Invariants of MacWilliams identity: weight and ideal distribution in poset spaces

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Abstract: The weight distribution of a poset space is determined by the distribution of ideals in the poset. In this paper, we prove the opposite relation: the distribution of ideals is completely determined by the weight distribution of a poset space.

Keywords: MacWilliams identity; poset spaces; weight distribution; weight enumerator.

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1 Introduction

Let $P = ([n], \preceq)$ be a partially ordered set (poset) on the underlying set $[n] = \{1, 2, ..., n\}$. A subset $I \subset P$ is an *ideal* of P if whenever $a \in I, b \in P$ and $b \preceq a$ then $b \in I$. Given $A \subseteq [n]$, we denote by $\langle A \rangle$ the smallest ideal containing A, called the *the ideal generated by* A.

Let $\mathbb{F}_q^n$ be the vector space of n-tuples over the finite field with q elements $\mathbb{F}_q$. Given $u = (u_1, u_2, ..., u_n) \in \mathbb{F}_q^n$, the support of the vector u, denoted by $supp(u)$, is the set of non-zero coordinates of u, i.e.,

$$supp(u) := \{i | u_i \neq 0\},$$

and the P-weight $\omega_P(u)$ of u is defined by

$$\omega_P(u) := |\langle supp(u) \rangle|,$$

where $|B|$ is just the cardinality of B. It is well known that for any $u, v \in \mathbb{F}_q^n$, $d_P(u, v) := \omega_P(u - v)$ is a metric on $\mathbb{F}_q^n$, called *P-metric*.

Let $C \subset \mathbb{F}_q^n$ be a linear code. The *weight enumerator* $W_C^P(X)$ of C is the polynomial

$$W_C^P(X) = \sum_{u \in C} X^{\omega_P(u)} = \sum_{i=0}^{n} A_i^P(C) X^i,$$

where, $A_i^P(C)$ is the number of vectors $u \in C$ with $\omega_P(u) = i$. We note that $W_C^P(X)$ depends also on the scalar field $\mathbb{F}_q$, and when it should be relevant, we should include the q index $W_C^{P,q}(X)$ and $A_i^{P,q}(C)$. When $C = \mathbb{F}_q^n$, actually, $A_i^P := A_i^P(\mathbb{F}_q^n)$ is the cardinality of the sphere with centre at zero vector and radius i. The weight enumerator is an important invariant of codes, and in many instances, when the MacWilliams identity holds (see, for example, Gutiérrez and Tapia-Recillas (1998) and Kim and Oh (2005)), it allows to establish the invariants of high dimensional codes considering only low dimensional ones, which are easier to handle.

It is well known and not difficult to prove (as stated in Brualdi, Graves and Lawrence (1995)) that

$$A_i^P = \sum_{j=1}^{i} \Omega_j^P(i)(q-1)^j q^{i-j}, \tag{1}$$

where $\Omega_j^P(i)$ is the number of ideals of P with cardinality i and exactly j maximal elements, which we call the *ideal distribution of* P. So, the invariants $\Omega_j^P(i)$, which depend uniquely on the poset P and not on the particular code C, determine the weight enumerator. In this work, we prove the converse, i.e. that the weight enumerator of $\mathbb{F}_q^n$ determines the ideal distribution of P.

2 Weight enumerator and ideal distribution

When considering two posets P and Q on $[n]$, they determine both the weight enumerators' coefficients $A_i^P$and A_i^Q and the ideal distributions $\Omega_j^P(i)$ and $\Omega_j^Q(i)$. From Eq. (1), we know that if $\Omega_j^P(i) = \Omega_j^Q(i)$ for every $j = 1, ..., i$ then $A_i^P = A_i^Q$. We will prove the converse of this statement, that is, that $A_i^P = A_i^Q$ for every $i = 1, 2, ..., n$ and every field $\mathbb{F}_q$ (or, at least n different ones) implies $\Omega_j^P(i) = \Omega_j^Q(i)$ for every $j = 1, 2, ..., i$ and every $i = 1, 2, .., n$.

We remark that neither the weight nor the ideal distribution determine the poset, that is, there are non-isomorphic posets having the same weight and ideal distribution. We demonstrate this with an example at the end of the work.

Theorem 1: *Let $P = (n, \preceq_P)$ and $Q = (n, \preceq_Q)$ be two posets. Suppose that $A_i^{P,q} = A_i^{Q,q}$ for every $i \in [n]$ and at least n different values of q. Then, $\Omega_j^P(i) = \Omega_j^Q(i)$ for every $i \in [n]$ and every $1 \leq j \leq i$.*

Proof: We consider the equality

$$A_i^{P,q} = \sum_{j=1}^{i} \Omega_j^P(i)(q-1)^j q^{i-j}$$

and expand $(q-1)^j$ into a sum

$$\sum_{k=0}^{j} \binom{j}{k} (-1)^k q^{j-k},$$

so that we can rewrite (1) as

$$A_i^{P,q} = \sum_{j=1}^{i} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \Omega_j^P(i) q^{i-k}. \tag{2}$$

Hence, considering $A_i^{P,q}$ as a polynomial $A_i^P(q)$ over the variable q, if $A_i^{P,q} = A_i^{Q,q}$ for at least $n+1$ different q's, we find that the polynomials $A_i^{P,q}(x)$ and $A_i^{Q,q}(x)$ are equal as polynomial, thus the coefficients must be the same. We shall consider $\Omega_j^P(i)$ as given and we need to prove that $\Omega_j^Q(i) = \Omega_j^P(i)$.

Let us denote $\Omega_j^Q(i) = x_{ji}$. With this notation, we are assuming that

$$A_i^P = \sum_{j=1}^{i} (-1)^k \binom{j}{k} \Omega_j^P(i) = \sum_{j=1}^{i} (-1)^k \binom{j}{k} x_{ji} = A_i^Q.$$

Thus, we obtain an homogeneous linear system

$$Ax^t = 0$$

where, $A = (a_{kl})_{i+1,i}$ with $a_{kl} = (-1)^{k-1}\binom{l}{k-1}$ and

$$x = (x_{1i} - \Omega_1^P(i), x_{2i} - \Omega_2^P(i), ..., x_{ii} - \Omega_i^P(i)).$$

Note that

$$A = (a_{kl})_{i+1\times i} = \begin{bmatrix} \binom{1}{0} & \binom{2}{0} & \cdots & \binom{i}{0} \\ -\binom{1}{1} & -\binom{2}{1} & \cdots & -\binom{i}{1} \\ \vdots & \vdots & \cdots & \vdots \\ (-1)^{i+1}\binom{1}{i} & (-1)^{i+1}\binom{2}{i} & \cdots & (-1)^{i+1}\binom{i}{i} \end{bmatrix}_{i+1,i}.$$

Since $\binom{l}{k-1} = 0$ for $l < k-1$, $\binom{l}{k-1} = 1$ and $l = k-1$, it follows that $a_{kl} = 0$ for $l < k-1$, $a_{kl} = (-1)^{k-1}$ and $l = k-1$. Thus, we find that the matrix A actually looks like

$$A = (a_{kl})_{i+1\times i} = \begin{bmatrix} \binom{1}{0} & \binom{2}{0} & \cdots & \binom{i}{0} \\ -1 & -\binom{2}{1} & \cdots & -\binom{i}{1} \\ 0 & 1 & \cdots & \binom{i}{2} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & (-1)^{i+1} \end{bmatrix}_{i+1,i}.$$

Note that the first line of the matrix A is a linear combination of the remaining, which are clearly linearly independent, thus we can remove from the system the first line, obtaining an equivalent system, with same solution set.

The new system is described by the matrix

$$B = (b_{kl})_{i\times i} = \begin{bmatrix} -1 & -\binom{2}{1} & \cdots & -\binom{i}{1} \\ 0 & 1 & \cdots & \binom{i}{2} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & (-1)^{i} \end{bmatrix}_{i,i}.$$

Since B is triangular, $\det(B) \neq 0$ and the system $Bx^t = 0$ has the unique solution $x = 0$. Therefore, given $i \in \{1, 2, ..., n\}$, we have $x_{ji} - \Omega_j^P(i) = 0$ for every $j \geq 2$ and we find that $x_{ji} = \Omega_j^Q(i) = \Omega_j^P(i)$ for every $j \geq 2$. For the remaining case $j = 1$, it follows from the fact that $\Omega_1^P(i)$ is completely determined by A_i^P and $\Omega_j^P(i)$ with $j > 1$, as can be seen in Eq. (2). □

As stated in the beginning of this section, we demonstrate that neither the weight distribution nor the ideal distribution (which is now known to be equivalent) determines the poset structure. It is done through an example.

Example 2: Consider over $[4] = \{1, 2, 3, 4\}$ the partial orders $N = \{1 \preceq_N 3, 2 \preceq_N 3, 2 \preceq_N 4\}$ and $P = \{1 \preceq_P 2, 2 \preceq_P 3, 1 \preceq_P 3\}$, which Hasse diagrams are illustrated in Figure 1. Those posets are clearly not isomorphic.

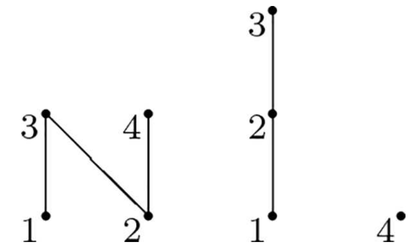

Figure 1 Hasse diagrams of N and P

Straight calculations show that the two posets have the same ideal distribution:

$$\Omega_1^N(1) = \Omega_1^P(1) = 2$$
$$\Omega_1^N(2) = \Omega_1^P(2) = 1$$
$$\Omega_1^N(3) = \Omega_1^P(3) = 1$$
$$\Omega_1^N(4) = \Omega_1^P(4) = 0$$
$$\Omega_2^N(2) = \Omega_2^P(2) = 1$$
$$\Omega_2^N(3) = \Omega_2^P(3) = 1$$
$$\Omega_2^N(4) = \Omega_2^P(4) = 1$$
$$\Omega_3^N(3) = \Omega_3^P(3) = 0$$
$$\Omega_3^N(4) = \Omega_3^P(4) = 0$$
$$\Omega_4^N(4) = \Omega_4^P(4) = 0$$

3 Conclusion

Despite the fact that the weight distribution of a poset space seems to be an invariant that is more refined than the distribution of ideals, we showed that those invariants are equivalent, in the sense that each one can be obtained from the other. It is well known that neither of those invariants is a characterisation of the poset: to find conditions to ensure such a characterisation remains as an open question.

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