Fuzzy fractional sliding mode observer design for a class of nonlinear dynamics of the cancer disease

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Abstract: The purpose of this paper is designing to optimise controller parameters for improvement of the performances of cancer dynamic by combining both TS fuzzy sliding mode method and fractional variable structure control. The fractional fuzzy controller matched is formulated here. This paper is proven the stability conditions based on linear matrix inequalities (LMIs) where the fractional-order \( \alpha \) belongs to \( 0 < \alpha < 1 \). Furthermore, this method is investigated TS fuzzy method to design fractional observer to estimate states of the fractional nonlinear dynamic of drug administration in cancer chemotherapy based on the sector nonlinearity method. This approach presented a mathematical model to expand the dynamics between tumour cells, normal cells, immune cells, chemotherapy drug concentration and drug toxicity. The stability of the closed loop system and the convergence of both the observer and the tracking error to zero are the main merits of the proposed method. The simulation results show the promising performance of both the observer and controller scheme.

Keywords: fractional TS observer; fractional system; cancer; sector nonlinearity approach; sliding mode theory.


Biographical notes: Saeed Rooka received his BSc and MSc in Control Engineering from IA University, respectively in 2013 and 2015. His research interests include fuzzy systems, fractional calculation and research about new dynamic model and also use fuzzy knowledge in flight systems.

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1 Introduction

In many real systems measure, all states are not possible therefore for design controller for this system need an observer to estimate. Many papers discussed about the subject and are available in the recent literature (Liu, 2009; Cacace et al., 2010; Busawon et al., 2009). Many types of observer indicated for integer systems same as, sliding mode observers (Rhif, 2012), high gain observers (Deza et al., 1992) and Kalman that different scientists have been expanded operation of observers. As previous part, there are many types of an observer but among them sliding mode observer is one of the best approach against uncertainty (Edwards and Spurgeon, 1998). To reduce this undesired effect, many solutions have been proposed. Sliding mode control is a special kind of robust control that allows to completely rejecting any disturbance acting. Since in this theory is considered uncertainty to any dynamic model so the best approach is sliding mode. Standard sliding mode theory has a problem it is chattering because control signal is discontinued and move around zero make a high frequency (Rhif, 2012). To destroying this phenomenon this paper is considered fuzzy approach against uncertainty. Since in this theory is considered uncertainty so it is used sliding mode. On the other hand, recently fractional calculus considered by many researchers and they have proven that many models with fractional property have better performance (Atanackovic et al., 2007; Ding and Te, 2009). Fractional systems have used in many realms and especially control theory (Monje et al., 2010; Magin, 2006; Rooka and Ghasemi, 2015a; N’Doye et al., 2014). In recent years, many scientists become interested to solve and implementation problems of control theory such as controllability (Matignon and d’Andrea-Novel, 1996), stability (Matignon, 1996; Wen et al., 2008) observability (Bettayeb and Djennoune, 2006) are extended in fractional order derivatives. In literature, there are many types of research with different fractional control method (Polubny, 1999; Rooka and Ghasemi, 2015b; Agrawal, 2004; Dadras and Momeni, 2010; Chen et al., 2004; Dzielski and Sierociuk, 2006). In Hartley and Lorenzo (2002) and Dadras and Momeni (2011) described a fractional observer for fractional systems. This paper is used a fractional dynamic of cancer that all of the states of a system is not able to measurement in other hands to estimate state cannot use typical observer so it needs to use fractional observer. One of the approaches the can prove accurate of any model is fractional calculus for example in the biological organism may describe the member of cells of the biological organism so they are considered is fractional type dynamic so this paper’s purpose is showed an exact theory to show any fractional dynamic. The main reason is that fractional-order equations naturally can depict an exact mode of the system. In addition, they are closely related to fractals which are also abundant in biological systems (Rooka and Ghasemi, 2015b). According to the World Health Organization, cancer is one of the major causes of morbidity and mortality. Many researchers for better performance worked on this subject (Lu and Cao, 2009) and one of the important realms is the dynamic model. The immune system is composed by a variety of organs, cells, and molecules that are doing their work for gaining a complete coordination. A different dynamic of cancer illness exist but a fractional model has better performance and many researchers worked on effects of the drug on cells and human. When the immune system does not work properly the results is a disease. In the case of a tumour, the immune system must recognise diseased cells and can kill them. Failure in this task results in an
uncontrolled growth of the tumour mass (Shahbazi et al., 2014). This paper presented designing observer for a fractional nonlinear system. One of the most problems in nonlinear systems is linearisation but we presented a fuzzy approach to solving this problem in Takagi and Sugeno (1985) and Sala et al. (2009). This representation is the so-called Takagi-Sugeno (T-S) model (Krokavec and Filasova, 2009). One of the advantages of using TS fuzzy approach spite of ordinary fuzzy is proved the stability of systems by many acclaimed methods same as linear matrix inequalities (LMIs), which can be solved via convex optimisation techniques (Dong et al., 2011; Khoygani et al., 2013). As previous section TS fuzzy method is one of a type representation of nonlinear dynamic and can demonstrate original nonlinear model has several nonlinear terms since the number of linear models is. Based on a nonlinear descriptor model (Khoygani et al., 2013, 2014), in Gao and Liao (2014) and Ghasemi (2009), the T-S descriptor model was introduced. There are several works with applications concerning this T-S representation (Ghasemi et al., 2013). In literature, many researchers worked on TS fuzzy method but we use fractional TS fuzzy observer to estimate the parameter. This paper presented a fuzzy approach to estimate fractional systems and prove convergence of error to zero. Also for proving and analyse the stability of the estimation error system use fractional order Lyapunov approach. It should be noted that the observer proposal is very simple and productive for practical applications. In addition, the use of fractional calculations, stability analysis method for dynamic error when the observer fractional apply.

The paper is organised as follows. Section 2 presents the TS models obtained by the sector nonlinearity approached and the stability conditions our method relies on. Section 3 presents fractional part; Section 4 presents the TS models of cancer; and in Section 5, the observer design conditions are derived; scheduling vector does not depend on the states to be estimated. The resulting conditions are summarised as LMI. Section 6 presents the simulation result and Section 7 concludes the paper.

2 TS fuzzy model

Equation (1) presented a nonlinear dynamic (Rooka and Ghasemi, 2015a)

\[ D^\alpha x = \chi(z)x + \sigma(z)(u + d) \]
\[ y = cx \]  

where \( \chi \) and \( \sigma \) are smooth nonlinear matrix functions, \( x \in \mathbb{R}^n \) is state vector, \( u \in \mathbb{R}^m \) is input vector and \( y \in \mathbb{R}^n \) is measurement vector, \( C_{yx} \) is compact set that all variables assumed to be bounded on it and \( a \) is the fractional order satisfying \( 0 < \alpha < 1 \). \( d \) is limit uncertainty. Let \( \theta_j \in [\psi_j, \psi_j] \) \( j = 1, 2, \ldots, p \) be the set of bounded nonlinearities in \( \chi \) and \( \sigma \). By using the sector nonlinearity approach and nonlinear parameter that chosen as scheduling variables weighting functions obtained as

\[ h_j^1(\cdot) = \frac{\psi_j - \theta_j(\cdot)}{\psi_j - \psi_j} \quad h_j^2(\cdot) = 1 - h_j^1(\cdot) \quad j = 1, 2, 3, \ldots \]  

where \( \psi_j \) and \( \psi_j \) presented the vector of scheduling variables, \( z = T[\mathbf{x}^T \mathbf{y}^T \mathbf{u}^T]^T, T \) is a constant matrix and determine by the membership functions as
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\[ \varphi_i(z) = \prod_{j=1}^{p} h_{ij}^j(z_j) \]  

(3)

with \( i = 1, 2, \ldots, 2^p, \) \( ij \in \{0, 1\} \). Note that these membership functions are normal, i.e., \( \varphi_i(z) \geq 0 \) and \( \sum_{i=1}^{r} \varphi_i(z) = 1, \) \( r = 2^p \), where \( r \) is the number of rules. By use the equation (3), equation (4) presented an exact model of equation (1) (Rooka and Ghasemi, 2015a):

\[ D^\alpha x = \sum_{i=1}^{r} \varphi_i(z)(A_i x + B_i(u + d)) \]

\[ y = C_i x \quad 0 < \alpha < 1 \]

\[ X(0) = x_0 \]  

(4)

with \( i = 1, 2, \ldots, 2^p, \) \( r \) is the number of local linear models, \( A_i, B_i, i = 1, 2, \ldots, r \) matrices of proper dimension and \( \varphi_i, i = 1, 2, \ldots, p \) defined as in equation (3).

For the system (4), there is some presumption, the matrices \( A_i, B_i, C_i \) are constant matrices with proper dimensions; the pair \( (A_i, B_i) \) is controllable, \( (A_i, C_i) \) is observable, and \( (B_i) \) is full column rank. The disturbance \( d(t) \) is unknown but bounded as \( |d(t)| \leq h \), where \( h > 0 \) is a known constant.

The rules of the TS system are constructed such that all the terms \( \theta_j, j = 1, 2, \ldots, p \), are taken into account, i.e., the rules have the form rule \( \theta_j \):

\[ \text{if } \theta_j(\cdot) \text{ is } M_{\theta_j} \text{ and } \ldots \theta_p(\cdot) \text{ is } M_{\theta_p} \text{ THEN } \]

\[ \begin{cases} D^\alpha x(t) = A_i x(t) + B_i(u + d) \\ y(t) = C_i x(t) \end{cases} \]

\( i = 1, 2, \ldots, r \)

The fuzzy sets corresponding to both weighting functions are defined on \([\psi_j, \varphi_j]\), i.e., the domain where \( \theta_j \) takes its values. These fuzzy sets are denoted in the sequel by \( M_{\theta_j} \).

3 Fractional system

3.1 Definition of the fractional derivative

In this paper, some symbol is considered as non-integer presentation (Magin, 2006)

\[ \ _{a}D^\alpha_t = \begin{cases} \frac{d^\alpha}{dt^\alpha} & \alpha > 0 \\ 1 & \alpha = 0 \\ \int_{a}^{t} (dr)^{-\alpha} & \alpha < 0 \end{cases} \]  

(6)

where \( \ _{a}D^\alpha_t \) represents initialised \( \alpha \)-th order differential integration from start point \( a \) to \( t \), the lower limit and upper limit of the integral operator, respectively. Equation (7) is presented a nonlinear dynamic.
The fractional calculations can be reviewed in two classes. Riemann-Liouville derivative and Grunward-Letnikov derivative, on one hand, defined as (Rooka and Ghasemi, 2015a):

\[
\frac{d^\alpha x}{dt^\alpha} = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{d\tau^n} \int_{\tau}^{t} \frac{f(\tau)}{(t - \tau)^{n-\alpha+1}} d\tau \quad n - 1 \leq \alpha < n
\]

Or the Caputo derivative on the other, defined as (Magin, 2006; Bettayeb and Djennoune, 2006)

\[
\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{d\tau^n} \int_{\tau}^{t} \frac{f(\tau)}{(t - \tau)^{n-\alpha+1}} d\tau \quad n - 1 \leq \alpha < n
\]

with \(n \in \mathbb{N}\) and \(\alpha \in \mathbb{R}_+\), where \(\Gamma(\cdot)\) is the gamma function. There are many different problems as fractional order in (Magin, 2006). Here and throughout, only the Caputo definition is used since its initial conditions take on the same form as for integer-order differential equations.

The Laplace transformation method is one of the acclaimed methods. Caputo definition of Laplace transform of the Caputo fractional derivative (9) is

\[
\int_{0}^{t} D^\alpha f(t)e^{-st} dt = s^\alpha \mathcal{L}(f(t)) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)_+ \quad \text{where} \quad s \in \mathbb{C} \quad \text{denotes the Laplace operator. That by taking zero initial conditions can be expressed equation (10) as follows equation}
\]

\[
\mathcal{L}\left(\frac{d^\alpha f(t)}{dt^\alpha}\right) = s^\alpha \mathcal{L}(f(t))
\]

Proper initialisation functions for solving differential fractional is very important. Equation (12) overview of the system offers.

\[
\epsilon \quad D^\alpha f(t) = \alpha D^\alpha f(t) + \alpha^\alpha D^\alpha f(t) \quad a \leq (c = 0) < t
\]

where \(\epsilon D^\alpha f(t)\) is \(\alpha^{th}\) derivative of \(f(t)\) starts at time \(a\) and continues at time \(t\). This means that (Rooka and Ghasemi, 2015b):

\[
\epsilon \quad d^\alpha f(t) = \frac{d}{dt} \left( \frac{1}{\Gamma(1-\alpha)} \frac{\epsilon f(\tau)}{(t-\tau)^{\alpha}} d\tau \right)
\]

and where \(\epsilon d^\alpha f(t) = \kappa(\alpha, f, a, 0, t)\) is a initialisation function. This means that
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\[ a f(t) = \kappa(\alpha, f, a, 0, t) = \frac{d}{dt} \left( \frac{1}{\Gamma(1-\alpha)} \int_{i=0}^{\alpha} f(t) (t-\tau)^{\alpha-1} d\tau \right) \]  \hspace{1cm} (14)

3.2 Definition of the two-parameter Mittag-Leffler function

The two-parameter Mittag-Leffler function is defined as follows (N’Doye et al., 2014):

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta > 0 \]  \hspace{1cm} (15)

where

\[ E_{\alpha,1}(z) = \int_{\alpha=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = E_{\alpha}(z) \]  \hspace{1cm} (16)

is the one-parameter Mittag-Leffler function. The Laplace transform of the two-parameter Mittag-Leffler function is

\[ \int_{0}^{\infty} e^{-st} t^{\alpha k + \beta - 1} E_{\alpha,\beta}(at^2) dt = \frac{s^{\alpha - \beta k}}{(s^\alpha - a)} \]  \hspace{1cm} (17)

where \( E_{\alpha,\beta} = \frac{dk}{dz} E_{\alpha,\beta} \).

Lemma 1 (N’Doye et al., 2014): If \( \alpha < 2, \beta \) is an arbitrary real number, \( \gamma \) is such that \( 0.5\alpha \pi < \gamma < \min[\pi, \pi\alpha] \) and \( C > 0 \) is a real constant, then \( |E_{\alpha,\beta}(z)| \leq C_0 \frac{1}{1 + z} \), \( \gamma \leq |\arg(z)| \leq \pi, \ z \neq 0 \) for the n-dimension matrix, we have the corollary as follows:

Corollary 1: if \( A \in \mathbb{C}^{n \times n} \) and \( \alpha < 2, \beta \) is an arbitrary real number, \( \gamma \) is such that \( \frac{\alpha \pi}{2} < \lambda < \min[\pi, \pi\alpha] \) and \( \theta > 0 \) is a real constant then

\[ \left\| E_{\alpha,\beta}(A) \right\| \leq \frac{\theta}{1 + \|A\|}, \gamma \leq |\arg(\lambda_i(A))| \leq \pi \quad i = 1, \ldots, n \]  \hspace{1cm} (18)

where \( \theta = \max(\lVert C \rVert, \lVert P^{-1} \rVert \lVert C \rVert), \lambda_i(A) \) denotes the \( i^{th} \) eigenvalue of matrix \( A \), \( P \) is a non-singular coordinate transformation giving the Jordan form of \( A \),

\[ \frac{C}{1 + \|A\|} \geq \max_{1 \leq i \leq n} \left| \frac{C_{ii}}{1 + \lambda_i} \right|, \quad \text{where} \ C \text{ and } C_{ii} \text{ are given positive constant.} \]

4 TS model of cancer

In literature, there are many different mathematical models of tumour-immune that discussed about the tumour illness to treatment and relation between drug and illness. The
model that we represent in our article has 4 states and as well as is showed the relationship between the drug and the patient’s cells. There are two different models. The first, in order to simplify the process of finding Lyapunov function, the dose-response dynamics is represented by mass-action term instead of exponentially decaying term and also the effects of drug on normal and immune cells have been neglected, in fact measure of toxicity of drugs is calculated in one term separately (Lu and Cao, 2009), the second, the model presented here includes a term of drug toxicity.

Equation (Agrawal, 2004) indicated a fractional model of cancer (Shahbazi et al., 2014)

\[
D^\alpha T = aT(1-bT) - cTN - K_TMT
\]

\[
D^\alpha N = s + \frac{\beta NT}{\beta + T} - cNT - d_1N - pTN - K_NM N
\]

\[
D^\alpha C = s_1 - \psi C - K_NMC
\]

\[
D^\alpha M = -SM + U
\]

(19)

In this section, equation (Shahbazi et al., 2014) presented the model of cancer illness that describing the interaction between \((T), (N), (C)\), respectively indicated the tumour cell, plus the effector-immune cell, with the circulation lymphocyte population, which measures the patient health. The third equations is represented the effect of chemotherapy to the three cells are shown by the terms consisting of the multiplication of each cell to the drug concentration with the constant rates \(M\) with the constant rates \(K_T, K_N, K_C\), respectively. All of the value indicated in Table 1. Forth equation is shown variation of drug concentration by an outside treatment source \(U\), and the term \(-SM\) that shows the drug decays out of the system. Detailed explanation (Shahbazi et al., 2014) is shown.

Table 1  Parameters of infection

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(4.31 \times 10^{-3})</td>
<td>Tumour growth rate</td>
<td>(\text{day}^{-1})</td>
</tr>
<tr>
<td>(b)</td>
<td>(1.02 \times 10^{-14})</td>
<td>(1/b_1) is tumour carrying capacity</td>
<td>(\text{cells}^{-1})</td>
</tr>
<tr>
<td>(c_1)</td>
<td>(3.41 \times 10^{-10})</td>
<td>Fractional immune cell kill by tumour cells</td>
<td>(\text{c}^{-1}\text{d}^{-1})</td>
</tr>
<tr>
<td>(d_1)</td>
<td>(4.12 \times 10^{-2})</td>
<td>Data rate of effector cell</td>
<td>(\text{day}^{-1})</td>
</tr>
<tr>
<td>(\beta)</td>
<td>(2.02 \times 10^1)</td>
<td>Steepness coefficient of the effector cell recruitment curve</td>
<td>(\text{c}^{-1}\text{d}^{-1})</td>
</tr>
<tr>
<td>(\rho)</td>
<td>(1.5 \times 10^{-3})</td>
<td>Max of coefficient of effector cell recruitment rate by tumour cell</td>
<td>(\text{day}^{-1})</td>
</tr>
<tr>
<td>(k_N, k_C)</td>
<td>(6 \times 10^{-3})</td>
<td>Fractional effector cell recruitment rate by drug</td>
<td>(\text{day}^{-1})</td>
</tr>
<tr>
<td>(k_T)</td>
<td>(8 \times 10^{-3})</td>
<td>Fractional tumour cell kill by chemotherapy</td>
<td>(\text{day}^{-1})</td>
</tr>
<tr>
<td>(s_1)</td>
<td>(1 \times 10^8)</td>
<td>Constant source of circulating lymphocytes</td>
<td>(\text{c}^{-1}\text{d}^{-1}\text{day}^{-1})</td>
</tr>
<tr>
<td>(s)</td>
<td>(1 \times 10^6)</td>
<td>Constant source of immune cells</td>
<td>(\text{c}^{-1}\text{d}^{-1}\text{day}^{-1})</td>
</tr>
<tr>
<td>(p)</td>
<td>(2 \times 10^{-3})</td>
<td>Effector cell inactivation rate by tumour cell</td>
<td>(\text{c}^{-1}\text{d}^{-1}\text{day}^{-1})</td>
</tr>
<tr>
<td>(\psi)</td>
<td>(9 \times 10^{-3})</td>
<td>Rate of chemotherapy drug</td>
<td>(\text{day}^{-1})</td>
</tr>
</tbody>
</table>

Source: Most of the parameters were taken from Shahbazi et al. (2014)
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Fuzzy model obtained by TS theory as follow:

\[ \theta_1 = (-bT - c_1 N - K_f M) \in [\psi_j, \overline{\psi_j}], \]

\[ \theta_2 = (-K_N M) \in [\psi_j, \overline{\psi_j}] \]

5 Design fractional observer and controller

As we all know, in practice there are many systems such that have unobservable states to measure there for the estimation problem has become more important. This section is provided a full order observer to estimate system’s states, and a sliding mode controller will also be designed.

Considered equation (4) a TS observer as form below (Rooka and Ghasemi, 2015b):

\[ D^a \dot{\hat{x}} = \sum_{i=1}^{r} \phi_i(\hat{z})([A_i \hat{x} + B_i (u + d(t))] + \sigma_i(\hat{z})(y - \hat{y})) \]

\[ \hat{y} = C\hat{x} \]

where \( \hat{x}(t) \in R^n \) indicative the estimation of \( x(t) \), and \( \sigma_i \in R^{m \times p} \) indicative the observer gain that we want to design it in optimal situation. In order to reduce the estimation error, we want to design a fractional sliding mode surface for each fuzzy rule as follows (Ghasemi, 2013)

\[ S_i(t) = \gamma_i \hat{x}_i(t) = \frac{1}{\Gamma(a)} \int_0^t \gamma_i (A_i \hat{x}_i + B_i (u + d(t)) + \beta(t) + \delta \gamma \hat{x}_i(t)) \, dt \]

\[ \hat{y} = C\hat{x} \]

The matrix \( \gamma_i \in R^{m \times m} \) is chosen such that satisfy \( \gamma_i B_i \) and be non-singular. \( \gamma_i \) have to satisfy \( \gamma_i B_i = I \). The sliding mode control law is designed as

\[ u_i = \sum_{i=1}^{r} \phi_i(z(t))(\Phi_i \hat{x}_i - [\zeta + \beta(t) + \delta] \text{sgn}(\hat{x}_i)) \]

where \( \beta(t) = \max_{\gamma \in \Omega} \|\|\sigma_i y(t)\|\| + \|\|\sigma_i C_i \hat{y}(t)\|\| \) and \( \zeta > 0 \) is a real constant. The observer (21) by applying the control law in Theorem 2 could be obtained.

Lemma 2: if \( u(t) \equiv 0, D(t) \equiv 0 \), system (4) is stochastically stable if and only if there exist a set of symmetric and positive definite matrices \( P_i, i \in \Omega \) such that satisfying equation (24)

\[ A_i^P P_i + P_i A_i + Q_i < 0 \]

\[ Q_i = \sum_{i=1}^{\Omega} \lambda_j P_j \]

Theorem 2: Sliding mode parameter \( \gamma_i \) has chosen such that satisfy \( \gamma_i B_i = I \). After that, the path of the observer (21) is driven on the sliding surface \( s(t) = 0 \).
Proof 1: Lyapunov function for each fuzzy rule considered as equation (25)

\[ V_i = \frac{1}{2} \sum_{r=1}^{r_i} \phi_i(z(t)) (s_i^T(t) s_i(t)) \]  

(25)

By combining equations (21) and (23) can be gained following equation:

\[ D^sS(t) = \gamma_i D^s\dot{x}(t) - \gamma_i (A_i + B_i \Phi_i) \dot{x}(t) \]

\[ = \gamma_i B_i [u_i(t) - \Phi_i \dot{x}(t) + d(t)] + \gamma_i \sigma_i y(t) - \gamma_i \sigma_i C_i \dot{x}(t) \]  

(26)

According to equation (26) and Lyapunov conditions, by taking the derivation of \( V(t) \) obtained

\[ D^a V(t) = \sum_{i=1}^{r} \phi_i(z(t)) \left( s^T(t) D^a s(t) \right) \]

\[ = \sum_{i=1}^{r} \sum_{r=1}^{r_i} \phi_i(z(t)) \phi_j(z(t)) \left[ s^T \gamma_i \sigma_i y(t) - \gamma_i \sigma_i C_i \dot{x}(t) + d(t) \right. \]

\[ -[\zeta + \beta(t) + \delta] \|s(t)\| \]

\[ \leq O \]

\[ O = -\zeta \|s(t)\| = -\sqrt{2} \zeta V^{1/2}(t) \]

Equation (27) is shown \( t^* = \sqrt{2} \sqrt{\zeta} (0) / \zeta \), such that \( V(t) = 0 \) (equivalently, \( x(t) = 0 \) and \( t^* \) is constant when \( t \geq t^* \)). Thus, the system path is driven on the sliding surface in a finite time.

After this part of paper we want to study stability of close loop system. Let \( e(t) = x(t) - \hat{x}(t) \) denote the estimation error.

By using equations (4) and (21) can be gained error dynamics (Rooka and Ghasemi, 2015a)

\[ D^a e(t) = \sum_{i=1}^{r} \sum_{r=1}^{r_i} \phi_i(z(t)) \phi_j(z(t)) \left[ (A_i - \sigma_i C_i) e(t) \right] \]

(28)

Proposition 2: If there exists sets of symmetric and positive definite matrices \( X_i \) and matrices \( \Psi_i \) such that satisfying the following LMIs for each \( i \in \Omega \) the error dynamic system (28) is stable,

\[
\begin{bmatrix}
\rho_i(X) & O_i(X) \\
O^T_i(X) & -W_i(X)
\end{bmatrix} < 0
\]  

(29)

\[ \rho_i(X) = X_i A_i^T + X_i A_i - C_i^T \Psi_i^T - \Psi_i C_i + \lambda_i X_i \]

\[ O_i(X) = \left[ \sqrt{\lambda_i} X_i, \sqrt{\lambda_i} X_i, \ldots, \sqrt{\lambda_i} X_i, \sqrt{\lambda_{i+1}} X_i, \sqrt{\lambda_{i+1}} X_i, \ldots, \sqrt{\lambda_{N}} X_i \right] \]

\[ W_i(X) = \text{diag} [ X_i, X_2, \ldots, X_{i-1}, X_{i+1}, X_N ] \]

(30)
Proof 2: According to Lemma 2, the error dynamic will be stable if and only if there exist a set of symmetric and positive definite matrices $P_i$, $i \in \Omega$ such that satisfy equation (31).

$$\left(A_i - \sigma_i C_i\right)^T P_i + P_i \left(A_i - \sigma_i C_i\right) + \sum_{j \in \Omega} \lambda_{ij} P_j < 0$$  \hfill (31)

In order to get the LMI form, let $X_i = P_i^{-1}$, pre- and post-multiply equation (31) by $X_i$ obtained

$$X_i A_i^T + X_i A_i - X_i C_i^T \sigma_i^T + \sigma_i C_i X_i + X_i \sum_{j=1}^{N} \lambda_{ij} \lambda_{ij}^{-1} X_j < 0$$ \hfill (32)

Pay attention to equation (30) then term \(X_i \sum_{j=1}^{N} \lambda_{ij} \lambda_{ij}^{-1} X_j\) could convert as equation (33).

$$X_i \left[\sum_{j=1}^{N} \lambda_{ij} \lambda_{ij}^{-1}\right] X_i = \lambda_{ii} X_i + O_i(X)W^{-1}(X)O_i^T(X)$$ \hfill (33)

If there are some appropriate matrices ensuring that the equation $C_i X_i = Y_i C_i$ holds for each $i \in \Omega$, then let $\Psi_i = \sigma_i X_i$, and equation (32) could be rewritten as

$$X_i A_i^T + A_i X_i - C_i^T \Psi_i^T + \Psi_i C_i + \lambda_{ii} X_i + O_i(X)W^{-1}(X)O_i^T(X) < 0$$ \hfill (34)

Let

$$\rho_i(X) = X_i A_i^T + A_i X_i - C_i^T \Psi_i^T + \Psi_i C_i + \lambda_{ii} X_i$$ \hfill (35)

and use the Schur complement to equation (34). Then, equation (29) could be obtained. According to the sliding mode control theory, when the system trajectories reach onto the sliding surface, it follows that $s(t) = 0$ and $D^p s(t) = 0$.

Thus, by $D^p s(t) = 0$, obtain the control law as

$$u_{eq} = \sum_{i=1}^{r} \phi_i \left(\Phi_i \tilde{x}(t) - [\gamma_i \sigma_i y(t) - \gamma_i \sigma_i C_i \tilde{x}(t) + d(t)]\right)$$ \hfill (36)

Substituting equation (36) into equation (4), obtained get the closed-loop system dynamics (Rooka and Ghasemi, 2015b).

$$D^\alpha \tilde{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} \phi_i(\zeta(t)) \phi_j(\zeta(t)) \left[\left(A_i + B_i \Phi_i\right) x(t) - \left(B_\gamma \Phi_i + B_\gamma \gamma_i \sigma_i C_i\right) e(t)\right]$$ \hfill (37)

Combing the above equation with the error dynamics (28), we could get

$$D^\alpha \begin{bmatrix}X \\ e\end{bmatrix} = \sum_{i=1}^{r} \sum_{j=1}^{r} \phi_i(\zeta(t)) \phi_j(\zeta(t)) \begin{bmatrix}A_i + B_i \Phi_i & -(B_\gamma \Phi_i + B_\gamma \gamma_i \sigma_i C_i) \\ 0 & A_i - \sigma_i C_i\end{bmatrix} \begin{bmatrix}X \\ e\end{bmatrix}$$  \hfill (38)

The performance of the closed-loop system will be analysed in the following context, and some results will be given.
Theorem 3: For some appropriate matrices $\Phi_i$ and $\sigma_i$, where $i \in \Omega$, if there exist symmetric and positive-definite matrices $X_i$ and $Q_i$, and matrices $R_i, N_i$ satisfying the following LMIs, for each $i \in \Omega$:

$$\begin{bmatrix} Y_i(P) & P_iB_i & P_iB_i^T \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} < 0$$  \hspace{1cm} (39)$$

$$\begin{bmatrix} Y_i(Q) & \Phi_i & C_i^T \sigma_i^T \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} < 0$$  \hspace{1cm} (40)$$

where

$$Y_i(P) = A_i^T P_i + P_iA_i + \Phi_i^T B_i^T P_i + P_iB_i\Phi_i + \sum_{j=1}^{N} \lambda_j P_j$$  \hspace{1cm} (41)$$

$$Y_i(Q) = A_i^T Q_i + Q_iA_i - C_i^T \sigma_i^T Q_i - Q_i\sigma_iC_i + \sum_{j=1}^{N} \lambda_j Q_j$$  \hspace{1cm} (42)$$

Then the system (4) is stabilised.

$\sigma_i$ and $\Phi_i$ form equations (28) and (29) cannot be able directly, as they are nonlinear for designing parameters $\sigma_i$ and $\Phi_i$. So we need to transform the stability conditions to the LMI formalism.

Let $X_i = P_i^{-1}$ and $\Theta_i = Q_i^{-1}$, pre- and post-multiply and let $Y_i = \Phi_iX_i$, $\sigma_iC_i = C_iM_i$, $N_i = M_i\Theta_i$, and $R_i = \Phi_i\Theta_i$, we could get

$$\begin{bmatrix} X_iA_i^T + A_iX_i + Y_i^T B_i^T + B_iY_i + B_iB_i^T + B_i^T\gamma_i^T B_i^T + \sum_{j=1}^{N} \lambda_j X_iX_jX_i \end{bmatrix} < 0$$  \hspace{1cm} (43)$$

$$\begin{bmatrix} \Theta_iA_i^T + A_i\Theta_i - N_i^T C_i^T - C_iN_i + R_iR_i^T + N_i^T C_i^T C_iN_i + \sum_{j=1}^{N} \lambda_j \Theta_i\Theta_i \end{bmatrix} < 0$$  \hspace{1cm} (44)$$

Applying Schur complement to equations (43) and (44), we could obtain the following theorem.

Theorem 4: If there exist symmetric and positive-definite matrices $X_i$ and $\Theta_i$ and matrices $R_i, N_i$ satisfying the following LMIs, for each $i \in \Omega$:

$$\begin{bmatrix} \rho_i(X) & O_i(X) \\ O_i^T(X) & -W_i(X) \end{bmatrix} < 0$$  \hspace{1cm} (45)$$

$$\begin{bmatrix} \rho_i(Q) & R_i^T & N_i^T C_i^T & O_i(Q) \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -\varphi_i(Q) \end{bmatrix} < 0$$  \hspace{1cm} (46)$$
Then system (4) could be stochastically stabilised by the observer-based output feedback controller formed in equations (21) and (23). The parameters of the controller could be calculated from

\[
\Phi_i = Y_i X_i^{-1} \\
\sigma_i = C_i N_i \Theta_i^{-1} C_i^{-1}
\]

(47)

Remark 1: Notice the LMIs of Theorem 4 are independent, so that for each \( i \in \Omega \), we could solve equation (45) to get \( \Phi_i \), and solve equation (46) to obtain \( \sigma_i \). As \( C_i \) is not square matrices, so \( C_i^{-1} \) represents the generalised inverse of \( C_i \).

6 Simulation result

In this section, we simulated dynamic mode of cancer-based close loop controller. In our study, this method is combined with fractional order concept-based state feedback approach, resulting in the fractional fuzzy observer method. This new method can control and estimate the cancer concentration more effectively.

In this part first the system of equations with initial condition of \( I(0) = 1, T(0) = 0.25 \) and \( N(0) = 1 \) in absence of chemotherapy has been solved.

Figure 1  Behaviour of the system equations in presence of chemotherapy-based sliding mode (see online version for colours)
In addition, the system of equations for the patient with aforementioned initial conditions but this time in the presence of chemotherapy and using of drug administration condition.

As shown in Figure 1 solution of the system of equations are guided toward tumour free equilibrium point in which after approximately 40 days all cancerous cells are destroyed and healthy cells are at desirable situation.

In Figure 3, the method of drug administration to patient using of condition resulted from LMI theorem is shown.
Fuzzy fractional sliding mode observer design for a class of nonlinear dynamics

One could see that for three days and approximately within days 2, 4 and 5, drug is given to patient. As system stands in attraction region of tumour free there is no need for drug administration and solution of equations goes spontaneously toward tumour free equilibrium point. Figures 1 and 2 presented a better speed of the fractional system-based sliding mode approach. We considered free virus on body as uncertainty.

7 Conclusions

In this paper, we presented a fractional fuzzy observer design for TS fuzzy systems obtained by the sector nonlinearity method and also presented a general framework for the design of LMI approach based on sliding mode controller. One could see with the use of this method of drug administration, dynamic solution always tends to tumour free equilibrium point. It could be observed that with use of extracted drug administration condition, there is no need for calculation of attraction regions at free tumour equilibrium point. Drug administration will be stopped after system enters attraction regions of free tumour point. Both the convergence of the observer error to zero and the stability of the closed loop system are the main merits of the proposed method.

References


