1)\log(\frac{x}{\gamma}) - (\frac{x}{\gamma})^{\beta}$. Using the form of FRV valued polynomial function if we denote, $\psi(\tilde{x}) = \log(\frac{\beta}{\gamma}) + (\beta-1)\log(\frac{\tilde{x}}{\gamma}) - (\frac{\tilde{x}}{\gamma})^{\beta}$ then $\psi(\tilde{x})$ is called the FRV valued Weibull function.

It can be seen from the definition that FRV valued Weibull function consists of one constant (parameter of Weibull function), logarithmic function and one power function. Hence, according to the theorem of FRV valued logarithmic function and power function (Zhong and Wang, 1993), we can write the following theorems.

Theorem 3.1: $\psi(\tilde{x})$ has the following property:

1 $\psi(\tilde{x})(\omega) \in \mathbb{F}_0(\mathbb{R})$ for every $\omega \in \Omega$

2 for any $\alpha \in (0, 1]$,

$$\begin{aligned}\left(\psi(\tilde{x})\right)_{\alpha} &= \log\left(\frac{\beta}{\gamma}\right) + (\beta-1)\log\left(\frac{\tilde{x}_{\alpha}}{\gamma}\right) - \left(\frac{\tilde{x}_{\alpha}}{\gamma}\right)^{\beta},\\ &= \left[\psi\left(x_{\alpha}^{-}\right), \psi\left(x_{\alpha}^{+}\right)\right].\end{aligned}$$

3 for any $\alpha \in (0, 1]$,

$$\begin{aligned}&\left[\log\left(\frac{\beta}{\gamma}\right) + (\beta-1)\log\left(\frac{\left[x_{\alpha}^{-}, x_{\alpha}^{+}\right]}{\gamma}\right) - \left(\frac{\left[x_{\alpha}^{-}, x_{\alpha}^{+}\right]}{\gamma}\right)^{\beta}\right]\\ &= \log\left(\frac{\beta}{\gamma}\right) + (\beta-1)\left[\log\left(\frac{x_{\alpha}^{-}}{\gamma}\right) - \log\left(\frac{x_{\alpha}^{+}}{\gamma}\right)\right] - \left[\left(\frac{x_{\alpha}^{-}}{\gamma}\right)^{\beta} - \left(\frac{x_{\alpha}^{+}}{\gamma}\right)^{\beta}\right],\\ &= \log\left(\frac{\beta}{\gamma}\right) + \left[(\beta-1)\log\left(\frac{x_{\alpha}^{-}}{\gamma}\right) - \left(\frac{x_{\alpha}^{-}}{\gamma}\right)^{\beta}, (\beta-1)\log\left(\frac{x_{\alpha}^{+}}{\gamma}\right), \left(\frac{x_{\alpha}^{+}}{\gamma}\right)^{\beta}\right].\end{aligned}$$

$$\left[\psi\left(x_{\alpha}^{-}\right), \psi\left(x_{\alpha}^{+}\right)\right]$$
$$= \log\left(\frac{\beta}{\gamma}\right) + \left[(\beta-1)\log\left(\frac{x_{\alpha}^{-}}{\gamma}\right) - \left(\frac{x_{\alpha}^{-}}{\gamma}\right)^{\beta}, (\beta-1)\log\left(\frac{x_{\alpha}^{+}}{\gamma}\right) - \left(\frac{x_{\alpha}^{+}}{\gamma}\right)^{\beta}\right].$$

4 $\psi(\tilde{x}) = \log\left(\frac{\beta}{\gamma}\right) + (\beta-1)\log\left(\frac{\tilde{x}}{\gamma}\right) - \left(\frac{\tilde{x}}{\gamma}\right)^{\beta}$, for every $\tilde{x} \in \mathbb{FR}(\Omega)$.

Now, following the similarity behaviour of the FRV valued logarithmic function and power function (Zhong and Wang, 1993), Weibull function can be also included as a member of the fuzzy probability space. Consequently, Weibull function also holds the property of a fuzzy membership function.

4 FRV valued Gaussian function

In this section, the concept of FRV-valued Gaussian function is introduced and we shall present its fundamental properties. Prior to bring some theorem, we start with the following definition.

Definition 4.1: The mapping $f : \tilde{D} \to \mathbb{FR}(\Omega)(\tilde{x} \mapsto f(\tilde{x}))$ is called a fuzzy random variable-valued function defined on $\tilde{D}$, the domain of definition for a function $f(\tilde{x})$ and represents any non-empty subset of $\mathbb{FR}(\Omega)$ and $\tilde{x}$ is called the independent variable. It is also understood that $f(\tilde{x})$ is also a FRV in $\mathbb{FR}(\Omega)$ for every $\tilde{x} \in \tilde{D}$.

The definition and fundamental properties of FRV-valued exponential function is described in detail elsewhere in Zhong and Wang (1993). Similar strategy is followed while presenting the FRV valued Gaussian function. If $\tilde{a} \in \mathbb{FR}(\Omega)$ be a given FRV then $f_1(\tilde{x}) = \tilde{a}\tilde{x}$ and $f_2(\tilde{x}) = \tilde{a} + \tilde{x}$ for any $\tilde{x} \in \tilde{D}$ are two fuzzy random variable-valued functions defined on $\tilde{D}$. It is understood that we may structure maps from $\tilde{D}$ to $\mathbb{FR}(\Omega)$ by all sorts of methods. We shall use the extension principle to define the FRV valued Gaussian function.

4.1 *FRV valued Gaussian function* $\exp\left(-\frac{\tilde{x}^2}{2}\right)$

Lemma 4.1.1: Suppose that *f*(*x*) $(x \in \mathbb{R})$ is an ordinary real valued continuous function, $\tilde{a} \in \mathbb{F}_0(\mathbb{R})$. Defining $f(\tilde{a}) \triangleq \bigcup_{\alpha \in (0,1]} \alpha f(\tilde{a}_\alpha)$, we can write

1 for any $\alpha \in (0, 1]$, $(f(\tilde{a}))_\alpha = f(\tilde{a}_\alpha) = [\wedge f(x), \vee f(x)], x \in \tilde{a}_\alpha$

2 $f(\tilde{a}) \in \mathbb{F}_0(\mathbb{R})$.

Proof of the Lemma 4.1.1 can be found elsewhere (Zhong and Wang, 1993). It can be pointed here that $f(\tilde{a})$ is also a bounded closed fuzzy number in $\mathbb{F}_0(\mathbb{R})$. Following the definition of an exponential function, we can write the following expression exp(x) = e^x, $x \in \mathbb{R} = (-\infty, \infty)$. Hence, for $\tilde{x} \in \mathbb{FR}(\Omega)$, $\exp(\tilde{x}) \triangleq \bigcup_{\alpha \in (0,1)} \alpha \exp(\tilde{x}_\alpha)$, where, $\exp(\tilde{x}_\alpha) = \{\exp(\tilde{x}) \mid x \in \tilde{x}_\alpha = [x_\alpha^-, x_\alpha^+]\}$, $\alpha \in (0, 1]$ and $\exp(\tilde{x})$ is called as the FRV valued exponential function with respect to base e. One of the important properties of $\exp(\tilde{x})$ at its α-level representation is $(\exp(\tilde{x})_\alpha) = \exp(\tilde{x}_\alpha) = [\exp(\tilde{x}_\alpha^-), \exp(\tilde{x}_\alpha^+)]$, for any $\alpha \in (0, 1]$ (Nather, 2001). For every $\tilde{x} \in \mathbb{FR}(\Omega)$ using Lemma 4.1.1 and this property we can say that x_α^- and x_α^+ are two random variables in the fuzzy probability space for any $\alpha \in (0, 1]$.

Definition 4.2: Let $\exp = (-\frac{\tilde{x}^2}{2}) = e^{-x^2/2} = \psi(\tilde{x})$, (where $x \in \mathbb{R} = (-\infty, \infty)$) be an ordinary Gaussian function. For any $\tilde{x} \in \mathbb{FR}(\Omega)$, we can define $\exp(-\frac{\tilde{x}.\tilde{x}}{2}) \triangleq \alpha \exp(-\frac{\tilde{x}_\alpha \tilde{x}_a}{2})$, where $\exp(-\frac{\tilde{x}_\alpha \tilde{x}_\alpha}{2}) = \{\exp(\frac{x^2}{2}) \mid x \in \tilde{x}_\alpha = [x_\alpha^-, x_\alpha^+]\}, \alpha \in (0,1]$. The expression $\exp(-\frac{\tilde{x}.\tilde{x}}{2})$ is called the FRV valued Gaussian function.

Let $\tilde{y} = \frac{\tilde{x}.\tilde{x}}{2}$, therefore $\exp(-\frac{\tilde{x}.\tilde{x}}{2}) = \exp(-\tilde{y})$. So, we can write

$$\begin{aligned}\tilde{y}_\alpha &= \frac{1}{2}(\tilde{x}.\tilde{x})_\alpha,\\ &= \frac{1}{2}[\tilde{x}_\alpha].[\tilde{x}_\alpha],\\ &= \frac{1}{2}[x_\alpha^-, x_\alpha^+].[x_\alpha^-, x_\alpha^+],\\ &= \frac{1}{2}[x_\alpha^-.x_\alpha^+, x_\alpha^-.x_\alpha^+].\end{aligned}$$

Theorem 4.1: For $\tilde{y} \in \mathbb{FR}(\Omega)$,

1 $(\exp(-\tilde{y}))(\omega) \in \mathbb{F}_0(\mathbb{R})$ for every $\omega \in \Omega$

2 $(\exp(-\tilde{y}))_\alpha = \exp(-\tilde{y}_\alpha) = [\exp(-y_\alpha^-), \exp(-y_\alpha^+)]$ for any $\alpha \in (0, 1]$.

3 $\exp(-\tilde{y}) \in \mathbb{FR}(\Omega)$.

Proof of this theorem can be found elsewhere (Zhong and Wang, 1993) and hence the same is not further repeated here.

Theorem 4.2: Let $\tilde{x}_1, \tilde{x}_2 \in \mathbb{FR}(\Omega)$. Then,

$$\exp\left(-\left[\frac{\tilde{x}_1.\tilde{x}_1}{2} + \frac{\tilde{x}_2.\tilde{x}_2}{2}\right]\right) = \exp\left(-\frac{\tilde{x}_1.\tilde{x}_1}{2}\right).\exp\left(-\frac{\tilde{x}_2.\tilde{x}_2}{2}\right).$$

Proof: Since for any, $\alpha \in (0, 1]$,

$$\left(\tilde{x}_1^2+\tilde{x}_2^2\right)_\alpha = \left(\tilde{x}_1^2\right)_\alpha + \left(\tilde{x}_2^2\right)_\alpha ,$$

$$= \left\{\left[\tilde{x}_1^-, \tilde{x}_1^+\right]_\alpha . \left[\tilde{x}_1^-, \tilde{x}_1^+\right]_\alpha + \left[\tilde{x}_2^-, \tilde{x}_2^+\right]_\alpha . \left[\tilde{x}_2^-, \tilde{x}_2^+\right]_\alpha\right\},$$

$$= \left\{\left[\tilde{x}_1^- . \tilde{x}_1^- + \tilde{x}_2^- . \tilde{x}_2^-\right]_\alpha , \left[\tilde{x}_1^+ . \tilde{x}_1^+ + \tilde{x}_2^+ . \tilde{x}_2^+\right]_\alpha\right\}.$$

Hence,

$$\left(\tilde{x}_1^2+\tilde{x}_2^2\right)_\alpha^- = \tilde{x}_{1\alpha}^- \tilde{x}_{1\alpha}^- + \tilde{x}_{2\alpha}^- \tilde{x}_{2\alpha}^-$$

$$\left(\tilde{x}_1^2+\tilde{x}_2^2\right)_\alpha^+ = \tilde{x}_{1\alpha}^+ \tilde{x}_{1\alpha}^+ + \tilde{x}_{2\alpha}^+ \tilde{x}_{2\alpha}^+$$

Thus, by Theorem 4.1, we can write the followings:

$$\left(\exp\left(-\frac{\tilde{x}_1^2+\tilde{x}_2^2}{2}\right)\right)_\alpha = \exp\left(-\left(\frac{\tilde{x}_1^2+\tilde{x}_2^2}{2}\right)_\alpha\right)$$

$$= \left[\exp\left(-\left(\frac{\tilde{x}_1^2+\tilde{x}_2^2}{2}\right)_\alpha\right)^- , \exp\left(-\left(\frac{\tilde{x}_1^2+\tilde{x}_2^2}{2}\right)_\alpha\right)^+\right]$$

$$= \left[\exp\left(-\left(\frac{\tilde{x}_1^2}{2}\right)_\alpha\right)^- \exp\left(-\left(\frac{\tilde{x}_2^2}{2}\right)_\alpha\right)^- , \right.$$

$$\left. \exp\left(-\left(\frac{\tilde{x}_1^2}{2}\right)_\alpha\right)^+ \exp\left(-\left(\frac{\tilde{x}_2^2}{2}\right)_\alpha\right)^+\right]$$

$$= \left[\exp\left(-\left(\frac{\tilde{x}_1^2}{2}\right)_\alpha\right)^- , \exp\left(-\left(\frac{\tilde{x}_1^2}{2}\right)_\alpha\right)^+\right]$$

$$\left[\exp\left(-\left(\frac{\tilde{x}_2^2}{2}\right)_\alpha\right)^- , \exp\left(-\left(\frac{\tilde{x}_2^2}{2}\right)_\alpha\right)^+\right]$$

$$= \left[\exp\left(-\frac{\tilde{x}_1^2}{2}\right)_\alpha\right]\left[\exp\left(-\frac{\tilde{x}_2^2}{2}\right)_\alpha\right]$$

$$= \left[\exp\left(-\frac{\tilde{x}_1^2}{2}\right)_\alpha \exp\left(-\frac{\tilde{x}_2^2}{2}\right)_\alpha\right], \text{ for any } \alpha \in (0,1].$$

Consequently, $\exp(-\frac{\tilde{x}_1^2+\tilde{x}_2^2}{2}) = \exp(-\frac{\tilde{x}_1^2}{2})\exp(-\frac{\tilde{x}_2^2}{2})$.

Theorem 4.3: Let $\tilde{x}_1 \le \tilde{x}_2$, then for any $\alpha \in (0, 1]$, $x_1^- \le x_2^-, x_1^+ \le x_2^+$. Hence, $0 \le (\exp(-\frac{\tilde{x}_1^2}{2}))_\alpha = [\exp(-\frac{x_1^2}{2})_\alpha^-, \exp(-\frac{x_1^2}{2})_\alpha^+] \le [\exp(-\frac{x_2^2}{2})_\alpha^-, \exp(-\frac{x_2^2}{2})_\alpha^+] = (\exp(-\frac{\tilde{x}_2^2}{2}))_\alpha$.

This shows that $0 \le \exp(-\frac{\tilde{x}_1^2}{2}) \le \exp(-\frac{\tilde{x}_2^2}{2})$.

5 FRV valued first derivative of Gaussian function

Here in this section let us introduce the concept of a new FRV valued function and its fundamental properties using the extension principle. This new fuzzy random variable valued function is based on the first derivative of Gaussian function.

Definition 5.1: Let $\tilde{x}$ be a FRV. Analytical expression of the first derivative of Gaussian function is $-x\exp(-x^2/2)$. In order to have the simplicity the negative sign will be dropped out for the further consideration of the algebraic structure of the first derivative of Gaussian function. For any $x \in \mathbb{FR}(\Omega)$, we define,

$$xe^{-x^2/2} \triangleq \bigcup_{\alpha\in(0,1)} \alpha\left\{x\exp\left(-x^2/2\right) \mid x \in x_\alpha\right\}.$$

If we denote $f(\tilde{x}) = \tilde{x}e^{-x^2/2}$, then $f(\tilde{x})$ is called the FRV valued first derivative of Gaussian function.

Theorem 5.1: The FRV $f(\tilde{x})$ has the following properties:

1 $f(\tilde{x})(\omega) \in \mathbb{F}_0(\mathbb{R})$ for every $\omega \in \Omega$

2 $f(\tilde{x})_\alpha = \tilde{x}_\alpha e^{-\tilde{x}_\alpha^2/2} = [f(x_\alpha^-), f(x_\alpha^+)] = [x_\alpha^- e^{-(\tilde{x}_\alpha^2)^-/2}, x_\alpha^+ e^{-(\tilde{x}_\alpha^2)^+/2}]$ for any $\alpha \in (0, 1]$

3 $\tilde{x}e^{-\tilde{x}^2/2} \in \mathbb{FR}(\Omega)$ for every $\tilde{x} \in \mathbb{FR}(\Omega)$.

Following the proof of Theorem 4.2 one can easily proof Theorem 5.1.

Theorem 5.2: Let $\tilde{x}, \tilde{y} \in \mathbb{FR}(\Omega)$. Then $(\tilde{x}\exp(-\tilde{x}^2/2))(\tilde{y}\exp(-\tilde{y}^2/2)) = \tilde{x}\tilde{y}\exp(-(\tilde{x}^2 + \tilde{y}^2)/2)$.

Proof:

$$\begin{aligned}
&\left(\tilde{x}_\alpha \exp\left(-\tilde{x}_\alpha^2/2\right)\right)\left(\tilde{y}_\alpha \exp\left(-\tilde{y}_\alpha^2/2\right)\right)\\
&=\left[x_\alpha^-, x_\alpha^+\right]\left[y_\alpha^-, y_\alpha^+\right]\times\left[\exp\left(-\left[x_\alpha^-, x_\alpha^+\right].\left[x_\alpha^-, x_\alpha^+\right]/2\right)\right]\times\left[\exp\left(-\left[y_\alpha^-, y_\alpha^+\right].\left[y_\alpha^-, y_\alpha^+\right]/2\right)\right]\\
&=\left[x_\alpha^-, x_\alpha^+\right]\times\left[\exp\left[\left(-x_\alpha^-.x_\alpha^-\right)/2\right], \exp\left[\left(-x_\alpha^+.x_\alpha^+\right)/2\right]\right]\\
&\quad\times\left[y_\alpha^-, y_\alpha^+\right]\times\left[\exp\left[\left(-y_\alpha^-.y_\alpha^-\right)/2\right], \exp\left[\left(-y_\alpha^+.y_\alpha^+\right)/2\right]\right]\\
&=\left[x_\alpha^- \exp\left(-\left[x_\alpha^-.x_\alpha^-\right]/2\right), x_\alpha^+ \exp\left(-\left[x_\alpha^+.x_\alpha^+\right]/2\right)\right]\\
&\quad\times\left[y_\alpha^- \exp\left(-\left[y_\alpha^-.y_\alpha^-\right]/2\right), y_\alpha^+ \exp\left(-\left[y_\alpha^+.y_\alpha^+\right]/2\right)\right]\\
&=\left(\tilde{x}\tilde{y}\right)_\alpha\left[\exp\left(-\frac{\tilde{x}^2+\tilde{y}^2}{2}\right)\right]_\alpha, \text{ for every } \alpha \in (0,1].
\end{aligned}$$

The first derivative of the Gaussian function is a combination of linear function and Gaussian function itself. Hence, one can have its log transformed definition as presented in Definition 5.2.

Definition 5.2: Let $f(x) = xe^{-x^2/2}$. Therefore, $\log(f(x)) = \log(x) - \frac{x^2}{2}$. Let us now define, $\phi(x) = \log(x) - \frac{x^2}{2}$. If we denote $\phi(\tilde{x}) = \log(\tilde{x}) - \frac{\tilde{x}^2}{2}$, then $\phi(\tilde{x})$ is called the FRV valued first derivative of Gaussian function.

Log transformed representation of the first derivative of Gaussian function consists of a logarithmic function and power function. Hence, according to the theorem of fuzzy random variable valued logarithmic function and power function (Zhong and Wang, 1993), we have the following theorems.

Theorem 5.3: The function $\phi(\tilde{x})$ has the following properties:

1 $\phi(\tilde{x})(\omega) \in \mathbb{F}_0(\mathbb{R})$ for every $\omega \in \Omega$.

2 for any, $\alpha \in (0, 1]$

$$\begin{aligned}\phi(\tilde{x})_\alpha &= \left[\log(\tilde{x}) - \frac{\tilde{x}^2}{2}\right]_\alpha \\ &= \left[\log\left(\left[x_\alpha^-, x_\alpha^+\right]\right) - \left[x_\alpha^-, x_\alpha^+\right]\left[x_\alpha^-, x_\alpha^+\right]/2\right], \\ &= \left[\log x_\alpha^-, \log x_\alpha^+\right] - \left[x_\alpha^- . x_\alpha^- / 2, x_\alpha^+ . x_\alpha^+ / 2\right], \\ &= \left[\log x_\alpha^- - x_\alpha^+ x_\alpha^+ / 2, \log x_\alpha^+ - x_\alpha^- x_\alpha^- / 2\right], \\ &= \left[\phi\left(x_\alpha^-\right), \phi\left(x_\alpha^+\right)\right].\end{aligned}$$

3 $\phi(\tilde{x}) = \log(\tilde{x}) - \frac{\tilde{x}^2}{2}$, for every, $\tilde{x} \in \mathbb{FR}(\Omega)$.

Theorem 5.3 physically signifies that the FRV valued first derivative of the Gaussian function is an algebraic combination (subtraction) of the fuzzy random variable valued logarithmic function and FRV valued half the power function. Therefore, all the fundamental properties of FRV valued logarithmic and power function will hold good for the FRV valued first order derivative of Gaussian function. This fact can be seen using the following definitions and example.

Definition 5.3: Let $\tilde{x} \in \mathbb{FR}(\Omega)$ such that $\tilde{x} \geq 0$ or $\tilde{x} < 0$. Let $r \in \mathbb{R}$. So, we can write,

1 $\tilde{x}^0 \triangleq 1$ if $\tilde{x} \neq 0$

2 if r is a positive integer, then $\tilde{x}^r \triangleq \tilde{x}.\tilde{x}.\tilde{x} \cdots$ upto r times

3 if r is a negative integer, then $\tilde{x}^r \triangleq 1/(\tilde{x}^{-r})$

4 if r is any real number, then $\tilde{x}^r \triangleq \bigcup_{\alpha \in (0,1]} \alpha\{\tilde{x}^r \mid x \in \tilde{x}_\alpha\}$.

where $f(\tilde{x}) = x^r \mid x \in \mathbb{R}$ is the ordinary power function and we require that $(-1)^{k-1}$ is a real number when $\tilde{x} < 0$.

So, using the Definition 5.3, we can write the following properties for $r = 2$:

a $\tilde{x}_\alpha^2 = \{x^2 \mid x \in \tilde{x}_\alpha\}$, for any $\alpha \in (0, 1]$

b if $\tilde{x} > 0$, then for any $\alpha \in (0, 1]$,

$$(\tilde{x}^2)_\alpha = \begin{cases} \left[(x_\alpha^-)^2, (x_\alpha^+)^2\right] & \text{for any } r > 0; \\ \left[(x_\alpha^+)^2, (x_\alpha^-)^2\right] & \text{for any } r < 0. \end{cases}$$

c if $\tilde{x} < 0$, then for any $\alpha \in (0, 1]$,

$$(\tilde{x}^2)_\alpha = \begin{cases} \left[(x_\alpha^-)^2, (x_\alpha^+)^2\right] & \text{if } (-1)^{2-1} \text{ for any } r > 0; \\ \left[(x_\alpha^+)^2, (x_\alpha^-)^2\right] & \text{if } (-1)^{2-1} \text{ for any } r < 0. \end{cases}$$

d $\tilde{x}^2 \in \mathbb{FR}(\Omega)$.

We can also say that as the ordinary power function $f(x) = x^r$ where, $x \in \mathbb{R}$, $x > 0$ is differentiable, $\frac{df(x)}{dx} = r.x^{r-1}$, $r > 0$, $\frac{df(x)}{dx} > 0$. Therefore, it can be stated that $f(x)$ is a continuous monotone increasing function. Using the above property (a), for general value of r we can write $(\tilde{x}^r)_\alpha = [(x_\alpha^-)^r, (x_\alpha^+)^r]$, for any $\alpha \in (0, 1]$.

Further details of the behaviour of power function in the FRV space can be found in Zhong and Wang (1993). It is easy to conclude that $\tilde{x}^r$ is the ordinary power function, when we confine, $\tilde{x} \in \mathbb{R}$.

Example 1: $\tilde{x}^2 = \exp(2\log(\tilde{x})), \phi(x) = -[\log(\tilde{x}) - x^2 / 2] = -\log(\tilde{x}) - \exp(2\log(\tilde{x})) / 2].$

6 FRV valued second derivative of Gaussian function

In this section, the second derivative of Gaussian function along with its fundamental properties is presented.

Definition 6.1: Let $f(x) = e^{x^2/2}$ be a Gaussian function. The second derivative of this Gaussian function can be written as, $\phi(x) = (x^2 - 1)e^{x^2/2}$. Therefore, $\log(\phi(x)) = \log(x^2 - 1) - \frac{x^2}{2}$. Let us now define, $\psi(x) = \log(\phi(x)) = \log(x^2 - 1) - \frac{x^2}{2}$. If we denote $\psi(\tilde{x}) = \log(\tilde{x}^2 - 1) - \frac{\tilde{x}^2}{2}$, then $\psi(\tilde{x})$ is called the FRV valued second derivative of Gaussian function.

It can be seen from the definition that FRV valued the second derivative of Gaussian function consists of one logarithmic function and one power function. Hence, according to the theorem of FRV valued logarithmic function and power function (Zhong and Wang, 1993), we have the following theorems.

Theorem 6.1: $\psi(\tilde{x})$ has the following properties:

1 $\psi(\tilde{x})(\omega) \in \mathbb{F}_0(\mathbb{R})$, for every $\alpha \in (0, 1]$.

2 for any $\alpha \in (0, 1]$,

$$\begin{aligned}\psi(\tilde{x})_\alpha &= \left[\log\left(\tilde{x}^2 - 1\right) - \frac{\tilde{x}^2}{2}\right]_\alpha, \\ &= \left[\log\left(\left[x_\alpha^-, x_\alpha^+\right].\left[x_\alpha^-, x_\alpha^+\right] - 1\right) - \left[x_\alpha^-, x_\alpha^+\right].\left[x_\alpha^-, x_\alpha^+\right]/2\right], \\ &= \left\{\log\left(\left[x_\alpha^-.x_\alpha^-, x_\alpha^+.x_\alpha^+\right] - [1, 1]\right) - \left[x_\alpha^-.x_\alpha^- / 2, x_\alpha^+.x_\alpha^+ / 2\right]\right\}, \\ &= \left[\left[\log\left(x_\alpha^-.x_\alpha^- - 1\right) - x_\alpha^+.x_\alpha^+ / 2\right], \left[\log\left(x_\alpha^+.x_\alpha^+ - 1\right) - x_\alpha^-.x_\alpha^- / 2\right]\right], \\ &= \left[\psi\left(x_\alpha^-\right), \psi\left(x_\alpha^+\right)\right].\end{aligned}$$

3 $\psi(\tilde{x}) = \log(\tilde{x}^2 - 1) - \dfrac{\tilde{x}^2}{2}$, for every, $\tilde{x} \in \mathbb{FR}(\Omega)$.

Now, following the similarity behaviour of the FRV valued logarithmic function and power function, the second derivative of the FRV valued Gaussian function and its subsequent higher order derivatives can be also included as a member of the fuzzy probability space. Consequently, as Gaussian function holds the property of a fuzzy membership function (Johnson and Kotz, 1970; Liu, 2008), derivatives of that will also hold the property of fuzzy membership function.

7 Conclusions

Concept of the fuzzy probability space is formulated. The entities of this kind of probability space are basically fuzzy numbers which are characterised by the fuzzy probability distribution function (FPDF). In this paper we have shown that Gumbel, Weibull, Gaussian and the derivative of Gaussian function can also be considered as FPDF. Reliability analysis of any system in presence of epistemic uncertainty can be carried out using the corresponding FPDF. Fuzzy randomness concept can provide a new tool to the decision makers to make decisions under uncertain environment. The outcome of the present research on the FRV valued new FPDF opens the scope for carrying out further research to establish the randomness-fuzziness consistency principle.

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