On the nature of the voting paradox as a basis of economic analysis

Yasumi Matsumoto

School of Political Science and Economics,
Waseda University,
1-6-1, Nishi-waseda, Shinjuku-ku,
Tokyo 169-8050, Japan
Email: shohon-hobi-48@waseda.jp

Abstract: Economics has pursued the deduction of rational conclusions based on a standard of maximising behaviour such as profit maximisation. This approach inevitably makes preference rankings over alternatives satisfying some rational consistency. But there is a case which does not lead to rational social preference relations in reality. A typical case of irrational preference relations is represented by the voting paradox, n-way deadlock in general. Since there are actually so many social conflicts caused by the voting paradox, it is important to study how to reach a social conclusion in such cases. In this paper, we will examine how to reach an agreement all individuals can accept in the case of a voting paradox. Different from the traditional approach, we do not force a deduction of rational preference relations but try to find a way to reach a social agreement. Although this approach requires all participants to give up their preferences to some extent, it may be possible to decrease social conflicts peacefully because the dignity of the individuals concerned is respected.

Keywords: voting paradox; three-way deadlock; cyclical chain; irrational social choice; simple majority decision rule.


Biographical notes: Yasumi Matsumoto is a Professor of Theoretical Economic Policy at the School of Political Science and Economics, Waseda University, Tokyo, Japan. He holds a DPhil in Economics. His major subjects are collective choice theory, economic policy, computer science and gene science. He now concentrates on clarifying the reasons for individual preference in terms of biology, culture, language and religion, among others.

This paper is a revised and expanded version of a paper entitled ‘On the nature of the voting paradox as a basis of economic analysis’ presented at Business & Economics Society International Conference 2014, Florence, Italy, 6–9 July 2014.
1 Introduction: irrational social preference relations as a matter of course

As is quite often experienced, much social decision-making falls into deadlock because it is difficult to choose the best alternative in reality. Such problems contain logical conflicts in general. In this paper, we try to find a way to reach a socially acceptable conclusion in a logically conflicting problem.

Let us first make clear our framework of study. Throughout this paper, our discussion is confined in a framework where all individuals have their own preference orderings over a set of alternatives and the social preference relations are deduced somehow by collecting individual orderings. If the social preference relations satisfy some rationality, they may be acceptable in society. If social preference relations always satisfy some rationality, it is not difficult to get the best alternative from the set of alternatives and, therefore, there is no need for social choice theory. The fact that social choice theory is established as a science indicates that there are many irrational social preference relations caused by different opinions in society. This situation may quite often happen if Arrow’s impossibility theorem proving the non-existence of social preference orderings is recognised. If irrational preference relations come from completely different recognition among people, it is really difficult to reach an agreement all individuals can accept. Since a typical example of such a case is the three-way deadlock, or the voting paradox, we will try to examine social conflicts caused by the voting paradox in this paper.

Before going further, let us examine the theoretical reason for social conflicts. Arrow’s impossibility theorem concludes that there is no dictator all of whose pairwise preferences become social preference relations (Arrow, 1963). Here, it should be noted that if the weak Pareto condition is replaced with the reverse weak Pareto condition in Arrow’s framework, we can arrive at the conclusion that there is no reverse dictator such that all social preference relations are opposite to his/her preference relations (Wilson, 1972; Matsumoto, 2013). As with the definition of dictator, it is impossible to have a reverse dictator in reality. The conclusions of these theorems are universal because the proofs are no-existence ones. From the above results, we conclude that everybody has to give up at least one preferred alternative in pairwise comparison to be realised and has at least one preference relation to be accepted in the society. Even if Arrow’s social welfare function giving a social preference ordering is weakened to a social decision function, the social situation is not dramatically improved in terms of individual freedom and equality, though a small number of individuals become weak dictators with veto power (Gibbard, 1969; Sen, 1970; Schwartz, 1972; Mas-Colell and Sonnenschein, 1972; Guha, 1972; Blau, 1976 among others).

If we actually look at decision-making issues in society, there are many conflicts of opinion resulting in a strict preference cycle (Linhart, 1998). Of course, we can recognise that human beings have developed various means to manage irrational social preference relations somehow in order to reach an agreement in society. Since, however, there are so many various types of conflicts of opinion in society, it may be necessary to survey if there exist other ways to liquidate social conflicts.

Here, if we look back at the history of social choice theory, it may be said that the social choice theorist has so far tried to deduce social preference relations satisfying some rationality by changing or modifying conditions imposed on the choice function in order to avoid the emergence of irrational social preference relations such as a voting paradox. But the approach to daringly induce logically consistent social preference relations by introducing arbitrary conditions may actually create a new problem because it tends to
bring forth a disagreeable sense of incongruity in individual minds. It is easy to find an enormous number of such troubles if you briefly look back at human history.

First of all, it is not assured in advance that we can reach social preference relations satisfying some rationality. Secondly, the introduction of arbitrary conditions inevitably leads to a special case and, therefore, decreases the universality of the conclusion. A compromise is necessary to induce an irrational but agreeable social conclusion from a voting paradox, in more generally \(n\)-way deadlock, because it is inevitable that we must introduce plural standards of evaluation to make a social decision, as Arrow’s impossibility theorem indicates.

We, however, investigate an approach which is different from the traditional one in social choice theory in order to resolve social conflicts. Before going further, let us investigate the fist game, because its structure is the same as those of the voting paradox we often see in our daily life and its fundamental idea behind the game seems to be effective to resolve social conflicts when we are at a loss in choosing the best alternative.

2 The fist game reflecting social conflicts

2.1 The basic structure of ‘jan ken’

Here, the rule of the fist game is introduced to explain the nature of deadlocked social decision issues. There are various variations in the fist game and the simplest one is known as ‘jan ken’ (the game of paper, stone and scissors or rock-paper-scissors)\(^2\), a game of three-way deadlock. The three-way deadlock is a situation in which all three parties cannot move because each of them has superior as well as inferior counterparts among the three. The voting paradox is a kind of three-way deadlock. The latter does not care about each party’s individual preference relations but focuses only on the result of the decision as a group. The numbers of participants and alternatives can be expanded to more than three. In this paper, we are concerned with social conflicts with three or more alternatives but do not consider the case with two alternatives.

Linhart\(^3\) (1998) analyses the importance of ‘jan ken’ in terms of a social decision-making model as follows in his book (written in Japanese).

There is no need to explain to Japanese how much value ‘jan ken’ has in their daily decision-making. The thought of the three-way deadlock behind ‘jan ken’ offers a model of power structure in Japanese society. An example quite often mentioned in Japan is a three-way deadlock among politicians, business circles and bureaucrats. Since business circles give money to politicians, the former is superior to the latter. Since many politicians come from among bureaucrats, the former is superior to the latter. Since bureaucrats have the power of administrative guidance over business circles, the former is superior to the latter. It seems that the social model based on a three-way deadlock can explain more properly today’s internet society than social models such as ‘vertically-structured society’ or ‘middle class society’. If it is true, it may be said that the social philosophy of the three-way deadlock is contained in ‘jan ken’. Since ordinary Japanese who make decisions by ‘jan ken’ in daily life have the thought of a three-way deadlock at the back of their minds without consciousness, they can easily apply the thought of a three-way, four-way or five-way deadlock to social relations and can surpass the simple structured social model. In other words, it seems that various social and human relations in society can be better explained by a model of ‘\(n\)-way deadlock’ taking a
serious view of mutual dependency than a simple model concerning only a vertical relation [Linhart, (1998), pp.2–3]. It is quite common to find a case of three-way deadlock in reality. Let us consider the case such that: people are stronger than politicians, who are stronger than bureaucrats, who are stronger than people. Politicians are weaker than people because the result of the election crucially depends on people’s votes, but stronger than bureaucrats because they have power to force the latter to execute policies. Meanwhile, bureaucrats are stronger than people because they govern people. This case is recognised when we look back at recent Japanese economic stagnation continuing more than two decades. Let us pick three political as well as economic issues,

a the increase in VAT
b financial reconstruction
c the issue of national loans.

They were main disputes when we discussed how to revitalise the Japanese economy. The preference relations among them are $a > b > c$ for bureaucrats, $b > c > a$ for people and $c > a > b$ for politicians. (Here, $>$ means a strict preference relation). Bureaucrats think as follows: since it is easy to get a huge amount of money in a short time from (a), it is effective for escaping economic stagnation. (b) requires the bureaucracy making a huge spending cut, which takes time and bureaucrats are reluctant to tie themselves fast. Since the collapse of the financial system directly leads to the collapse of the economy, (c) must be avoided at any cost. People think as follows: since there is so much useless expenditure in administration, the government has to improve its administrative efficiency and use the saved money for economic revitalisation. In case of a shortage of money, (c) should be taken. The rich and big business can buy national loans. (a) is out of the question. Politicians think as follows: a huge amount of money is necessary for the recovery of the economy but they do not want to make an enemy of the people. Therefore, they put pressure on bureaucrats to take (c). If the money is not enough, it is unavoidable to take (a) because the administration will not work if they put bureaucrats into a corner.

There are many cases and ideas of a three-way deadlock in economics and business management. For example, a main paradigm in the theory of industrial organisation is:

(industrial structure) $>$ (firm’s strategy) $>$ (achievements of industry) $>$ (industrial structure). It is also easy to have a three-way deadlock in sport competitions. The food chain of our lives is also recognised as a case of $n$-way deadlock because the number of interested parties increases but there is no party who is stronger than the others.

2.2 How to reach a social agreement in the case of a voting paradox

There are various ways to reach a social agreement such as trade-off and reconciliation, among others, in a voting paradox/three-way deadlock. Similar to a lottery, ‘jan ken’ is often used as a fair and easy means to make a decision. But ‘jan ken’ is used rather to decide the order of individuals to start making choices. For example, the winner of ‘jan ken’ has the right to start the decision-making process by expressing his/her preference
relation to a pair of alternatives in a voting paradox. However, the commonly shared background reason of 'jan ken' among concerned individuals works well in a serious social voting paradox, where all of them are ready to go into a negotiation of trade-offs before playing 'jan ken'. This recognition, not actually recognised by many people, is similar to the tacit agreement among voters that they will accept the result by majority decision rule based on their votes before voting (Rousseau, 1754).

Here, we do not take an approach to construct rational social rankings from irrational ones but employ a practical way of reaching a solution by leaving irrational rankings as they are. There is a maxim in Japan: ‘in a quarrel, both parties are to blame’. This thought is still popular among court judges in Japan even today. This approach may seem unfair or ambiguous to many people, especially to westerners, but a quarrel normally has reasons on both sides and, therefore, it is sometimes very difficult to detect the original cause. This means that the direct cause of the quarrel may be just a trigger for the outbreak of conflict in many cases. Therefore, a one-sided conclusion may sometimes cause revenge from the loser against the winner, which provokes counter-revenge from the latter against the former and both sides go into an endless repetition of revenge. For example, a famous case is one involving the Tokugawa Shogunate in 1703. The case is called the *Genroku Ako Incident*. Since the trial of the Tokugawa Shogunate ascribed 100% responsibility to Asano Takuminokami and no blame to Kira Kozukenosuke, against the traditional rules of judgment, Ako’s 47 samurai, the former men, took revenge and killed the latter. This revenge still wins the strong sympathy of many Japanese, which indicates how firmly and deeply the standard of ‘in a quarrel, both parties are to blame’ occupies Japanese minds. From this example, it may be understood that there is a way to conclude a social conflict by forcing all individuals concerned to take some responsibility. A good and important point of this approach is that the dignity of all individuals is respected, which may calm their angry feelings.

### 2.3 The fist game and the westerner

There are many games similar to the fist game in many countries but the origin of the game came from countries of polytheism such as Japan, India and China and countries of monotheism imported the game later (Linhart, 1998; Murata 2000). According to Murata (2000), the three-way deadlock is a typical idea of people who believe in polytheism, showing a sharp contrast to the idea of people who believe in monotheism relying on an absolute being. He also insists that western culture based on monotheism has the power to force people to make rankings among people and alternatives and makes people strongly wish to be at the top of the ranking (see also Matsumoto, 2007). The westerner may be one who does not allow the existence of a three-way deadlock and somehow forces rankings to be made among the alternatives of the deadlock. If Murata’s opinion is right, it may be difficult for the westerner to accept the coexistence of various different types of thought.

In the following, we try to investigate the structure of and deduce the necessary and sufficient conditions of the voting paradox. Here, we employ an informal explanation in the following for readers who are not familiar with mathematical logic. The following discussion is a generalisation of Nicholson’s (1965) discussion.
3 The structure of voting paradox

As was explained above, there are many actual cases in which the structure of the voting paradox, more generally an \( n \)-way deadlock is held. Among them, the case of an election is often cited as a typical one. It is actually easy to construct a voting paradox from an actual election result by supposing a cyclical chain of individual preferences (the definition is given later) based on votes for candidates (Matsumoto, 2003). It is, however, impossible to fully know voters’ preferences.

3.1 The case of three voters and three alternatives

The voting paradox (Arrow, 1963; Black, 1958) is normally explained based on the left-hand table of Table 1. Here, \( N_1 \sim N_6 \) are sets of individuals (suppose each set contains only one individual for the moment). Let all individuals have a quasi-ordering over all alternatives. The numbers of each line indicate rankings of the left-hand side individual. If we check binary comparison by the simple majority decision rule, we have: \( x_1: x_2 = 2:1, x_2: x_3 = 2:1, x_3: x_1 = 2:1 \). From here, we have an infinite cycle of social preference relations: \( x_1 \xRightarrow{x_2} x_3 \xRightarrow{x_1} \ldots \) This is a voting paradox.

<table>
<thead>
<tr>
<th>Alternative set of voters</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>Alternative set of voters</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_1 )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>( N_4 )</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( N_2 )</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>( N_5 )</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>( N_3 )</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>( N_6 )</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 1 Voting paradox (case of three alternatives)

What are individual preference relations when social preference relations have a voting paradox: \( x_1 P x_2 P x_3 P x_1 \ldots \)? If all individuals or two of them have the same quasi-ordering, a voting paradox never happens. Therefore, they must have different individual preference relations from each other, which are given in the left-hand table of Table 1. Then, binary comparisons of two alternatives are: \( x_1: x_2 = 2:1, x_2: x_3 = 2:1, x_3: x_1 = 2:1 \). It should be noticed that individual preferences becomes \( x_1 P x_2 P x_3 P x_1, x_2 P x_3 P x_2, x_3 P x_2 P x_3 \) by calling individuals 1, 2 and 3 in \( N_1, N_2 \) and \( N_3 \) respectively. Then, we get a preference cycle of \( x_1 \xRightarrow{x_2} x_3 \xRightarrow{x_1} \ldots \) by chaining all individual preference relations. We call this a cyclical chain of individual preferences. We also recognise from Table 1 that there exist at least three alternatives and three individuals for a voting paradox.

However, the individual preference relations of \( N_1, N_2 \) and \( N_3 \) are only a part of the combinations of three alternatives. The rest are given in the right-hand table of Table 1. As is easily understood, this table is formally the same as those of the left-hand table. The only difference is that we have a reverse cycle of \( x_1 \xRightarrow{x_2} x_3 \xRightarrow{x_1} \ldots \) among individual preference relations. Therefore, we have a voting paradox: \( x_3 P x_2 P x_1 P x_3 \ldots \) from \( x_1: x_2 = 1:2, x_2: x_3 = 1:2, x_3: x_1 = 1:2 \). If, however, all of \( N_1 \sim N_6 \) contains only one individual respectively, all alternatives become socially indifferent and, therefore, a voting paradox does not emerge. Hence, the number of individuals of either side must be 0 to make a voting paradox. Namely, for a voting paradox: \( x_1 P x_2 P x_3 P x_1 \ldots \), it is necessary to have: \( |N_1| = |N_2| = |N_3| = 1 \) and \( |N_4| = |N_5| = |N_6| = 0 \). Of course, for a voting paradox: \( x_3 P x_2 P x_1 P x_3 \ldots \), \( |N_1| = |N_2| = |N_3| = 0 \) and \( |N_4| = |N_5| = |N_6| = 1 \) have to hold.
107

The above formal rule seems to be applied to the case containing four or more alternatives.

3.2 The case of four alternatives

Let us apply the discussion on the voting paradox in the previous section to the case of four alternatives. All quasi-orderings consisting of four alternatives total 24 combinations as given in Table 2. Since we suppose here all of $N_1 \sim N_{24}$ contain only one individual respectively, the total number of individuals is 24. In Table 2, variations of individual preferences are divided into 4 groups for easy understanding.

Table 2  Voting paradox (case of four alternatives)

<table>
<thead>
<tr>
<th>Block 1</th>
<th>Block 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$N_2$</td>
<td>1</td>
</tr>
<tr>
<td>$N_3$</td>
<td>1</td>
</tr>
<tr>
<td>$N_4$</td>
<td>1</td>
</tr>
<tr>
<td>$N_5$</td>
<td>1</td>
</tr>
<tr>
<td>$N_6$</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Block 3</th>
<th>Block 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>$N_{14}$</td>
<td>3</td>
</tr>
<tr>
<td>$N_{15}$</td>
<td>3</td>
</tr>
<tr>
<td>$N_{16}$</td>
<td>3</td>
</tr>
<tr>
<td>$N_{17}$</td>
<td>3</td>
</tr>
<tr>
<td>$N_{18}$</td>
<td>3</td>
</tr>
</tbody>
</table>

Following the discussion in the previous section, let us examine a cyclical chain of individual preferences. Then, we have:

- **case 1**: $x_1 P_1 x_2 P_2 x_3 P_3 x_4$ (block 1 $N_1$), $x_3 P_3 x_4 P_4 x_2 P_2 P_1 x_1$ (block 2 $N_{10}$), $x_3 P_3 x_4 P_4 x_2 P_2 P_1 x_1$ (block 3 $N_{13}$), $x_5 P_5 x_3 P_3 x_4 P_4 x_2 P_2 P_1 x_1$ (block 4 $N_{19}$)

- **case 2**: $x_3 P_3 x_4 P_4 x_2 P_2 x_1$ (block 4 $N_{24}$), $x_1 P_1 x_2 P_2 x_3 P_3 x_4$ (block 1 $N_6$), $x_3 P_3 x_4 P_4 x_2 P_2 P_1 x_1$ (block 2 $N_8$), $x_3 P_3 x_4 P_4 x_2 P_2 P_1 x_1$ (block 3 $N_{15}$).

If both cases exist, all alternatives become indifferent and a voting paradox does not exist. If individuals 1, 10, 17 and 19 exist and none of individuals 6, 8, 15 or 24 exists, a voting paradox: $x_1 P_1 x_2 P_2 x_3 P_3 x_4$... appears. If the situation is upside down, we have a voting paradox: $x_3 P_3 x_4 P_4 x_2 P_2 P_1 x_1$... These conclusions are formally the same as those in the case of three alternatives.

Let us consider individual preferences when a voting paradox: $x_4 P_4 x_3 P_3 x_2 P_2 P_1 x_1$... exists. If all individuals have the same quasi-ordering, we do not have a voting paradox. If three individuals have the same quasi-ordering, this preference becomes the social preferences and we do not have a voting paradox, either. If two individuals have the same
preference, there is no social voting paradox because these two individuals’ preference relations are never overturned in the society. Therefore, four individuals should have different preference relations from each other, namely, \( x_1; x_2 = x_2; x_3 = x_3; x_4 = x_4; x_1 = 3:1 \) by the simple majority decision rule. Therefore, the only possible combinations are: 

\[ x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4 \]

which makes a cyclical chain of individual preferences.

Although it is possible to make four cases of a cyclical chain of individual preferences in Table 2, neither of them makes a voting paradox. Let us see, for example, the following case.

- **case 3:** \( x_1P_1x_2P_2x_3P_4x_4 \) (block 1 \( N_2 \)), \( x_1P_1x_2P_2x_3P_4x_4 \) (block 2 \( N_2 \)), \( x_1P_1x_2P_2x_3P_4x_4 \) (block 3 \( N_2 \)), \( x_1P_1x_2P_2x_3P_4x_4 \) (block 4 \( N_2 \)).

Since we have: \( x_1; x_2 = 3:1, x_2; x_3 = 2:2, x_3; x_4 = 1:3, x_4; x_1 = 2:2 \) in this case, a voting paradox does not emerge.

### 3.3 The minimum cyclical chain of individual preferences assuring a voting paradox

Here, let us examine if it is possible to have a voting paradox when one individual preference relation is excluded from case 1. Then, we have the following four cases:

- **case 1–1:** \( x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4 \)
- **case 1–2:** \( x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4 \)
- **case 1–3:** \( x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4 \)
- **case 1–4:** \( x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4 \)

When we try binary comparison by the simple majority decision rule, we have a voting paradox: \( x_1P_1x_2P_2x_3P_4x_4 \)… in any case in the above. Of course, we suppose the number of individuals in case 2 is 0. In a similar way, if we suppose the number of individuals in case 1 is 0, we have a voting paradox: \( x_1P_1x_2P_2x_3P_4x_4 \)…

But we do not have a voting paradox in the other cases. In case 3, for example, check:

\( x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4, x_1P_1x_2P_2x_3P_4x_4 \). Then, we have a quasi-ordering from:

- \( x_1P_1x_2P_2x_3P_4x_4 \) and \( x_1P_1x_2P_2x_3P_4x_4 \)
- \( x_1P_1x_2P_2x_3P_4x_4 \) and \( x_1P_1x_2P_2x_3P_4x_4 \)
- \( x_1P_1x_2P_2x_3P_4x_4 \) and \( x_1P_1x_2P_2x_3P_4x_4 \)
- \( x_1P_1x_2P_2x_3P_4x_4 \) and \( x_1P_1x_2P_2x_3P_4x_4 \)

If we rule out one alternative in case 1, we do not have a voting paradox because there are two individuals sharing the same preference relations.

### 3.4 The necessary and sufficient conditions of the voting paradox

#### 3.4.1 The nature of the voting paradox: the case of three alternatives

Let us suppose the number of individuals in \( N_1 \sim N_6 \) in Table 1 is an arbitrary natural number or 0. In order to get a voting paradox: \( x_1P_1x_2P_2x_3P_4x_4 \), we have to get by the simple majority decision rule: \( x_1P_2x_1, x_2P_3x_2, \ldots \). Therefore, \((|N_1| + |N_2| + |N_3|) > (|N_1| + |N_2| + |N_3|)\), \((|N_1| + |N_2| + |N_3|) > (|N_1| + |N_2| + |N_3|)\), \((|N_1| + |N_2| + |N_3|) > (|N_1| + |N_2| + |N_3|)\) must hold. From this, we have: \((|N_1| + |N_2| + |N_3|) > (|N_1| + |N_2| + |N_3|)\). Here, we have a cyclical chain of individual preferences: \( \forall i \in N_1 \), \( x_1P_1x_2P_3x_4 \), \( \forall i \in N_2 \), \( x_1P_1x_2P_3x_4 \), \( \forall i \in N_3 \), \( x_2P_2x_3P_4x_5 \) in \( N_1 \), \( N_2 \) and \( N_3 \). If \( |N_1| = |N_2| = |N_3| \) and \( |N_4| = |N_5| \) all alternatives
become indifferent and there is no voting paradox. Therefore, \(|N_1| > |N_4|, |N_2| > |N_6|\) and \(|N_3| > |N_5|\). In addition, a cyclical chain of individual preferences in \(N_1, N_2\) and \(N_3\), namely, \(x_1P_1x_2P_3x_3\), \(x_2P_5x_5P_3x_3\) and \(x_3P_3x_5P_y\) should be dealt with as a set, which may be easily recognised from the explanation in Section 3.1. Otherwise, a voting paradox is not sure to emerge. Thus, the set must be multiplied by any natural number \(n\). Therefore, \(|N_1| - |N_4| = |N_2| - |N_6| = |N_3| - |N_5| = n\) must hold.

If a voting paradox: \(x_1Px_2, x_2Px_3\) and \(x_3Px_1\) respectively satisfy

\[
\begin{align*}
(N_1 & + N_3) + (N_4 + N_6) > (N_2 + N_5) + (N_3 + N_4) > (N_3 + N_5) + (N_2 + N_4) > (N_1 + N_2 + N_4) + (N_3 + N_5 + N_6) \quad (1) \\
N_1 & + N_3 + N_5 + N_4 + N_6 > N_2 + N_4 + N_5 + N_6 + N_3 + N_2 > N_3 + N_5 + N_6 + N_4 \quad (2) \\
N_1 & + N_3 + N_5 + N_4 + N_6 > N_2 + N_3 + N_4 + N_6 + N_3 + N_2 > N_3 + N_4 + N_5 + N_6 \quad (3) \\
N_1 & + N_3 + N_5 + N_4 + N_6 > N_2 + N_3 + N_4 + N_5 + N_6 + N_2 \quad (4)
\end{align*}
\]

From (1) + (2) + (3) + (4),

\[
\begin{align*}
(N_1 & + N_3 + N_5 + N_4 + N_6) > (N_2 + N_4 + N_5 + N_6 + N_3 + N_2) > (N_3 + N_4 + N_5 + N_6 + N_2) \quad (5)
\end{align*}
\]

Individual preference relations in this case are:

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\forall i \in N_1):</td>
<td>(\forall i \in N_6):</td>
</tr>
<tr>
<td>(x_1P_3x_5P_3x_5P_5x_4)</td>
<td>(x_1P_3x_5P_3x_5P_5x_2)</td>
</tr>
<tr>
<td>(\forall i \in N_4):</td>
<td>(\forall i \in N_3):</td>
</tr>
<tr>
<td>(x_2P_3x_5P_3x_5P_5x_1)</td>
<td>(x_2P_3x_5P_3x_5P_5x_4)</td>
</tr>
<tr>
<td>(\forall i \in N_6):</td>
<td>(\forall i \in N_1):</td>
</tr>
<tr>
<td>(x_3P_3x_5P_3x_5P_5x_2)</td>
<td>(x_3P_3x_5P_3x_5P_5x_4)</td>
</tr>
<tr>
<td>(\forall i \in N_2):</td>
<td>(\forall i \in N_9):</td>
</tr>
<tr>
<td>(x_4P_3x_5P_3x_5P_5x_3)</td>
<td>(x_4P_3x_5P_3x_5P_5x_3)</td>
</tr>
</tbody>
</table>
By supposing $|N_1| - |N_2d| = |N_{10}| - |N_6| = |N_{17}| - |N_8| = |N_{19}| - |N_{18}| = n$ ($n$: any natural number), the voting paradox: $x_1P_{x2}P_{x3}P_{x4}P_{x1}\ldots$ is surely given by a cycle: $N_1 \rightarrow N_{19} \rightarrow N_{17} \rightarrow N_{10} \rightarrow N_1$ made by a cyclical chain of individual preferences (case 1).

When a voting paradox: $x_1P_{x2}P_{x3}P_{x4}P_{x1}\ldots$ exists, individual preference relations have to satisfy (1), (2), (3) and (4) and therefore (5). In order to assure the existence of a voting paradox, $|N_1| - |N_2d| = |N_{10}| - |N_6| = |N_{17}| - |N_8| = |N_{19}| - |N_{18}| = n$ ($n$: any natural number) has to hold by excluding the influence of the left-hand term in equation (5).

### 3.4.3 The voting paradox from any three sets of individuals

Similarly to Section 3.3, it is easy to recognise a voting paradox: $x_1P_{x2}P_{x3}P_{x4}P_{x1}\ldots$ when we pick any three sets of individuals in case 1. As is understood from the explanation in Section 3.3, $|N_1| - |N_2d| = |N_{10}| - |N_6| = |N_{17}| - |N_8| = |N_{19}| - |N_{18}| = n$ ($n$: any natural number) must hold.

### 3.4.4 The voting paradox from any number of alternatives

The discussion so far for the cases of 3 and 4 alternatives is applied to a case of any finite number of alternatives because the strategy of analysis is formally the same.

Suppose the set of all alternatives: $S = \{x_1, x_2,\ldots, x_m\}$ ($3 \leq m < +\infty$). In order to get a voting paradox: $x_1P_{x2}P_{x3}P_{x4}P_{x1}\ldots$, we need to have the following groups of individual preference relations.

<table>
<thead>
<tr>
<th>General case 1</th>
<th>General case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall i \in S_{a1}$: $x_1P_{x2}P_{x3}x_4\ldots x_{m-1}P_{x_m}$</td>
<td>$\forall i \in S_{a2}$: $x_1P_{x2}P_{x3}x_4\ldots x_{m-1}P_{x_m}$</td>
</tr>
<tr>
<td>$\forall i \in S_{a2}$: $x_2P_{x3}P_{x4}x_5\ldots x_{m-1}P_{x_1}$</td>
<td>$\forall i \in S_{a2}$: $x_2P_{x3}P_{x4}x_5\ldots x_{m-1}P_{x_1}$</td>
</tr>
<tr>
<td>$\forall i \in S_{a3}$: $x_3P_{x4}x_5\ldots x_{m-1}P_{x_1}P_{x_2}$</td>
<td>$\forall i \in S_{a3}$: $x_3P_{x4}x_5\ldots x_{m-1}P_{x_1}P_{x_2}$</td>
</tr>
<tr>
<td>$\forall i \in S_{a4}$: $x_4P_{x5}x_6\ldots x_{m-1}P_{x_1}P_{x_2}P_{x_3}$</td>
<td>$\forall i \in S_{a4}$: $x_4P_{x5}x_6\ldots x_{m-1}P_{x_1}P_{x_2}P_{x_3}$</td>
</tr>
</tbody>
</table>

In order to get a voting paradox here, $|S_{a1}| - |S_{a2}| = |S_{a2}| - |S_{a3}| = \ldots = |S_{a4}| - |S_{a5}| = n$ ($n$: any natural number) must hold. If a voting paradox: $x_1P_{x2}P_{x3}x_4\ldots x_{m-1}P_{x_1}P_{x_2}\ldots$ holds, individual preference relations have to satisfy: $|S_{a1}| - |S_{a2}| = |S_{a2}| - |S_{a3}| = \ldots = |S_{a4}| - |S_{a5}| = n$ ($n$: any natural number) to assure the voting paradox.

Let us show that any three sets of $m$ sets of individual preference relations are enough to give a voting paradox: $x_1P_{x2}P_{x3}\ldots x_{m-1}P_{x_m}P_{x1}\ldots$ in general case 1.

From any three sets of individual preference relations $S_{aj}$, $S_{aj}$ and $S_{ak}$ ($1 \leq I < j < k \leq m$), the following are induced.

$\forall i \in S_{aj}$: $x_1P_{x2}x_3\ldots x_{j-1}P_{x_j}x_{j+1}\ldots x_{m-1}P_{x_m}x_1\ldots P_{x_{j-2}}P_{x_{j-1}}$

$\forall i \in S_{ak}$: $x_1P_{x2}x_3\ldots x_{k-1}P_{x_k}x_{k+1}\ldots x_{m-1}P_{x_m}x_1\ldots P_{x_{k-2}}P_{x_{k-1}}$
By examining binary comparison based on the simple majority decision rule, we have: 
\[ x_{i-1}; x_i = 2:1, x_{j-1}; x_j = 2:1 \text{ and } x_{k-1}; x_k = 2:1 \text{ and in the other cases: } x_l; x_l+1 = 3:0 \quad (l = 1, \ldots, i-2, i, \ldots, j-2, j, \ldots, k-2, k, \ldots m). \]
Obviously, a voting paradox: \( x_1P_2P\ldots x_{m-1}Px_1\ldots \) holds. Of course, \( |S_{a1}| - |S_{m1}| = |S_{a2}| - |S_{j1}| = \ldots = |S_{an}| - |S_{rn-1}| = n \quad (n: \text{any natural number}). \)

When a voting paradox: \( x_1P_2P\ldots x_{m-1}Px_1\ldots \) holds, it is assured if the relations among individual preference relations satisfy: \( |S_{a1}| - |S_{m1}| = |S_{a2}| - |S_{j1}| = \ldots = |S_{an}| - |S_{rn-1}| = n \quad (n: \text{any natural number}). \)

### 3.5 The necessary and sufficient conditions of the voting paradox

From the above explanation, the necessary and sufficient conditions assuring a voting paradox are summarised as follows:

- set of all alternatives: \( S = \{x_1, x_2, \ldots, x_m\} \quad (3 \leq m < +\infty) \)
- set of all individuals: \( N = \{1, 2, \ldots, n\} \quad (3 \leq n < +\infty) \)
- rule of individual preference relation: each individual has a quasi-ordering over all alternatives
- social choice function: social preference relations make a voting paradox consisting of all alternatives under the simple majority decision rule imposed on individual preference relations.

Then, the necessary and sufficient conditions for a voting paradox: \( x_1P_2P\ldots x_{m-1}Px_1\ldots \) are as follows.

For any two general cases 1 and 2 in Section 3.4.5, any three sets of individuals \( S_{ai}, S_{aj} \) and \( S_{ak} \) (1 \( \leq i, j, k \leq m; i, j \) and \( k \) are natural numbers and different from each other), \( |S_{ai}| - |S_{aj-1}| = |S_{aj}| - |S_{j-1}| = |S_{ak}| - |S_{k-1}| = n \quad (n: \text{any natural number}) \) is to hold.

Here, \( n \) is any natural number but it automatically becomes common for any combination of three sets of individuals in general cases 1 and 2.

### 4 Concluding remarks

As is often experienced, there are many social preferences which fall into a deadlock in the sense that there is no best element to choose. These issues are normally caused by a strict social preference cycle, a voting paradox (Linhart, 1998). This means that there are many social conflicts caused by different opinions in society. The appearance of a social conflict is theoretically expected from the result of Arrow’s impossibility theorem. It is difficult to reach a conclusion all concerned parties can accept in a social conflict.

Different from the traditional approach of social choice theory trying to seek rational social preference relations even in the case of irrational social ones, we examined in this paper another way to resolve a voting paradox by using the deep sense of ‘jan ken’, the simplest fist game. The reason why ‘jan ken’ is examined is because its value is really effective in daily decision-making, the thought of the three-way deadlock behind ‘jan ken’ offering a model of power structure in society which seems to explain more properly today’s internet society than social models such as the ‘vertically-structured society’ or
the ‘middle class society’ and, therefore, various social and human relations in society can be well explained by a model of ‘n-way deadlock’ coupled with the common consciousness of mutual dependency contained in ‘jan ken’ (Linhart, 1998). This consciousness is well represented by a maxim: ‘in a quarrel, both parties are to blame’. Since a quarrel normally has causes stemming from all sides concerned, it may be difficult to judge which side is to be blamed because all parties defend themselves. If we are serious about maintaining society peacefully, it may be better to make all the individuals concerned accept their own responsibility because such a judgment may not only prevent a similar conflict from happening later but also respect all parties’ dignity. It is very important to respect all individuals’ dignity and, therefore, it is not necessary to impose equal responsibility on all parties.

Meanwhile, clarification of the necessary and sufficient conditions of the voting paradox may be useful in recognising the emergence of a voting paradox or a social conflict in advance during the policy-making process. In this sense, it is necessary to collect individual preference relations in the traditional approach of social choice theory. The different point from tradition is that it is not necessary to daringly deduce rational social preference relations from originally irrational social ones. Instead, we are required to make a balance among individuals with conflicting opinions according to the importance of the quarrel, which is not discussed in this paper.

References


Notes

1 It may be reasonable to recognise these two impossibility theorems as twins.

2 ‘Jan ken’ is a game among children invented in Japan about 200 years ago (Linhart, 1998; Osada and Suyama, 1977; Kikuchi, 1905). This game uses three forms of hand, ‘guh’, ‘choki’ and ‘pah’. ‘Guh’ is shown by a fist, meaning a stone, ‘choki’ is shown by a V sign, meaning scissors and ‘pah’ is shown by the palm, meaning a sheet of paper. Since a stone cannot be cut by scissors but is wrapped by a sheet of paper, a stone is stronger than scissors but weaker than a sheet of paper. Scissors can cut a sheet of paper and, therefore, it is stronger than the latter.

3 Linhart is an Austrian Japanologist at the University of Vienna. His book on the cultural history of fist games (Linhart, 1998) seems to be the sole academic study on the fist game. According to his research, fist games had already existed in China in the ninth century. Japan introduced them from China in the ninth or tenth century. There have been many variations of fist games which were invented or developed in China and Japan. These games seem to have been introduced to Europe by Dutch traders in the 18th century. Although they are various and complicated games, they are fundamentally based on a three-way deadlock. Only ‘jan ken’ has survived until today because the rule is the simplest among fist games. Mathematicians are interested in the game and a famous mathematician, Nishimura Toosato, explained fist games in his book *Suugaku Yawa* (interesting topics in mathematics), published in 1761.

4 Another case is: right wingers are stronger than the police, which are stronger than the mafia. The police can compulsorily investigate the mafia and can arrest members by using various laws against the mafia, but are weaker than right wingers, who reveal police scandals to the public. Since right wingers can hold the effectiveness of their appeal backed by the power of the mafia, they are weaker than the mafia. Yasuo Takei once made the biggest non-bankin Japan, ‘Takefuji’, based on the delicate balance of this three-way deadlock, though the company finally went into bankruptcy by failing to maintain a balance among the three.

5 When Asano Takuminokami slashed Kira Kozukenosuke using a short sword in fury against the latter’s vicious treatment of the former at Edo Castle, the Tokugawa Shogunate held the former responsible for the incident and immediately forced him to commit *seppuku*, ritual suicide, on the same day but did not blame the latter at all. This judgment by the Tokugawa Shogunate was against the traditional rule ‘in a quarrel, both parties are to blame’. Ako’s 47 *samurai* were infuriated by this uneven ruling and carefully planned their revenge, assassinating *Kira Kozukenosuke* a year and half later.

6 This mind is quite rare in the westerner, which may be one of the serious factors behind the repetition of historical revenge: France imposed a huge amount of compensation on Germany in the Treaty of Versailles after the First World War as revenge for defeat in the Franco-Prussian War and this harsh requirement may well have been a trigger leading to Hitler’s dictatorship in Germany.

7 We are discussing here the past influence of Christianity and Judaism, which still remains deep in the psyche of westerners. Since, however, the Pope is the chairman of the World Conference of Religions for Peace and admits the existence of religions other than Christianity today, it may be said that Christianity is gradually throwing away its strict ideas of exclusive monotheism.
According to Black (1958), the voting paradox was first discussed by Condorcet in 1785. For any two alternatives \( x, y \), \( xPy \) means \( x \) is judged to be better than \( y \). Quasi-transitivity means that if \( xPy \) and \( yPz \) holds for any three alternatives \( x, y, z \), then \( xPz \) holds as well. The situation in which preference relations always satisfy quasi-transitivity coupled with reflexivity is called quasi-ordering.

A cyclical chain of individual preferences is easily understood by rolling up the left-hand side of Table 1 like a cylinder. This is easily recognised by supposing a cyclical chain of individual preferences, say, for any individuals \( i, j, k, h, x_1P_ix_2P_jx_3, x_1P_jx_2P_kx_3, x_2P_jx_3P_kx_1 \) and \( x_3P_kx_1P_jx_2 \). A voting paradox is not given because of \( x_1x_3 = 2:2 \) in this case.