On convergence of difference schemes for Dirichlet IBVP for two-dimensional quasilinear parabolic equations

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Abstract: For Dirichlet initial boundary value problem (IBVP) for two-dimensional quasilinear parabolic equations, a monotone linearised difference scheme is constructed. The uniform parabolicity condition $0 < k_1 \leq k_\alpha(u) \leq k_2, \alpha = 1, 2$ is assumed to be fulfilled for the sign alternating solution $u(\mathbf{x}, t) \in \bar{D}(u)$ only in the domain of exact solution values (unbounded nonlinearity). On the basis of the proved new corollaries of the maximum principle not only two-sided estimates for the approximate solution y but its belonging to the domain of exact solution values are established. We assume that the solution is continuous and its first derivatives $\frac{\partial u}{\partial x_i}$ have discontinuities of the first kind in the neighbourhood of the finite number of discontinuity lines. No smoothness of the time derivative is assumed. Convergence of approximate solution to generalised solution of differential problem in the grid norm L_2 is proved.

Keywords: Convergence in the grid norm L2; Dirichlet IBVP; monotone linearised difference scheme; sign alternating solution; uniform parabolicity condition; domain of exact solution values; corollaries of the maximum principle; discontinuities of the first kind; no smoothness of the time derivative; generalised solution; initial boundary value problem; 2D quasilinear parabolic equation; unbounded nonlinearity; two-sided estimates.

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1 Introduction

When the solution of the given problem is smooth enough in the theory of finite difference schemes a fairly complete study of convergence has been performed and the estimates of accuracy have been obtained in the corresponding metrics. However, real physical processes, as a rule, take place in heterogeneous environments when various solution domains have various physical characteristics. Nowadays study of the computational methods accuracy depending on the smoothness of the sought solution is actual and in demand.

In Samarskii et al. (1987), a new apparatus for obtaining the estimates of accuracy of the method in which an order of the convergence rate is consistent with the smoothness of the given differential problem has been suggested

$$||y-u||_{W_2^s(\omega)} \leq M|h|^{k-s}||u||_{W_2^k(\Omega)},$$

where k, s are integer non-negative numbers, $k > s$, and $||\cdot||_{W_2^s(\omega)}, ||\cdot||_{W_2^k(\Omega)}$ are Sobolev norms on the set of functions of a discrete and continuous argument. In Lazarov et al. (1984) this estimate was generalised for real non-negative numbers k and s.

We consider some results on convergence of difference schemes with non-smooth input data for linear non-stationary problems of mathematical physics. In Ladyzhenskaya (1973) has proved a convergence of the solution of difference scheme for a linear hyperbolic equation of the second order to the generalised solution of the mixed differential problem not defining the order of convergence rate but assuming the existence of discontinuities of the first kind in the first derivatives of the solution. In study of the difference schemes with discontinuous coefficients the most important results have been obtained by Samarskii (2001). In Moskal'kov (1974), for linear wave equation under weak discontinuities of the solution Moskal'kov (1974) has proved the convergence in average to the generalised solution with the rate $O(\sqrt{h}+\sqrt{\tau})$. In Samarskii et al. (2002), the convergence of difference schemes with weights for linear parabolic equation with generalised solutions has been proved. In particular, the estimate of accuracy of the form of

$$||y-S_x^2 u|| \leq \left(\tau \left|\left|\frac{\partial u}{\partial t}\right|\right|_{L_2(Q_T)} + h^2 \left|\left|\frac{\partial^2 u}{\partial x^2}\right|\right|_{L_2(Q_T)}\right)$$

has been obtained. Here

$$S_x^2 v(x,t) = \frac{1}{h^2} \int\limits_x^{x+h} \int\limits_{\xi-h}^{\xi} v(\eta,t)\, d\eta\, d\xi.$$

Much more complicated case is investigation of an accuracy of the method for nonlinear and non-stationary mixed problems of mathematical physics (Akrivis et al., 1999; Amosov and Zlotnik, 1987, 1986). There are two approaches. The first one is used when certain properties imposed on nonlinear coefficients (boundedness and continuity, uniform ellipticity of the quasilinear operator, etc.) are fulfilled for all values $u \in \mathbb{R}$. In this case the solution is a parameter. We have the so-called problems with bounded nonlinearity. The convergence analysis of difference schemes for the problems of such type is considerably simplified since it coincides in essence with the investigation of the analogous problems in linear case. Unfortunately, nonlinearity of such type does not allow to consider a series of interesting applied problems (problems of gas dynamics, heat conductivity etc.).

Among the works on the investigation of the convergence of difference schemes approximating the problems with unbounded nonlinearity we would like to point out the works by Abrashin (1976a, 1976b) and Yakovlev (1983b, 1983a). Under the nonlinearities of such kind one can prove that a grid solution belongs to the neighbourhood of the exact solution values $\bar{D}_{\varepsilon_0}$ which can be small enough. This even for quasilinear equations leads to the necessity to investigate the method accuracy in the uniform norm not only for the solution of the difference scheme but for its first derivatives. In Matus (1985) and Matus and Stanishevskaya (1991), it was shown that such analysis demands extreme smoothness of the solution and does not allow to make a corresponding investigation of the generalised solutions. In Jovanović et al. (1999), for the quasilinear heat conductivity equation with unbounded nonlinearity a consistent estimate of accuracy for the solution of the difference scheme on generalised solutions from the class $W_2^{2,1}(Q_T)$ was obtained.

One of the main results of the given paper is the determination of remarkable property of difference schemes satisfying the maximum principle. As it appears, solution of such difference problems belongs to the domain of the values of exact solution or its small neighbourhood even in the class of generalised solutions of the given differential problem. Here we do not need to obtain a priori estimates of the method accuracy in the norm L_∞. The maximum principle allows not only to prove uniqueness and continuous dependence on the input data of the solution of the initial-boundary problems for parabolic and elliptic equations, but also, in contrast to the method of energy inequalities, to obtain a priori (upper) estimates of the solution in the uniform norm for such problems in arbitrary dimension with non-selfadjoint elliptic operator (Vladimirov, 1981, p.500). In the theory of difference schemes, the maximum principle is also used for studying the stability of a difference solution with respect to input data and its convergence to an exact solution of the problem in a uniform norm. Finite-difference methods that satisfy grid maximum principle are usually called monotone (Samarskii, 2001, p.228). Different classes of monotone difference schemes are developed and investigated for multidimensional linear convection-diffusion equations [see, e.g., the monograph (Samarskii and Vabishchevich, 1999, p.35)]. Monotone schemes play an important role in computations, since they allow to obtain a numerical solution without non-physical oscillations, even in the case of non-smooth solutions (Godunov, 1959).

Equally important are lower or, in general, two-sided estimates of the solutions of differential-difference problems. Such estimates are especially important in the study of the properties of computational methods for problems with unbounded nonlinearity, since in this case it is necessary to prove that discrete solution belongs to a neighbourhood of the exact solution (Matus, 2014; Matus et al., 2014). For linear problems, these estimates allow to find the boundaries of the change of the values of the desired solution through the input data of the problem (the coefficients of the equation and the right-hand side, initial and boundary conditions). In the nonlinear case, such estimates permit to prove the non-negativity of the exact solution, important in physical problems, and also to find conditions on the input data when the nonlinear problem is parabolic or elliptic.

The derivation of non-trivial estimates for the solutions of initial-boundary value problems is based on a special technique first applied by Ladyzhenskaya (1958) [see also the monograph (Ladyzhenskaya et al., 1968, p.22)]. It consists in a parameter-containing change of variable with subsequent minimisation or maximisation of certain functions with respect to this parameter. These extremal values give the corresponding estimates of the solution. Naturally, such estimates are required in computational algorithms, which give an approximate solution of initial-boundary value problems. In the theory of difference schemes (Samarskii, 2001, p.229) there is well-designed for linear problems apparatus of the grid maximum principle, which allows obtaining two-sided estimates of an approximate solution. Note, that these estimates for solutions of difference problems are less precise (Farago and Horvath, 2006) than the corresponding estimates for the solution of differential problems (Ladyzhenskaya et al., 1968, p.22). For linear problems and problems with bounded nonlinearity, similar estimates for the finite element method were obtained in works of I. Farago and co-authors (see, e.g., Farago et al., 2012). In Matus and Poliakov (2017), for linearised difference scheme on uniform grid that approximates the

Dirichlet problem for multidimensional quasilinear parabolic equation with unbounded nonlinearity two-side pointwise estimates of the solution consistent with the corresponding estimates for the differential problem are established. Note that the proven two-sided estimates do not depend on the diffusion coefficients. Such estimates are directly used for proving of the convergence of the considered difference scheme in the grid norm L_2. An example of the calculation by the Crank-Nicolson difference scheme is given; it shows that the violation of consistency conditions of differential and difference estimates leads to non-monotonic numerical solutions. On uniform grids for the one-dimensional convection-diffusion equation, the simplest approximations of first-order derivatives lead to difference schemes that use upwind differences. Such schemes are certainly monotone, but, as a rule, they have only first order of approximation. On the other hand, difference schemes are constructed using the central difference relations to approximate the convective term. In spite of the second order of approximation, the monotonicity of these schemes is satisfied under constraints on the spatial grid. Using the regularisation principle (Samarskii et al., 1996, 1999), unconditionally monotone difference schemes of the second order of approximation for convection-diffusion problems were constructed on uniform grids.

In Matus et al. (2017) the previous results are generalised for construction of unconditional monotone difference scheme of second-order of local approximation on uniform grids in space for the 1D quasilinear parabolic equation with unbounded nonlinearity. The obtained results are generalised for the case of non-uniform spatial grids.

This article is devoted to the development of the technique of Ladyzhenskaya (1958) and its application to difference schemes for 2D quasilinear parabolic equation with Dirichlet boundary conditions with unbounded nonlinearity and obtaining two-sided estimates of their solutions completely consistent with estimates of the solutions of the corresponding differential problem. Note that the proven two-sided estimates do not depend on the value of diffusion coefficients. We assume that the exact solution generalised solution satisfying the balance equation in the non-stationary case. In particular, $u(\mathbf{x},t) \in C(Q_T)$ is continuous and its first derivatives $\frac{\partial u}{\partial x_\alpha}$ have discontinuity of the first kind in the neighbourhood of the finite number of discontinuity lines. No smoothness of the time derivative is assumed. On the basis of the proved new corollaries of the maximum principle the convergence of the approximate solution to the generalised solution of differential problem in the grid norm L_2 is proved.

2 Two-sided estimates of the solution of initial-boundary value problems for parabolic equations with unbounded nonlinearity.

In a parallelepiped $\bar{Q}_T = \bar{\Omega} \times [0,T], \quad \Omega = \{\mathbf{x} : 0 < x_\alpha < l_\alpha, \quad \mathbf{x} = (x_1, x_2), \ \alpha = 1,2\}$ let us consider the following problem for a quasilinear diffusion equation:

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{2} \frac{\partial W_\alpha}{\partial x_\alpha} + f(\mathbf{x},t), \quad \mathbf{x} \in \Omega, \quad t \in (0,T], \tag{1}$$

with the initial condition

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{2}$$

and the Dirichlet boundary conditions

$$u(\mathbf{x},t) = \mu(\mathbf{x},t), \quad (\mathbf{x},t) \in \partial\Omega \times [0,T], \tag{3}$$

where $W_\alpha = k_\alpha(u)\dfrac{\partial u}{\partial x_\alpha}$, $\alpha = 1, 2$, functions f, u_0, μ, are continuous on Q_T and the corresponding matching conditions are fulfilled.

Let $u(\mathbf{x}, t)$ be a solution of problem (1)–(3), and let $D_u = [m_1, m_2]$ be a segment containing the set of its values: $m_1 \le u(\mathbf{x}, t) \le m_2$. The functions $k_\alpha = k_\alpha(u) > 0 \in C^2(D_u)$, $\alpha = 1, 2$, for $u \in D_u$ are sufficiently smooth. Suppose that the function $u(\mathbf{x}, t)$ is continuous in the domain $\bar{Q}_T$, has continuous derivatives from equation (1) in Q_T, and satisfies (1), initial (2) and boundary (3) conditions in Q_T. Moreover, let $u(\mathbf{x}, t) \in C^{4,2}(Q_T)$. Let $t_1 \le T$ and $Q_{t_1} = \{(\mathbf{x}, t) \in Q_T : t \le t_1\}$.

The theorem below is of a big importance for our further investigations.

Theorem 1 (Ladyzhenskaya, 1958): For the solution $u(\mathbf{x}, t)$ of the problem (1)–(3) at any point $(\mathbf{x}, t_1) \in \bar{Q}_T$ the following two-sided estimate is valid:

$$u(\mathbf{x}, t_1) \ge m_1$$
$$= \sup_{\lambda > \lambda_0} \left(e^{\lambda t_1} \min \left\{ 0, \min_{(\mathbf{x},t)\in Q_{t_1}} e^{-\lambda t} \{\mu(\mathbf{x}, t), u_0(\mathbf{x})\}, \min_{(\mathbf{x},t)\in Q_{t_1}} \frac{f(\mathbf{x}, t)e^{-\lambda t}}{\lambda} \right\} \right), \quad (4)$$

$$u(\mathbf{x}, t_1) \le m_2$$
$$= \inf_{\lambda > \lambda_0} \left(e^{\lambda t_1} \max \left\{ 0, \max_{(\mathbf{x},t)\in Q_{t_1}} e^{-\lambda t} \{\mu(\mathbf{x}, t), u_0(\mathbf{x})\}, \max_{(\mathbf{x},t)\in Q_{t_1}} \frac{f(\mathbf{x}, t)e^{-\lambda t}}{\lambda} \right\} \right). \quad (5)$$

We will give a sketch of the proof of the upper bound (5) in a convenient for us form, since analogous arguments will be used below in the difference case. Let us take the auxiliary function $v(\mathbf{x}, t) = u(\mathbf{x}, t)e^{-\lambda t}$, where $\lambda > \lambda_0$ is a parameter. Let $(\mathbf{x^0}, t^0)$ be the maximum point of the function v in the parallelepiped Q_{t_1} and $v^0 = v(\mathbf{x^0}, t^0)$. Only the following three possibilities exist:

1 The maximum v^0 is non-positive (then $v(\mathbf{x}, t) \le 0$, $(\mathbf{x}, t) \in Q_{t_1}$)).

2 The point $(\mathbf{x^0}, t^0)$ is on the boundary of the parallelepiped Q_{t_1} (then the following inequality is valid: $v(\mathbf{x}, t) \le \max_{(\mathbf{x},t)\in Q_{t_1}} e^{-\lambda t} \{\mu(\mathbf{x}, t), u_0(\mathbf{x})\}$, $(\mathbf{x}, t) \in Q_{t_1}$).

3 The maximum v^0 is positive, and the point $(\mathbf{x^0}, t^0)$ is the interior point of the parallelepiped Q_T.

In case 3 at the maximum point $(\mathbf{x^0}, t^0)$ the following relations are fulfilled

$$\frac{\partial v(\mathbf{x}^0, t^0)}{\partial t} \ge 0, \quad \frac{\partial v(\mathbf{x}^0, t^0)}{\partial x_\alpha} = 0, \quad \frac{\partial^2 v(\mathbf{x}^0, t^0)}{\partial x_\alpha^2} \le 0, \quad \alpha = 1, 2.$$

Therefore from the equation

$$\frac{\partial v}{\partial t} e^{\lambda t} + \lambda v e^{\lambda t} = e^{\lambda t} \sum_{\alpha=1}^{2} \frac{\partial^2 v}{\partial x_\alpha^2} k_\alpha(v e^{\lambda t}) + e^{2\lambda t} \sum_{\alpha=1}^{2} \frac{\partial k_\alpha}{\partial u} \left(v e^{\lambda t}\right) \left(\frac{\partial v}{\partial x_\alpha}\right)^2 + f$$

it follows that the inequality $\lambda v^0 e^{\lambda t} \le f$ is valid, that yields to the following estimate for the auxiliary function v:

$$v(\mathbf{x}, t) \le \max_{(\mathbf{x},t)\in Q_{t_1}} \frac{f(\mathbf{x}, t)e^{-\lambda t}}{\lambda}, \quad (\mathbf{x}, t) \in Q_{t_1}.$$

Combining cases 1–3 and returning to the original function u, we obtain the upper bound (5). Analogous arguments for the minimum point give a lower estimate (4).

3 Definition of generalised solution

According to Karchevsky and Pavlova (2008), function $u(\mathbf{x}, t)$ is called a generalised solution of problem (1)–(3), if for each infinitely differentiable function with compact support $\varphi(\mathbf{x}, t)$ the equality

$$\iiint\limits_{Q_T} \left[\frac{\partial u}{\partial t} - \sum_{\alpha=1}^{2} \frac{\partial W_\alpha}{\partial x_\alpha} - f \right] \varphi dx_1 dx_2 dt = 0 \tag{6}$$

takes place.

Such solutions, an existence of the derivative $\frac{\partial u}{\partial t}$ of which is not required, are often called a weak generalised solution. However, (6) contains implicitly an information about derivative (Karchevsky and Pavlova, 2008), i.e.

$$\frac{\partial u}{\partial t} \in L_2(0, T; H^{-1}(\Omega)).$$

If $u(\mathbf{x}, t) \in C(\bar{Q}_T)$, $\frac{\partial u}{\partial x_\alpha}$, $\quad \alpha = 1,\ 2$ are piecewise continuous functions then requirement (6) is equivalent to that along any surface C constraining the volume $Q^{'} \in \bar{Q}_T$,

$$\iint\limits_{\partial Q_T} u dx_1 dx_2 - k_1(u) \frac{\partial u}{\partial x_1} dt dx_2 - k_2(u) \frac{\partial u}{\partial x_2} dt dx_1 = \iiint\limits_{Q_T} f dx_1 dx_2 dt. \tag{7}$$

In this sence we will use the concept “generalised solution.”

4 The maximum principle for difference schemes with variable sign input data

To obtain a difference analogue of differential estimates, we will use the maximum principle for difference schemes with variable sign input data Matus et al. (2017). Let in the n-dimensional Euclidean space a finite number of points of the grid Ω_h be given. To each point $x \in \Omega_h$ we put one and only one stencil $\mathcal{M}(x)$ – a subset of Ω_h, containing this point. The set $\mathcal{M}'(x) = \mathcal{M}(x) \setminus x$ is called *neighbourhood* of the point x. Let the functions $A(x)$, $B(x, \xi)$, $F(x)$ be given at $x \in \Omega_h$, $\xi \in \Omega_h$ and they take real values. Next, to each point $x \in \Omega_h$ corresponds equation (Samarskii, 2001, p.226)

$$A(x)\, y(x) = \sum_{\xi \in \mathcal{M}'(x)} B(x, \xi)\, y(\xi) + F(x), \quad x \in \Omega_h, \tag{8}$$

which is called a *canonical form* of the difference scheme at a point x. Together with the grid Ω_h, we will consider its arbitrary subset $\overline{\omega}_h$ and we will denote

$$\overline{\Omega}_h = \bigcup_{x \in \omega_h} \mathcal{M}(x). \tag{9}$$

We will assume the fulfilment of the following positivity conditions for the coefficients

$$B(x,\xi) \geq 0, \quad \xi \in \mathcal{M}'(x), \quad x \in \Omega_h, \tag{10}$$

$$D(x) = A(x) - \sum_{\xi \in \mathcal{M}'(x)} B(x,\xi) > 0, \quad x \in \Omega_h. \tag{11}$$

They guarantee the unique solvability of the difference scheme (8), as well as its monotonicity and stability (in the linear case) in the uniform norm with respect to small perturbation of input data. Let us note that if conditions (10)–(11) are met, then coefficients $A(x) > 0, \quad x \in \Omega_h$.

Theorem 2 (Matus et al., 2017): Suppose that the positivity conditions for coefficients (10)–(11) are fulfilled. Then the maximal and minimal values of the solution of the difference scheme (8) belong to the value interval of the input data

$$\min_{x \in \Omega_h} \frac{F(x)}{D(x)} \leqslant y(x) \leqslant \max_{x \in \Omega_h} \frac{F(x)}{D(x)}. \tag{12}$$

5 Finite-difference scheme and two-sided estimates of its solution

In the domain $\bar{\Omega}$ we introduce an uniform grid $\bar{Q}_T$

$$\bar{\omega}_{h_\alpha} = \{x_{\alpha,i_\alpha} = i_\alpha h_\alpha, i_\alpha = 0,1,...,N_\alpha; h_\alpha N_\alpha = l_\alpha\}, \quad \alpha = 1,2,$$

$$\bar{\omega}_\tau = \{t_n = n\tau, n = 0,1,...,N_0; \tau N_0 = T\}, \quad \bar{\omega}_\tau = \omega_\tau \cup \{t_{N_0} = T\},$$

$$\bar{\omega} = \bar{\omega}_{h_1} \times \bar{\omega}_{h_2} \times \bar{\omega}_\tau, \quad \omega_{t_n} = \{(\mathbf{x},t) \in \bar{\omega} : t \leq t_n\},$$

for which $\sum\limits_{i_\alpha=1}^{N_\alpha} h_\alpha^{i_\alpha} = l_\alpha, \quad \alpha = 1,2$. We denote by $\hat{\omega}_h$ the set of internal nodes of the grid $\hat{\bar{\omega}}_h$.

On a uniform grid $\omega = \hat{\omega}_h \times \hat{\omega}_\tau$, we use the linearised finite-difference scheme

$$y_t = (a_1(y)\hat{y}_{\bar{x}_1})_{x_1} + (a_2(y)\hat{y}_{\bar{x}_2})_{x_2} + \hat{f}, \tag{13}$$

$$y(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \bar{\omega}_h, \tag{14}$$

$$y = \mu. \tag{15}$$

As usual, the stencil functionals

$$a_\alpha(y) = 0.5(k_\alpha(y_{i_\alpha-1}) + k_\alpha(y_{i_\alpha})), \quad \alpha = 1,2,$$

are chosen from the second-order consistency condition

$$(a_\alpha(u)\hat{u}_{\bar{x}_\alpha})_{x_\alpha} - \frac{\partial}{\partial x_\alpha}\left(k_\alpha(u)\frac{\partial u}{\partial x_\alpha}\right) = O(h_\alpha^2 + \tau).$$

for the elliptic operator with respect to the spatial variables. Here and in what follows, we use the standard notation [2, p.65]

$$y = y^n_{i_1 i_2} = y(x_{1,i_1}, x_{2,i_2}, t_n), \quad y_t = \frac{\hat{y} - y}{\tau}, \quad \hat{y} = y^{n+1}_{i_1 i_2},$$
$$v_{\bar{x}_\alpha} = \frac{v_{i_\alpha} - v_{i_\alpha - 1}}{h_\alpha}, \quad v_{x_\alpha} = \frac{v_{i_\alpha + 1} - v_{i_\alpha}}{h_\alpha}.$$

of the theory of finite-difference schemes.

6 Two-sided and a priori estimates of approximate solution.

Theorem 3: For the solution $y(\mathbf{x}, t)$ of the problem (13)–(15) at any point $(\mathbf{x},\ t_n) \in \omega$ the following two-sided estimate is valid:

$$y(\mathbf{x}, t_n) \geq m_{1,\tau}$$
$$= \sup_{\lambda > 0} \left(e^{\lambda t_n} \min \left\{ 0, \min_{(\mathbf{x},t) \in \omega_{t_n}} e^{-\lambda t} \left\{ \mu(\mathbf{x}, t), u_0(\mathbf{x}) \right\}, \min_{(\mathbf{x},t) \in \omega_{t_n}} \frac{\tau f(\mathbf{x}, t) e^{-\lambda t}}{e^{\lambda \tau} - 1} \right\} \right) \tag{16}$$

$$y(\mathbf{x}, t_n) \leq m_{2,\tau}$$
$$= \inf_{\lambda > 0} \left(e^{\lambda t_n} \max \left\{ 0, \max_{(\mathbf{x},t) \in \omega_{t_n}} e^{-\lambda t} \left\{ \mu(\mathbf{x}, t), u_0(\mathbf{x}) \right\}, \max_{(\mathbf{x},t) \in \omega_{t_n}} \frac{\tau f(\mathbf{x}, t) e^{-\lambda t}}{e^{\lambda \tau} - 1} \right\} \right) \tag{17}$$

Proof: Let us prove the upper bound (17). We take the auxiliary function $z = z(\mathbf{x}, t_n) = y(\mathbf{x}, t_n) e^{-\lambda t_n}$, where $\lambda > 0$ is a parameter. Let $(\mathbf{x^0}, t^0)$ be the maximum point of the function z in the grid domain ω_{t_n} and $z^0 = z(\mathbf{x^0}, t^0)$. Only the following three possibilities exist:

1 The maximum z^0 is non-positive (then $z(\mathbf{x}, t) \leq 0, \quad (\mathbf{x}, t) \in \omega_{t_n}$).

2 The point $(\mathbf{x^0}, t^0)$ is on the boundary of the grid domain ω_{t_n} (then the following inequality is valid: $z(\mathbf{x}, t) \leq \max\limits_{(\mathbf{x},t) \in \omega_{t_n}} e^{-\lambda t_n} \left\{ \mu(\mathbf{x}, t), u_0(\mathbf{x}) \right\}, \quad (\mathbf{x}, t) \in \omega_{t_n}$);

3 The maximum z^0 is positive, and the point $(\mathbf{x^0}, t^0)$ is the interior point of the grid domain ω_{t_n}.

In case 3 at the maximum point $(\mathbf{x^0}, t^0)$ the following relations are fulfilled:

$$\frac{\hat{z} e^{\lambda \tau} - z}{\tau} = (a_1(y) \hat{z}_{\bar{x}_1})_{x_1} e^{\lambda \tau} + (a_2(y) \hat{z}_{\bar{x}_2})_{x_2} e^{\lambda \tau} + e^{-\lambda t_n} \hat{f}.$$

We rewrite this equation in the canonical form

$$C^n_{i_1 i_2} z^{n+1}_{i_1 i_2} = A^n_{1,i_1 i_2} z^{n+1}_{i_1 - 1 i_2} + B^n_{1,i_1 i_2} z^{n+1}_{i_1 + 1 i_2} + A^n_{2,i_1 i_2} z^{n+1}_{i_1 i_2 - 1}$$
$$+ B^n_{2,i_1 i_2} z^{n+1}_{i_1 i_2 + 1} + K^n_{i_1 i_2} z^n_{i_1 i_2} + F^n_{i_1 i_2},$$

$$C^n_{i_1i_2} = \frac{e^{\lambda\tau}}{\tau} + \frac{e^{\lambda\tau}}{h_1^2}(a_{1,i_1+1i_2} + a_{1,i_1i_2}) + \frac{e^{\lambda\tau}}{h_2^2}(a_{2,i_1i_2+1} + a_{2,i_1i_2}),$$

$$A^n_{1,i_1i_2} = \frac{e^{\lambda\tau}}{h_1^2}a_{1,i_1i_2}, \quad B^n_{1,i_1i_2} = \frac{e^{\lambda\tau}}{h_1^2}a_{1,i_1+1i_2}, \quad A^n_{2,i_1i_2} = \frac{e^{\lambda\tau}}{h_2^2}a_{2,i_1i_2},$$

$$B^n_{2,i_1i_2} = \frac{e^{\lambda\tau}}{h_2^2}a_{2,i_1i_2+1}, K^n_{i_1i_2} = \frac{1}{\tau}, \quad F^n_{i_1i_2} = f^{n+1}_{i_1i_2}e^{-\lambda t_n},$$

and introduce the coefficients $D^n_{i_1i_2}$, $i_\alpha = \overline{1, N_\alpha}$, $\alpha = 1, 2$, as follows:

$$D^n_{i_1i_2} = C^n_{i_1i_2} - A^n_{1,i_1i_2} - A^n_{2,i_1i_2} - B^n_{1,i_1i_2} - B^n_{2,i_1i_2} - K^n_{i_1i_2},$$
$$i_\alpha = \overline{1, N_\alpha - 1}, \quad \alpha = 1, 2.$$

Note that for $t_n = 0$ $y \in D_u$. We carry out the proof by induction over time layers. Since

$$D^n_{i_1i_2} = \frac{\tau}{e^{\lambda\tau} - 1} > 0$$

for $\lambda\tau > 0$, we see that the assumptions of Theorem 2 are satisfied for $n = 1$ n = 1 and the estimate

$$z^n_{i_1i_2} \le \frac{\tau}{e^{\lambda\tau} - 1} \max_{(\mathbf{x},t)\in\omega_{t_n}} fe^{-\lambda t},$$

holds by inequality (10). We combine cases 1–3 and obtain the inequality

$$z \le \max\left\{0, \max_{(\mathbf{x},t)\in\omega_{t_n}} e^{-\lambda t}\{\mu(\mathbf{x}, t), u_0(\mathbf{x})\}, \frac{\tau}{e^{\lambda\tau} - 1} \max_{(\mathbf{x},t)\in\omega_{t_n}} f(\mathbf{x}, t)e^{-\lambda t}\right\}.$$

Now we return to the original function y and obtain the upper bound (24). Similar computations for the minimum give the lower bound (23). We have proved the induction assumption. Note that, by the results obtained above $y^1 \in D_u$ and the stencil functionals $a_1(y^1),\ a_2(y^1)$ satisfy the conditions of parabolicity on the solution (positivity). The argument for the inductive step differs from that for the induction assumption only in the notation of indices. The proof of the theorem is complete. □

7 Problem for the method error $z = y - u$

By virtue of nonlinearity of the investigated scheme this problem is not trivial. Really subtracting from the difference equation the consistency error equation

$$u_t = \sum_{\alpha=1}^{2}(a_\alpha(u)\hat{u}_{\bar{x}_\alpha})_{x_\alpha} + f - \psi,$$

we get two equivalent forms for the method error equation

$$z_t = \sum_{\alpha=1}^{2}((a\alpha(y)\hat{z}_{\bar{x}_\alpha})_{x_\alpha} + ((a_\alpha(y) - a_\alpha(u))\hat{u}_{\bar{x}_\alpha})_{x_\alpha}) + \psi, \tag{18}$$

$$z_t = \sum_{\alpha=1}^{2} ((a_\alpha(u)\hat{z}_{\bar{x}_\alpha})_{x_\alpha} + ((a_\alpha(y) - a_\alpha(u))\hat{y}_{\bar{x}_\alpha})_{x_\alpha}) + \psi. \tag{19}$$

The corresponding equations for initial and boundary conditions

$$z(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \bar{\omega}_h, \quad z_{\bar{\omega}\cap\partial Q_T} = 0, \quad t \in \omega_\tau, \tag{20}$$

should be added to the given equations. Although these problems are equivalent but in statement (19) we need a preliminary information about the local behaviour of the difference derivative of the approximate solutions. Obtaining of a priori estimates for the derivatives of approximate solution is not a simple task. Therefore further a priori estimates of the method accuracy we will get for the problem of the form of (18), (20). For this statement we do not need an information about the behaviour of the derivatives of the approximate solution.

Remark 1: In case of the fully implicit difference scheme we would have to prove the convergence of the iterational process and thus to analyse the behaviour of the derivatives of the approximate solution.

Further we will use usual scalar products and norms in the space of the grid functions

$$(u, v) = \sum_{i_1=1}^{N_1-1} \sum_{i_2=1}^{N_2-1} h_1 h_2 u_{i_1 i_2} v_{i_1 i_2}, \quad \|u\| = \sqrt{(u, u)},$$

$$(u, v)_\alpha = \sum_{i_\alpha=1}^{N_\alpha} \sum_{i_{3-\alpha}=1}^{N_{3-\alpha}-1} h_1 h_2 u_{i_1 i_2} v_{i_1 i_2}, \quad \|u\|_\alpha = \sqrt{(u, u)_\alpha}, \quad \alpha = 1, 2.$$

8 Main energetic identity

Taking the scalar product of the difference equation (18) and $2\tau\hat{z}$ and using the formula of summation by parts

$$(u, v_x) = -(u_{\bar{x}}, v] + u_N v_N - u_0 v_1,$$

the identity

$$z^{n+1} = \frac{1}{2}(z^{n+1} + z^n) + \frac{\tau}{2} z_t,$$

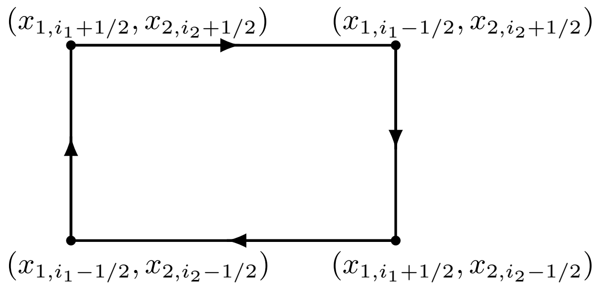
we get an energetic relation

$$\tau^2 ||z_t||^2 + ||z^{n+1}||^2 + \sum_{\alpha=1}^{2} 2\tau (a_\alpha(y), \hat{z}^2_{\bar{x}_\alpha}]$$
$$= ||z^n||^2 + \sum_{\alpha=1}^{2} 2\tau (a_\alpha(y) - a_\alpha(u), \hat{u}_{\bar{x}_\alpha} \hat{z}_{\bar{x}_\alpha}] + 2\tau(\hat{z}, \psi). \tag{21}$$

9 Presentation of the consistency error in the divergence form: estimate of the method accuracy

Lack of sufficient smoothness of the generalised solution of the differential problem requires use of the negative norms for the consistency error ψ both in space and time. For getting a divergent form of statement we will choose as a contour C in (7) an elementary cell (see Figure 1).

Figure 1 Elementary cell



We will get an expression

$$\left[\frac{1}{h_1h_2}\int\limits_{x_{1,i_1-\frac{1}{2}}}^{x_{1,i+\frac{1}{2}}}\int\limits_{x_{2,i_2-\frac{1}{2}}}^{x_{i_2+\frac{1}{2}}} u(\mathbf{x},t_n)dx_1dx_2\right]_{t_n} - \sum_{\alpha=1}^{2}\left[\frac{1}{\tau h_{2-\alpha}}\int\limits_{t_n}^{t_{n+1}}\int\limits_{x_{2-\alpha,i_{2-\alpha}-\frac{1}{2}}}^{x_{2-\alpha,i_{2-\alpha}+\frac{1}{2}}}\left(k_\alpha(u)\frac{\partial u}{\partial x_\alpha}\right)dt^{'}dx_{2-\alpha}\right]_{x,i_\alpha} = \frac{1}{\tau h_1h_2}\int\limits_{t_n}^{t_{n+1}}\int\limits_{x_{1,i_1-\frac{1}{2}}}^{x_{1,i+\frac{1}{2}}}\int\limits_{x_{2,i_2-\frac{1}{2}}}^{x_{i_2+\frac{1}{2}}} f dx_1dx_2dt.$$

Adding it to the expression for ψ, we obtain a divergent presentation for the consistency error

$$\psi^n = \eta_t + \sum_{\alpha=1}^{2}\xi_{\alpha,x_\alpha} + O(h_1^2 + h_2^2 + \tau),$$

$$\eta^n = \frac{1}{h_1h_2}\int\limits_{x_{1,i_1-\frac{1}{2}}}^{x_{1,i+\frac{1}{2}}}\int\limits_{x_{2,i_2-\frac{1}{2}}}^{x_{i_2+\frac{1}{2}}} u(\mathbf{x},t_n)dx_1dx_2 - u^n,$$

$$\xi_\alpha^n = a_\alpha(u)u_{\bar{x}_\alpha}^{n+1} - \frac{1}{\tau h_{2-\alpha}}\int\limits_{t_n}^{t_{n+1}}\int\limits_{x_{2-\alpha,i_{2-\alpha}-\frac{1}{2}}}^{x_{2-\alpha,i_{2-\alpha}+\frac{1}{2}}}\left(k_\alpha(u)\frac{\partial u}{\partial x_\alpha}\right)dt^{'}dx_{2-\alpha}.$$

A usage of the negative norms in space is carried out easily on the basis of the formulas of summation by parts or the Green's difference formula:

$$2\tau(\hat{z}, \xi_{\alpha,x_\alpha}) = -2\tau(\hat{z}_{\bar{x}_\alpha}, \xi_\alpha] \leq 2\tau\varepsilon_1||\hat{z}_{\bar{x}_\alpha}]|^2 + \frac{\tau\varepsilon_1^{-1}}{2}||\xi_\alpha]|^2. \tag{22}$$

The negative norms in time are based on the use of the more complicated combined norms. Now let us estimate in (21) the following scalar product:

$$\begin{aligned} 2\tau(\hat{z}, \eta_t) &= 2(\hat{z}, \hat{\eta}) - 2(z, \eta) - 2\tau(z_t, \eta) \leq 2(\hat{z}, \hat{\eta}) \\ &-2(z, \eta) + \tau^2||z_t||^2 + ||\eta||^2. \end{aligned} \tag{23}$$

Taking into account the inequalities $a_\alpha(y) \geq k_1$, (22), (23), from (21) we get the estimate

$$\begin{aligned} ||\hat{z}||_1^2 + \sum_{\alpha=1}^{2} 2\tau(k_1 - \varepsilon_1, \hat{z}_{\bar{x}_\alpha}^2] + \sum_{\alpha=1}^{2} 2\tau L(|z_{(0.5)}||\hat{u}_{\bar{x}_\alpha}|, |\hat{z}_{\bar{x}_\alpha}|] \leq ||z||_1^2 \\ + \sum_{\alpha=1}^{2} \frac{\tau\varepsilon_1^{-1}}{2}||\xi_\alpha]|^2 + ||\hat{\eta}||^2. \end{aligned} \tag{24}$$

Here $||z||_1^2 = ||z + \eta||^2$. In virtue of the assumptions on the smoothness of generalised solutions

$$|\hat{u}_{\bar{x}_\alpha}| \leq \frac{1}{h_\alpha} \int_{x_{\alpha,i_\alpha-1}}^{x_{\alpha,i_\alpha}} \left|\frac{\partial u}{\partial x_\alpha}\right| dx_\alpha \leq c_1, \quad c_1 = \text{const} > 0,$$

using the Cauchy inequality with ε for the following scalar product we get the estimate

$$\begin{aligned} &2\tau L(|z_{(0.5)}||\hat{u}_{\bar{x}_\alpha}|, |\hat{z}_{\bar{x}_\alpha}|] \leq 2\tau L c_1 \varepsilon_2 ||\hat{z}_{\bar{x}_\alpha}]|^2 \\ &\quad +0.5\tau L c_1 \varepsilon_2^{-1}||z||^2 \leq 2\tau L c_1 \varepsilon_2 ||\hat{z}_{\bar{x}_\alpha}]|^2 + \tau L c_1 \varepsilon_2^{-1}(||z||_1^2 + ||\eta||^2). \end{aligned}$$

Substituting the last inequality into the energetic relation (24) and choosing small enough $\varepsilon_1, \varepsilon_2$:

$$k_1 - \varepsilon_1 - c_1 L \varepsilon_2 \geq 0,$$

we come to the recurrent inequality

$$||z^{n+1}||_1^2 \leq (1 + \tau c_2)||z^n||_1^2 + \tau c_2||\psi^n||_{(-1)}^2, \tag{25}$$

where

$$||\psi^n||_{(-1)}^2 = \tau^{-1}(||\eta^n||^2 + ||\eta^{n+1}||^2) + \sum_{\alpha=1}^{2}||\xi_\alpha]|^2, \tag{26}$$

$c_2 > 0$ is a generalised constant independent of grid steps h, τ and grid functions y, z. For the estimation of the consistency error in the negative norm (26) we need the following lemma (Samarskii et al., 2002).

Lemma 4: If $f(x)$ is a piecewise continuous function over the interval $[a,\ b]$ then

$$\frac{1}{h}\int\limits_{x-h}^{x} f(\xi)d\xi = O(1), \quad x \in [a+h, b],$$

if $f(x)$ is a smooth function then

$$\frac{1}{h}\int\limits_{x-h}^{x} f(\xi)d\xi = f\left(x - \frac{h}{2}\right) + \Phi, \quad \Phi(x) = O(h^2).$$

Based on Lemma 4 at the points of smoothness of the integrands we have

$$\xi_\alpha^n = O(h_\alpha^2 + \tau), \quad \eta^n = O(h_1^2 + h_2^2),$$

and in the neighbourhood of the finite number of the discontinuity lines we have

$$\xi_\alpha^n = O(1), \quad \eta^n = O(h_1 + h_2).$$

In accordance with the structure η^n

$$||\eta^n||^2 = O(h_1^3 + h_2^3).$$

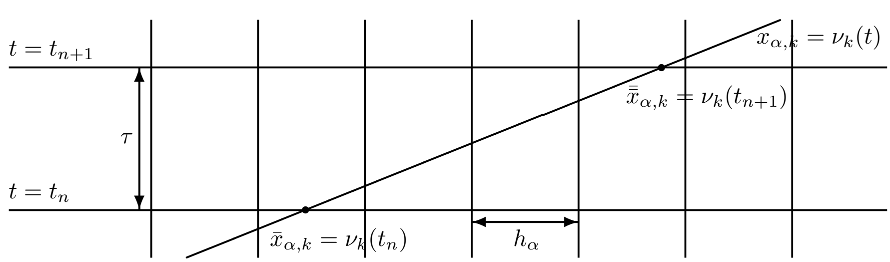
Likewise

$$||\eta^{n+1}||^2 = O(h_1^3 + h_2^3).$$

If the discontinuity line $x_{\alpha,k} = v_k(t)$ passes through the rectangle (see Figure 2), then based on Lemma 4 $\xi_\alpha = O(1)$. At each fixed $t = t_n$ the number of points for which $\xi_\alpha = O(1)$ in the neighbourhood of one discontinuity line of the derivative $\frac{\partial u}{\partial x_\alpha}$ can be estimated as follows:

$$m_1^k \le \frac{v_k(t_{n+1}) - v_k(t_n)}{h_\alpha} + 1 = \frac{v_k(t_{n+1}) - v_k(t_n)}{\tau}\frac{\tau}{h_\alpha} + 1.$$

Figure 2 Discontinuity line of the derivative $\frac{\partial u}{\partial x_\alpha}$



Since the number of the discontinuity lines of the derivative by hypothesis is finite, then the general number of points ω_p^n on the layer $t = t_n$, where $\xi_\alpha = O(1)$ is equal on the order of $O\left(\frac{h_\alpha+\tau}{h_\alpha}\right)$. Based on the foregoing, we have

$$||\xi_\alpha]|^2 = O(h_\alpha + \tau).$$

Finally, for the consistency error in the negative norm we get

$$||\psi||^2_{(-1)} = O(\tau^{-1}(h_1^3 + h_2^3) + h_1 + h_2 + \tau).$$

If

$$\tau \geq \max\{h_1, h_2\}, \tag{27}$$

then

$$||\psi||_{(-1)} = O(\sqrt{h_1 + h_2} + \sqrt{\tau}). \tag{28}$$

Else if

$$\tau \leq \max\{h_1, h_2\}, \tag{29}$$

then we again obtain (28).

Now using the difference analogue of the Gronwall lemma Samarskii et al. (2002), from (25) we find the estimate

$$||z^{n+1} + \eta^{n+1}||^2 \leq e^{\tau c_2}||z^n + \eta^n||^2 + \tau c_2||\psi||^2_{(-1)} \leq e^{\tau(n+1)c_2}||\eta^n||^2$$
$$+ \sum_{k=0}^{n} \tau e^{\tau(n-k)}||\psi^k||^2_{(-1)}.$$

Taking into account (28) provided (27) the final estimate of the method accuracy in the grid norm L_2 takes the form

$$||y^n - u^n|| \leq c_2(\sqrt{h_1 + h_2} + \sqrt{\tau}), \quad n = 0, 1, ..., N_0. \tag{30}$$

Theorem 5: Let the exact solution $u(\mathbf{x}, t) \in C(Q_T)$ of the problem (1)–(3) be continuous and its first derivatives $\frac{\partial u}{\partial x_\alpha}$ have discontinuity of the first kind in the neighbourhood of the finite number of discontinuity lines and satisfies (7). Then the solution of finite-difference scheme (13)–(15) convergences to the generalised solution of differential problem and a priory estimate (30) in the grid norm L_2 holds.

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