Lyapunov function-based non-linear control for two-wheeled mobile robots

Zareena Kausar*, Karl Stol and Nitish Patel

Department of Mechanical Engineering,
Faculty of Engineering,
The University of Auckland,
20 Symonds Street, Auckland, New Zealand
E-mail: zkau001@aucklanduni.ac.nz
E-mail: k.stol@auckland.ac.nz
E-mail: nd.patel@auckland.ac.nz
*Corresponding author

Abstract: This article presents a non-linear feedback control framework for two-wheeled mobile robots. The approach uses a constructive Lyapunov function which allows the formulation of a control law with asymptotic stability of the equilibrium point of the system and a computable stability region. The dynamic equations are simplified through normalisation and partial feedback linearisation. The latter allows linearisation of only the actuated coordinate. Description of the control law is complemented by the stability analysis of the closed loop dynamics of the system. The effectiveness of the method has been illustrated by its good performance and less control demand through simulations conducted for two control tasks: upright position stabilisation and velocity tracking for a statically unstable two wheeled mobile robot.

Keywords: Lyapunov function; non-linear control; stabilisation; two wheeled mobile robot.

Reference to this paper should be made as follows: Kausar, Z., Stol, K. and Patel, N. (2013) ‘Lyapunov function-based non-linear control for two-wheeled mobile robots’, *Int. J. Biomechatronics and Biomedical Robotics*, Vol. 2, Nos. 2/3/4, pp.172–183.

Biographical notes: Zareena Kausar received her Bachelor in Mechanical Engineering and Master of Engineering in Mechatronics from the University of Engineering and Technology, Lahore, Pakistan. She is currently working towards her PhD degree at the University of Auckland, New Zealand. She has held teaching appointments at the University of Engineering and Technology, Lahore and at the AIR University, Islamabad, Pakistan before pursuing her PhD degree. Her research interests include robotics, controls, system dynamics and mechatronics systems.

Karl Stol is a Senior Lecturer in the Department of Mechanical Engineering at the University of Auckland. He gained his PhD in Aerospace Engineering Science from the University of Colorado at Boulder, in developing structural dynamics models and control systems for modern wind turbines. Since joining the University of Auckland in 2004, he has supervised 98 project students and co-authored 70 refereed publications in the application of control techniques to wind energy and mobile robotics.

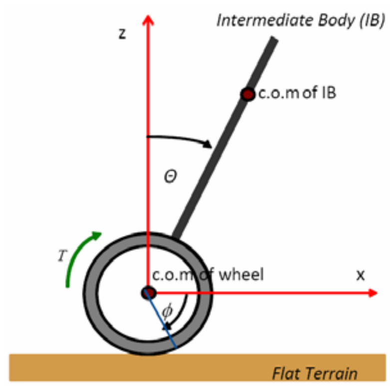
Nitish Patel received his BE degree from Mangalore University, India, and his PhD degree from the University of Auckland, New Zealand. He is currently a Lecturer at the University of Auckland. His research interests are in embedded systems, robotics, artificial neural networks and control systems. The focus of his work is on the hardware implementation of real-time systems for control. He is a member of the IEEE.

This paper is a revised and expanded version of a paper entitled ‘Nonlinear control design using Lyapunov function for two-wheeled mobile robots’ presented at International Conference in Mechatronics and Machine Vision in Practice (M2VIP), M2VIP 2012, Auckland, 28–30 November 2012.

1 Introduction

Control of two-wheeled mobile robots (TWRs) attracted attention of researchers in the last decade due to their inherent instability, non-linear and complex dynamics. These robots have wide applications in transportation and industry (Robonica, 2012; Hitachi, 2012; Segway, 2012). A TWR consists of two drive wheels, joined by an axle, and a robot body called the intermediate body (IB). The statically unstable (Salerno and Angeles, 2007) configuration of TWR, shown in Figure 1, is more challenging as compared to a statically stable (Kausar et al., 2010). A TWR has been termed statically unstable if the centre of mass of the body lies above the axle. The robot wheels are actuated by DC motors and the IB rotates freely. As there is no actuator dedicated directly to control the angular acceleration of vertical IB, the system is under actuated. The control of under actuated devices is theoretically more complex than that of fully actuated systems. The underlying control objective of this under actuated TWR is to drive the IB to upright position and to drive the wheels to rest or move at a desired velocity.

Figure 1 Schematic diagram of a TWR (see online version for colours)



Concerning the stability problems of the IB, ensuring the stationary wheels, many control algorithms have been proposed in literature. In particular, state space, optimal and intelligent controllers have been implemented (Salerno and Angeles, 2007; Kausar et al., 2010; Grasser et al., 2002; Nawawi et al., 2007; Kim et al., 2005; Chiu, 2010; Tsai et al., 2010; Pathak et al., 2005; Acar and Murakami, 2010). In Grasser et al. (2002), dynamic equations were linearised around an operating point to design a controller made up of two decoupled state-space controllers. A pole placement controller was proposed by Nawawi et al. (2007). But these controllers were non-robust with respect to un-modelled dynamic uncertainties. To avoid this problem a linear quadratic regulator (LQR) has been designed (Kim et al., 2005). Since the presence of non-linearities in the system limit the control authority and leads to limit cycles or even instability, non-linear controllers such as Fuzzy adaptive and neural network controls (Chiu, 2010; Tsai et al., 2010) have been designed to improve the stability control of TWRs. The upright position of the IB is naturally unstable and non-linear control with naturally unstable system is less effective. It is possible to use feedback to effectively change a non-linear control problem into a linear control problem. This technique is called feedback linearisation.

A non-linear controller based on feedback linearisation has been proposed in Pathak et al. (2005) but this lacks the robustness to parameter uncertainties. The robustness against modelling errors has been improved by Acar and Murakami (2010) presenting a back-stepping with sliding mode controller. These non-linear controllers improve the performance of the stability control but the issue of stability region is not yet addressed for TWRs in literature. Stability region tells from how far the system states can come back to their origin. In this paper, we propose a non-linear controller designed for the required stability region that enhances the stability of TWRs. The proposed controller is later compared to a conventional optimal linear controller.

The proposed non-linear control law is developed for a TWR using a Lyapunov function. The control law is able to compute stability region and the controller stabilises the closed loop system asymptotically around the unstable upright position of the IB. Although the advances in non-linear control systems theory (Kristic et al., 1995; Marino and Tomei, 1995; Freeman and Kokotovic, 1996; Sepulchre et al., 1997; Isidori, 1999) are at a stage of solving practical control design problems. The application of the direct Lyapunov method is still, in general, a difficult task as mentioned in Fontoni and Lozano (2001), Khalil (2002) and White et al. (2006, 2007). The robustness, however, is achieved starting from a control Lyapunov function (CLF) and redesign domination (Sepulchre et al., 1997; Jankovic et al., 1999). Lyapunov function-based (LFB) controllers have been used in many application such as ball on beam (Aguilar-Ibanez, 2009), turbo charged diesel engine (Jankovic and Kolmanovsky, 2000) and transportation systems (Ibañez et al., 2005). This paper contributes to the construction of a Lyapunov function and controller design for the statically unstable TWRs, which does not exist in literature. The asymptotic stability of the closed loop system is guaranteed, analysed using LaSalle invariance theorem.

In Section 2 of this article, the dynamics model of a statically unstable TWR is presented. The dynamic model is normalised to simplify the algebraic manipulation and then partially linearised to produce an equivalent feedback system in the affine form. Section 3 provides an insight to the construction of the Lyapunov function candidate, non-linear controller design to stabilise upright position of the IB and the robot speed tracking, corresponding stability and convergence analysis and stability region computation. In Section 4, the simulation results are presented for a TWR with a brief discussion to verify the effectiveness of the proposed control algorithm.

2 Dynamics model

This section presents the modelling of dynamics of a TWR. The equations of motion are formulated, simplified and partially feedback linearised to convert into a convenient form, suitable to design the non-linear controller. The TWR is modelled as an inverted pendulum on two wheels.

2.1 Dynamic equations

Schematic diagram of a statically unstable TWR is shown in Figure 1. In order to model the dynamics of a TWR, following were the assumptions:

1 the motion of the robot was restricted in an x-z plane

2 both the IB and wheels were assumed to be rigid bodies

3 the mass of each wheel, *m,* was alleged to be located at the centre of the wheel having a radius *r*

4 the mass of the IB, *M* was located at a distance *l* from the axle joining two wheels

5 the wheels rotate at an angle Ø without slipping

6 each wheel remains in contact with the ground at a single point

7 a differential drive was assumed such that motors apply total torque (T) on the wheels to navigate the robot a horizontal distance x along the x-plane

8 the IB rotates from the vertical, denoted by pitch angle, θ

9 the mass moment of inertia for the IB and wheels (including gearbox and transmission) were known and denoted as I_p and I, respectively.

The dynamics of the robot are modelled using Newton-Euler method. The system's equilibrium equations are manipulated and simplified resulting in equations of motion of the TWR. Using kinematic constraint of no-slip these equations of motion, on a horizontal terrain, are given by (1) and (2).

$$(Ml\cos\theta)\ddot{x} + \left(Ml^2 + I_p\right)\ddot{\theta} - Mgl\sin\theta = 0 \tag{1}$$

$$r(M+m)\ddot{x} + (Mlr\cos\theta)\ddot{\theta} - (Mlr\sin\theta)\dot{\theta}^2 + I\ddot{Ø} = T \tag{2}$$

In (1) and (2) g is acceleration due to gravity, $\ddot{x}$ is linear acceleration and $\ddot{Ø}$ is the rotational acceleration produced by the wheels while $\ddot{\theta}$ is rotational acceleration produced by the IB.

2.2 Model simplification

Equations (1) and (2) are normalised, following the approach used in Ibañez et al. (2005), to simplify algebraic manipulation of them in developing the affine form of the equations. The affine form of equations brings about the controller design easy. To achieve this normalisation, the following scaling transformations were introduced:

$$q = \frac{x}{l},\quad u = \frac{T}{Mlr},\quad \delta = \frac{m}{M},\quad a = \frac{I_p}{Ml^2},\quad \frac{d\tau}{dt} = \sqrt{\frac{g}{l}}$$

Substitution of the above scaling factors in (1) and (2) and algebraic manipulation gives the following simpler set of dynamic equations:

$$(\cos\theta)q'' + (1+a)\theta'' - \sin\theta = 0 \tag{3}$$

$$\left(1 + \delta + \frac{I}{r^2}\right)q'' + (\cos\theta)\theta'' - (\sin\theta)\theta'^2 = 0 \tag{4}$$

Due to normalisation, the differentiation in (3) and (4) and all following equations are with respect to the dimensionless time τ defined as $\tau = t\sqrt{\frac{g}{l}}$.

2.3 Partial feedback linearisation of the model

The above simplified equations (3) and (4) have strong inertial coupling. A linearisation of the system equations is proposed to separate the internal dynamics of the TWR. The outputs (pitch and displacement) are collocated with the input u, making input/output linearisation difficult. The simplified system is therefore partially feedback linearised. Considering the inertia matrix to be symmetric and positive definite, the equation (3) is manipulated to solve for θ'' and the resulting expression is substituted in equation (4). Using some standard calculus the new control variable (a scaling factor introduced during simplification), u, is written as (5):

$$u = \left(1 + \delta + b - \frac{\cos^2\theta}{1+a}\right)q'' + \sin\theta\cos\theta - \theta'^2\sin\theta \tag{5}$$

From (5) a partial feedback linearising controller can be defined as (6)

$$u = \left[1 + \delta + b - \frac{\cos^2\theta}{1+a}\right]v + \sin\theta\cos(\theta) - \theta'^2\sin\theta \tag{6}$$

The control input of the robot, u, will be computed from (6) given the input v, an additional control input yet to be defined. The feedback system equivalent to the normalised system of dynamic equations with new control input v is as following:

$$q'' = v \tag{7}$$

$$\theta'' = \frac{l.[\sin\theta - v.\cos(\theta)]}{1+a_1} \tag{8}$$

We see that the input/output system from v to q in (7) is linear and second order. The second equation (8) is non-linear and therefore represents the internal dynamics of the TWR system. This set of equations is now apparently easy to express in the affine form as

$$\dot{x} = f(x) + g(x)v \tag{9}$$

$x = (q', \theta, \theta')^T$, a vector of the state variables

If $v = 0$ and $\theta \in [0, 2\pi]$, the system has two equilibrium points. One is a stable equilibrium point as $x = (0, 0, 0)$ and the second is an unstable equilibrium point as $x = (0, \pi, 0)$. In the following section we will introduce a controller for this partially linearised system to stabilise the IB at its upright unstable equilibrium point.

3 Controller design

The control objective is to asymptotically stabilise the IB of the TWR to its upright position with velocity tracking. A Lyapunov method is applied to design the controller, assuming that the initial position of the IB is above the horizontal plane. In this section, a framework for designing a stabilising controller is presented. The main idea of the Lyapunov method is to propose a scalar, positive definite energy or energy-like function $V(t)$ for the system such that it has continuous partial derivatives. If its time derivative is provided to be at least negative semi definite, $V'(t) \le 0$, the system is stable in the sense of Lyapunov in that the system states (energy) can be constrained for all future time to lie inside a ball in state space. This ball is directly related to the size of the initial states of the system. Based on the designed controller, the stability region and the asymptotic stability of the closed loop system, shown in Figure 2, are imparted before concluding this section.

A CLF for a non-linear system of the form (9) is defined in Haddad and Chellaboina (2008) as a candidate Lyapunov function $V(x)$ with the property that for every fixed $x \neq 0$ there exists an admissible value u for the control such that $\nabla V(x) \cdot f(x, u) < 0$. In other words CLF is simply a candidate Lyapunov function whose derivative can be made negative by the choice of control values.

The CLF is shaped by means of the technique of added integration, presented by Lozano and Dimogianopoulos (2003). First, we introduced a Lyapunov function that represents the energy of the IB of the TWRs with certain constraint. The constraint bounds the states θ and θ'. This was used to develop the controller which stabilises the pitch dynamics of the IB, i.e., θ and θ'. Then an additional quadratic term is introduced for the dynamics of translational part of the robot which formulate a new Lyapunov function.

3.1 *Control synthesis for IB*

The control for IB includes a stabilisation of the pitch and pitch rate of IB of the TWR. We propose a Lyapunov function for this two state system as (10):

$$V_{c0}(\theta,\theta') = \frac{1}{2}\left(k_1\cos^2\theta - l\right)\theta'^2 + (1-\cos\theta)\frac{l}{1+a} \tag{10}$$

where $\frac{l}{1+a}$ is a parametric constant of the system and has a positive value. A constraint function $(k_1\cos^2\theta - l)$ is introduced in the proposed Lyapunov function (10) to bound the pitch θ of the IB. The proposed Lyapunov function, $V_{c0}(\theta,\theta')$, is positive definite for:

1. $k_1 > 1$, because $V_{c0}(\theta,\theta')$ is a locally convex function with a minimum at the origin if $k_1 > 1$. k_1 is a constant that controls the boundary of initial pitch as well as performance of the controller.
2. $|\theta| < \tilde{\theta}$, such that the initial conditions (θ_0, θ_0') with $|\theta_0| < \frac{\pi}{2}$ satisfies that $V_{c0}(\theta,\theta') < (1-\cos\tilde{\theta})$, where $\tilde{\theta} = \cos^{-1}\left(\frac{1}{\sqrt{k_1}}\right)$..

The time derivative of Lyapunov function (10) is given as

$$V_{c0}' = \theta'\cos\theta\left(k_1\beta(\theta,\theta') + v\gamma(\theta)\right) \tag{11}$$

where

$$\beta = \frac{\cos\theta\sin\theta}{1+a} - \sin\theta\theta'^2 = \sin\theta\left(\frac{\cos\theta}{1+a} - \theta'^2\right),$$

$$\gamma = -\frac{k_1\cos^2\theta}{1+a} + \frac{l}{1+a}.$$

3.2 *Control synthesis for the IB and the robot speed tracking*

To control the robot velocity q' as well as pitch θ and pitch rate θ', a quadratic term is added to the two states Lyapunov function (10). The CLF for $x = (q', \theta, \theta')$ is, therefore, proposed as

$$V_{c1}(q',\theta,\theta') = k_d V_{c0}(\theta,\theta') + \frac{1}{2}W^2 \tag{12}$$

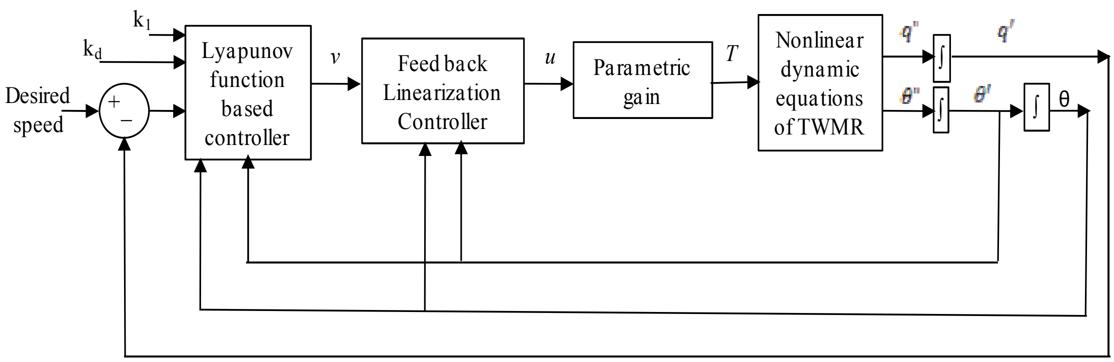
where W is an unknown auxiliary variable in the additional term. This is recommended that the time derivative of the additional term has the same of structure as (11). The auxiliary variable, a function of, q', θ, θ' with same structure as V_{c0}' satisfying

$$W'(\theta,\theta',q') = \theta'\frac{\partial W}{\partial \theta} + \sin\theta\frac{\partial W}{\partial \theta'} + \left(\frac{\partial W}{\partial q'} - \cos\theta\frac{\partial W}{\partial \theta'}\right)v$$

results into

$$W(\theta,\theta',q') = \frac{1}{1+a}\left(lq' + k_1\theta'\cos\theta\right). \tag{13}$$

Figure 2 Block diagram of the closed-loop system



Substituting (13) into (12), the new extended CLF is expressed as (14).

$$V_{c1} = k_d V_{c0}(\theta,\theta') + \frac{1}{2}\left[\frac{1}{1+a}\left(lq' + k_1\theta'\cos\theta\right)\right]^2 \quad (14)$$

Evidently, (14) is a positive definite function for all $|\theta| < \tilde{\theta} < \frac{\pi}{2}$ and $k_d > 0$. The time derivative of CLF (14) is

$$\begin{aligned} V_{c1}' &= k_d V_{c0}'(\theta,\theta') \\ &+ \frac{1}{1+a}\left(lq' + k_1\theta'\cos\theta\right) \\ &\left(lq'' + k_1\theta''\cos\theta - k_1\theta'^2\cos\theta\right) \end{aligned} \quad (15)$$

Substitution of (7), (8) and (11) in (15) and some algebraic manipulations lead to (16).

$$V_{c1}' = v\gamma(\theta)w' + k_1\beta(\theta,\theta')w' \quad (16)$$

such that

$$\beta = \frac{\cos\theta\sin\theta}{1+a} - \sin\theta\theta'^2 = \sin\theta\left(\frac{\cos\theta}{1+a} - \theta'^2\right),$$

$$\gamma = -\frac{k_1\cos^2\theta}{1+a} + \frac{l}{1+a}, \text{ and}$$

$$w' = \left(\frac{lq'}{(1+a)^2} + \frac{k_1\theta'\cos\theta}{(1+a)^2} + k_d\theta'\cos\theta\right)$$

To craft the derivative of Lyapunov function to be semi negative definite, $V_{c1}' \le 0$, $k_1\beta(\theta,\theta') + v\gamma(\theta)$ should be equal to $-w'$. The control law is then conveniently chosen as (17)

$$v = \frac{-1}{\gamma(\theta)}\left(w'\left(q',\theta,\theta'\right) + k_1\beta(\theta,\theta')\right) \quad (17)$$

The control law (17) produces $V_{c1}' = w'^2(q',\theta,\theta')$ and makes the variables q',θ,θ' to converge asymptotically to zero.

3.3 Stability region

In (10), V_{c0} becomes locally convex function with a minimum at origin if $k_1 > 1$. This can be easily checked by plotting the corresponding level curves. If initial conditions $(\theta,\dot{\theta})$ with $|\theta| < \frac{\pi}{2}$ satisfies that $V_{c0} < 1 - \cos\tilde{\theta}$, then we have $|\theta| < \tilde{\theta}$ and a set defined by (18) is a compact set.

$$\Omega_0 = \left\{(\theta,\theta') \in R^2 : V_{c0} < 1 - \cos\tilde{\theta}\right\} \quad (18)$$

This compact set represents the stability region of the system with two controlled states, θ and θ'. Similarly, we can define the stability region of the system with three controlled states q',θ,θ'. Equation (16) suggests the system has no singularity when $|\theta| < \tilde{\theta} < \frac{\pi}{2}$. To avoid singularity at $|\theta| = \pm\tilde{\theta}, |\theta_0| < \frac{\pi}{2}$ and $V_{c1} < K = k_d(1 - \cos\theta)$ should belong to the neighbourhood of the origin. In other words states are bounded with $|\theta| < \tilde{\theta}$ if $|\theta(t)| < \tilde{\theta}$ and $V_{c1} < K$. $V_{c1} < K$, is an outcome from the fact that V_{c1} is a non-increasing function as $V_{c1}' = -W'^2(q',\theta,\theta')$.. This also defines stability region, Ω_1, for the proposed closed loop system of the two-wheeled robot with three states (θ,θ',q').

$$\Omega_1 = \left\{(\theta,\theta',q'), |\theta| < \tilde{\theta} : V_{c1} < K\right\} \quad (19)$$

We conclude that the unstable equilibrium point of a TWR is stable with proposed controller in closed loop, in the sense of Lyapunov as $V_{c1}(x)$ is a positive definite function for all $x \in \Omega_1$ and $V_{c1}'(x)$ is negative semi definite for x.

3.4 Asymptotic stability

Since $V_{c1}(x)$ is a positive definite function for all $x \in \Omega_1$ and $V_{c1}'(x)$ is negative semi definite for x, stability exists in the sense of Lyapunov. We apply LaSall's theorem to prove the asymptotic stability of the equilibrium point of the TWR. We have considered equilibrium point $\theta = \theta' = 0$ and $q' = constant$. LaSall's theorem defines that if every solution starting in a compact set, and approaches the largest invariant set as $t \to \infty$ then the asymptotic stability is guaranteed.

Let D is a compact set defined for TWR as

$$D = \left\{ \begin{array}{l} x \in \Omega_1 : \left(\frac{l\dot{x}}{(1+a)^2} + \frac{k_1\theta'\cos\theta}{(1+a)^2} + k_d\theta'\cos\theta\right)^2 \\ = w'^2 = 0 \end{array} \right\} \quad (20)$$

and M is the largest invariant set in D. The Largest invariant set M is computed to ascertain if this is contained in set D.

From (20), it is evident that on D, w'=0. This tells that the auxiliary variable,

$$w = \left(\frac{lq}{(1+a)^2} + \left(\frac{k_1}{(1+a)^2} + k_d\right)\sin\theta\right) = \text{constant on set } D$$

$$w'' = 0.$$

Since, $w' = \left(\frac{lq'}{(1+a)^2} + \frac{k_1\theta'\cos\theta}{(1+a)^2} + k_d\theta'\cos\theta\right)$, its time derivative is

$$w'' = \left(\frac{lq''}{(1+a)^2} + \left(\frac{k_1}{(1+a)^2} + k_d\right)\left(\theta''\cos\theta - \theta'^2\sin\theta\right)\right)$$

Replacing θ'' and q'' from (7) and (8), we get

$$w'' = \frac{lv}{(1+a)^2}(1+\gamma) + \frac{k_1}{(1+a)^2} + k_d\beta \tag{21}$$

As $w'' = 0$, the above relation give rise to the equality

$$\beta = \frac{\dfrac{lv}{(1+a)^2}(1+\gamma)}{\left(\dfrac{k_1}{(1+a)^2} + k_d\right)} \tag{22}$$

The controller was selected such that

$$v\gamma + k_1\beta = -w'.$$

This relation then, on set *D*, provides

$$v\gamma + k_1\beta = 0 \tag{23}$$

Substituting (22) into (23) results into

$$v = \left[\gamma + \frac{\dfrac{l}{(1+a)^2}\left(k_1 + k_1\gamma\right)}{\left(\dfrac{k_1}{(1+a)^2} + k_d\right)}\right] = 0$$

As $k_1(1 + \gamma) > 0$, then $v = 0$ on set *D*. This zero control input clearly indicates the system is stable. When $v = 0$, $q'' = 0$ and q' is a constant on set *D*.

As w'=0, $\left[\dfrac{k_1}{(1+a)^2} + k_d\right]\theta'\cos\theta = 0.$ This leads to $\theta' = 0$ and $\theta'' = 0$ for $\theta < \dfrac{\pi}{2}$. Substitution of $\theta'' = 0$ and $v = 0$ in (8) leads to $\theta = 0$ on set *D*. Therefore, the largest invariant set *M* contained in the set *D* is the single unstable equilibrium point, $\theta = \theta' = 0$ and $q' = constant.$ Hence, LaSall's theorem proved that all the closed loop solutions starting in Ω_1 asymptotically converge to the largest invariant set *M* which is the unstable equilibrium point $\theta = \theta' = 0$ and $q' = constant.$

4 Simulations

In order to illustrate the effectiveness and limitations of the proposed control law, we carried out experiments in simulation using *Simulink* and MATLAB software. The simulation tests were conducted for the evaluation of velocity tracking and transient performance of the closed-loop system. Performance of the proposed controller is compared with an LQR designed for reference tracking, referred to as the baseline (BL) controller. The controllers were tuned keeping one of the performance metrics constant to allow fair comparisons. The performance was measured as transient response, integrated squared error (ISE) of pitch, speed tracking and torque demand to accomplish the task.

Both the controllers were designed with an assumption that all the states are measurable and available for feedback. The physical parameters of the system, used for all experiments, are given in Table 1. These parameters represent a real time two wheeled robotic platform system. The initial conditions of states were maintained at same values for both controllers.

Table 1 Physical parameters of the TWR

Parameter	*Value*	*Parameter*	*Value*
M (kg)	43.0	I_p (kg-m^2)	1.30
m (kg)	2.80	*l* (m)	0.19
I (kg-m^2)	0.09	*r* (m)	0.20

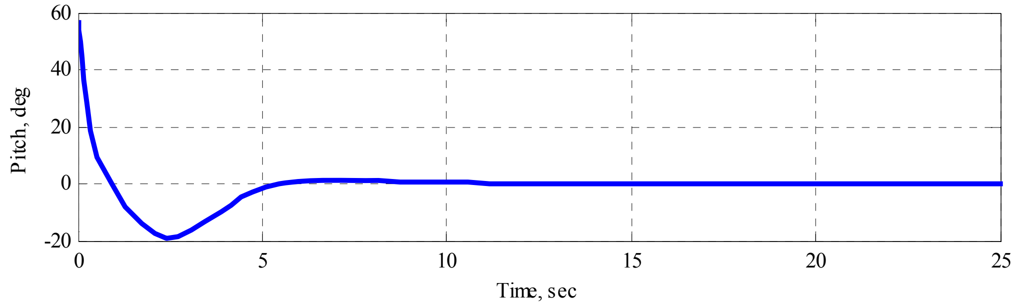
4.1 Controller performance

The purpose of the first experiment was to evaluate if the control algorithm brings the IB to upright unstable position effectively. The experiment was conducted with initial pitch, θ = 1 rad = 57 degrees. The second experiment was carried out to test the performance of the controller for TWR speed tracking. The experiment was repeated for reference speeds 0–5 m/s with initial pitch, θ = 1 rad = 57 degrees.

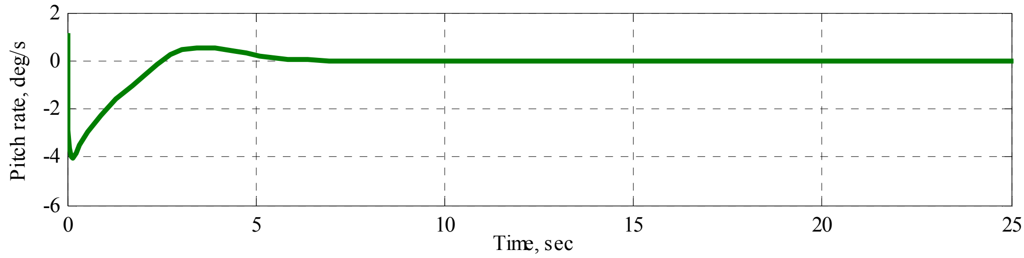
The initial pitch was selected such that $|\theta_0| < \tilde{\theta} = \cos^{-1}\left(\dfrac{1}{\sqrt{k_1}}\right)$ for both the experiments. The results of these experiments for the closed loop system with the proposed controller are shown in Figures 3(a) to 3(d). These results clearly indicate that all the states converge to the origin with a smooth speed tracking. The transient behaviour of the closed loop system portrays a good performance of the controller. All the states converge to steady state condition with an acceptable overshoot, settling within ten seconds and no steady state error. The first experiment results are presented in phase space form in Figure 3(e). The space shows the interrelationship between angular position and angular velocity of the IB. Even when the IB is initially located too far from the vertical top position, the proposed controller brings both the states very close to the unstable equilibrium states.

A few tests were conducted with variation of the mass, inertia and the height of centre of mass of the IB to test robustness of the controller to system parameters. The proposed control strategy proved robust for large mass range (50–100 kg), inertia range (0.05–1.35 kg-m^2) and for high centre of mass positions (0.1–1 m).

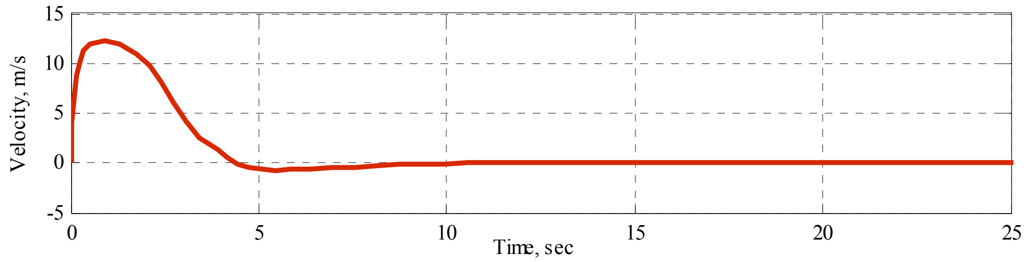
Figure 3 Performance of the LFB controller for the TWR, (a) transient response of pitch (b) transient response of pitch rate (c) transient response of velocity (d) velocity tracking (e) attraction of pitch and pitch rate to the equilibrium point (see online version for colours)



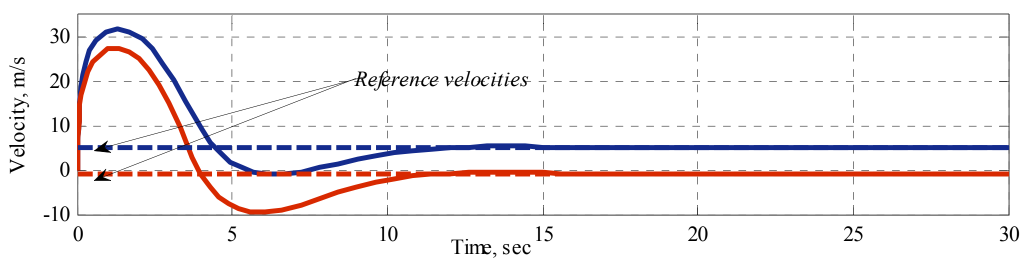
(a)



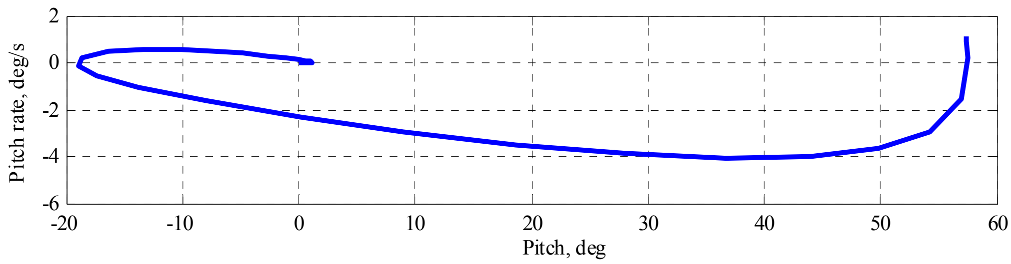
(b)



(c)



(d)



(e)

4.2 Controller comparison

The proposed controller is compared with a BL controller. The objective of the three comparative tests designed was to evaluate if the proposed control algorithm has a benefit over linear control. As the tests were conducted in different conditions, they give a confidence that the proposed algorithm may be used for variety of test conditions and may describe the limitations of the proposed controller.

The first test was conducted to compare the controller output and transient response such that the two closed loop systems have the same settling time. In this experiment, the controller parameters (gains) were tuned such that both the systems response has the same settling time. The objective was to compare the transient and steady state behaviour as well as the control demand by both the controllers to accomplish the task. The initial conditions of all the states are same for both the closed loop systems, i.e., all initiated from zero except pitch with 85 degrees, a pitch angle close to the boundary of initial pitch, calculated using $\tilde{\theta} = \cos^{-1}\left(\frac{1}{\sqrt{k_1}}\right)$. Figure 4 shows a comparison of the transient behaviour, pitch, pitch rate, ISE of pitch and control demand of the robot motion in the horizontal x-direction. Maximum settling time is aimed within ten seconds in this case and results are presented for comparison for both closed loop systems. The closed loop systems response, for both the controllers with same settling time, suggests that a LFB controller is better in sense of smooth transition of the states in comparison to the BL controller. This produces higher ISE of pitch but demands less control input to the system.

The second experiment was designed to compare the performance and control demand of TWR systems having the same closed loop polynomial but different controllers. In this experiment, the controller parameters were selected such that they produce same closed loop system polynomial. The objective was to see if both the controllers perform equally well close to the equilibrium states then how they perform at the initial conditions far from the equilibrium point. The parameters of the proposed control law were tuned and chosen such that the system shows same performance as in previous section. The control parameters of the BL controller were selected such that the characteristic polynomial of the closed loop system matches the characteristic polynomial computed for the closed loop system with proposed controller. The initial conditions were set at the origin except for the pitch of the IB. This was fixed as 28 degrees. The results shown in Figure 5 compare the transient behaviour of the IB position, ISE, control demand and the speed of the robot motion in longitudinal direction for two closed loop systems. When the system is made equivalent closed loop system the LFB controller shows good results while the BL controller could not stabilise the system and caused large oscillations.

Lastly, the transient performance and control demand were evaluated and compared with a condition that both the controllers produce the same initial torque at the same initial conditions. In this experiment we aimed to observe the response of the closed-loop systems if both produce the same control demand initially. For this purpose the controller gains were selected such that they produce same initial control input demand with both the controllers, for same initial conditions of the states. All the states were kept at zero initial condition except pitch. The pitch had an initial value of 86 degrees. The results shown in Figure 6 compare the transient behaviour of the IB position, rate of the IB pitch, ISE and the control demand for the robot motion in longitudinal direction for two closed loop systems. The results show although the LFB controller settles slowly it drops down the control demand over transition very smoothly and gradually as compared to the oscillated behaviour of the linear controller.

5 Conclusions

We proposed a non-linear LFB controller for stability control of TWRs and compared its performance with a linear controller in diverse scenarios. The stability region was computed and simulated. The asymptotic stability was proved as well. The conclusions from the presented results are following:

1. The proposed controller tracked the velocity and stabilised the IB position from far initial position within 10 seconds and with very low power demand.
2. The linear controller, as compare to the proposed controller, has a very high control demand that limits its practical use.
3. Velocity tracking performance of the linear controller is not to the desired standard.
4. Simulations with realistic data show that the proposed approach is effective. Further work is required to make the controller design robust with respect to parameter-uncertainties.

Figure 4 Comparison of performance and control input of baseline and LFB controllers for same settling time, (a) transient performance of pitch rate (b) transient performance of pitch (c) control input generated by each controller (d) ISE of pitch (see online version for colours)

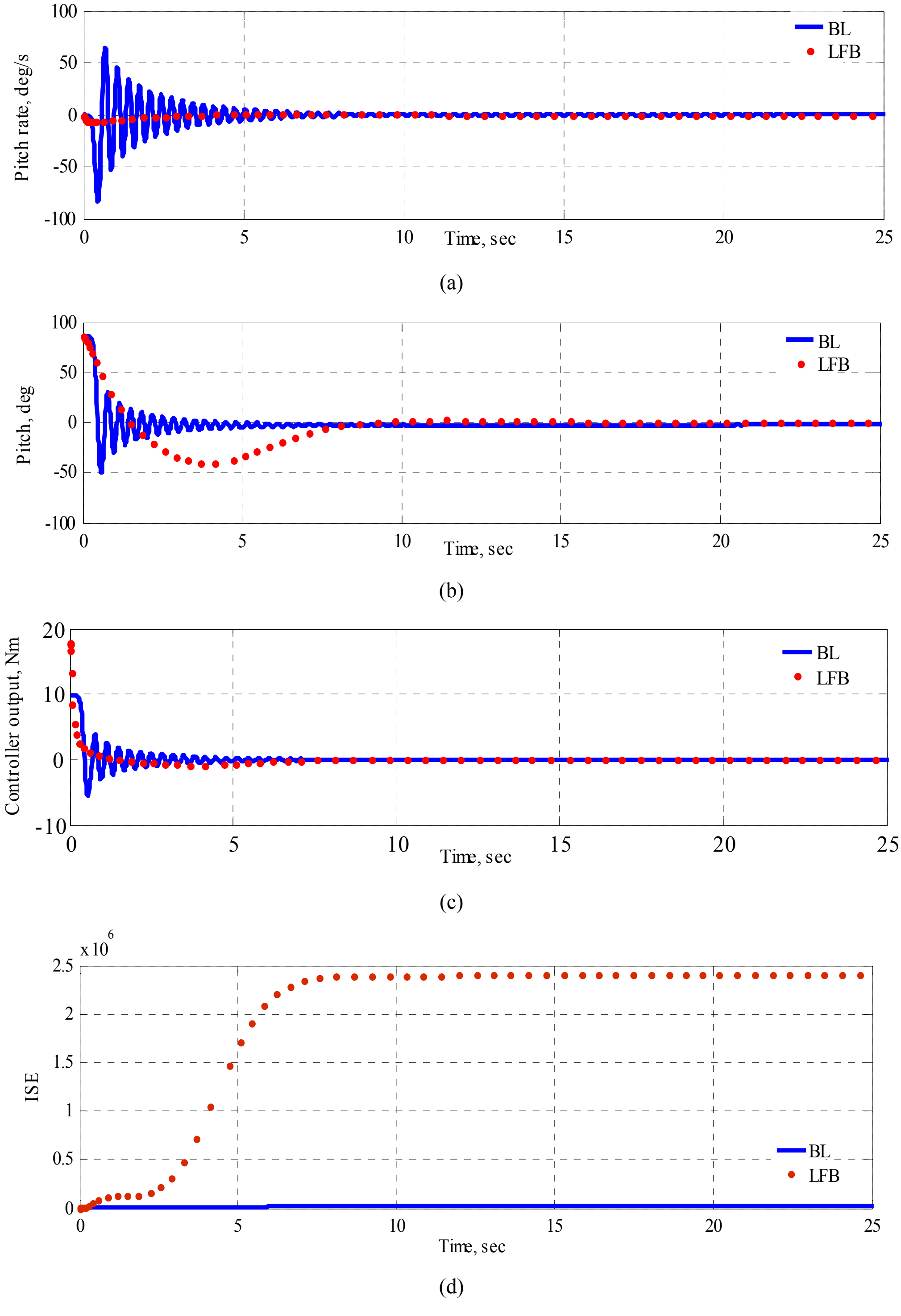


Figure 5 Comparison of performance and control input of baseline and LFB controllers for same closed loop polynomials, (a) transient performance of pitch rate (b) transient performance of pitch (c) control input generated by each controller (d) ISE of pitch (see online version for colours)

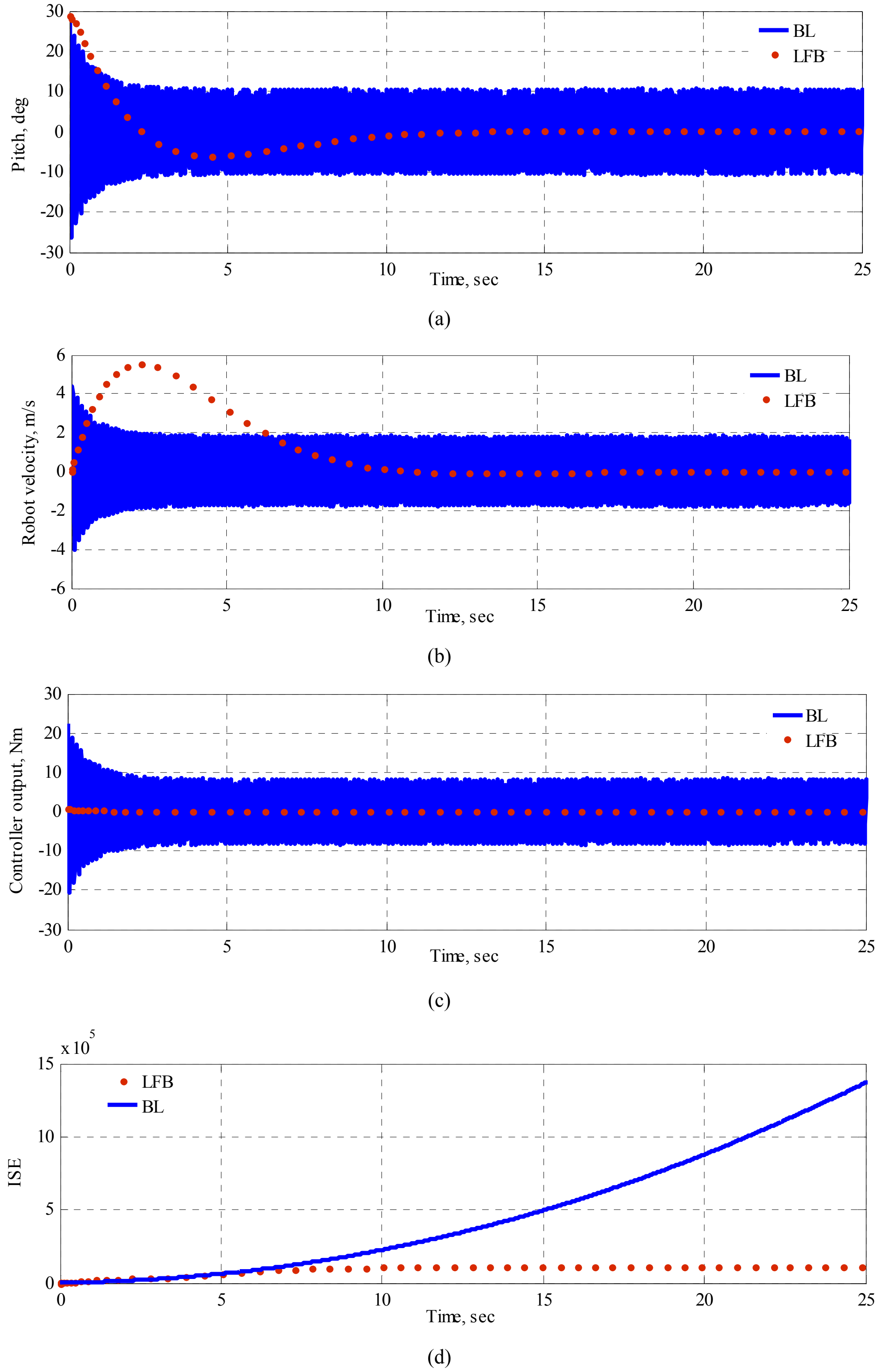
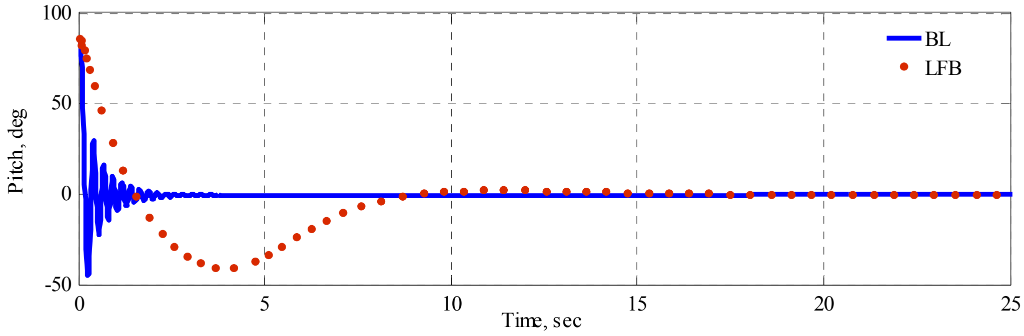
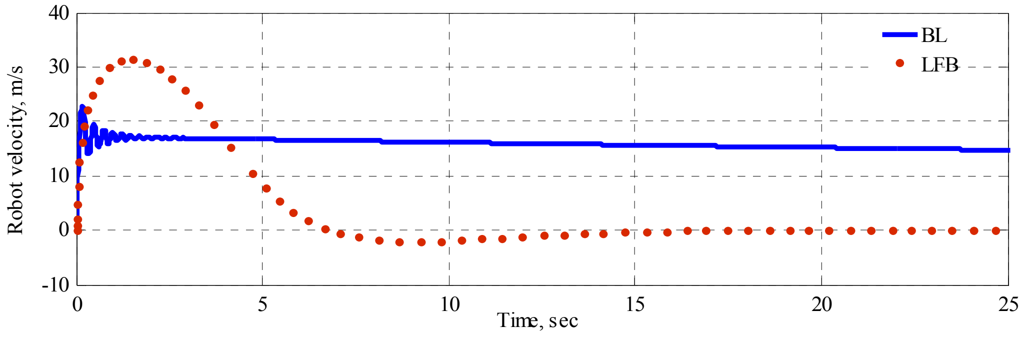


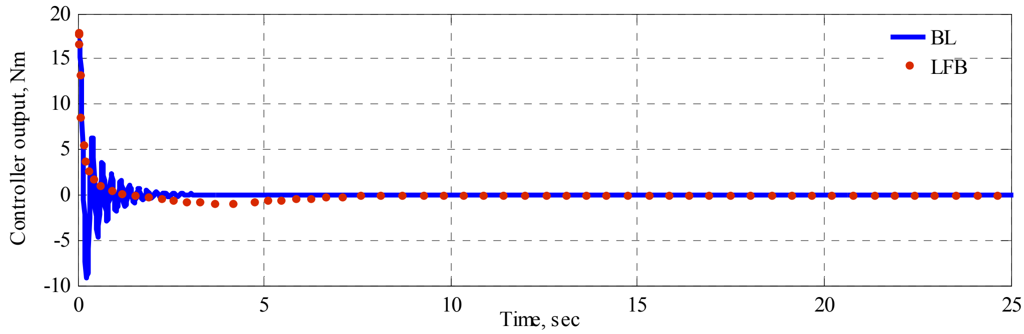
Figure 6 Comparison of performance and control input of baseline and LFB controllers for same initial control demand, (a) transient performance of pitch rate (b) transient performance of pitch (c) control input generated by each controller (d) ISE of pitch (see online version for colours)



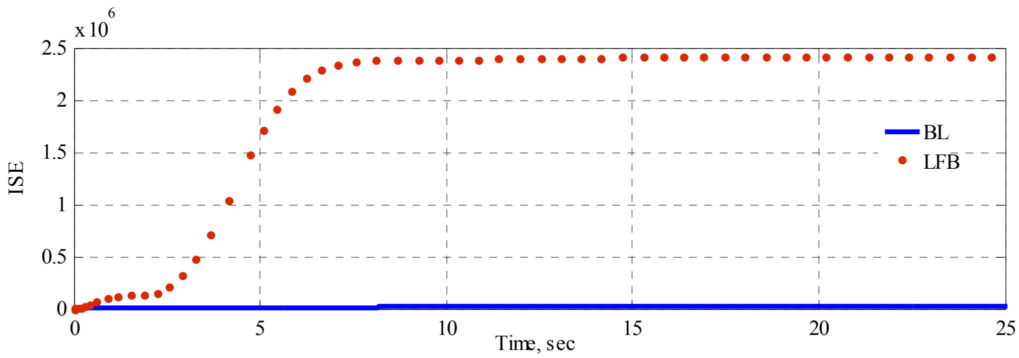
(a)



(b)



(c)



(d)

References

Acar, C. and Murakami, T. (2010) 'A robust control of two-wheeled mobile manipulator with underactuated joint by nonlinear backstepping method', *IEEJ Transactions on Industry Applications*, Vol. 130, No. 6, pp.742–751+745.

Aguilar-Ibanez, C. (2009) 'The Lyapunov direct method for the stabilisation of the ball on the actuated beam', *International Journal of Control*, Vol. 82, No. 12, pp.2169–2178.

Chiu, C.H. (2010) 'The design and implementation of a wheeled inverted pendulum using an adaptive output recurrent cerebellar model articulation controller', *IEEE Transactions on Industrial Electronics*, Vol. 57, No. 5, pp.1814–1822.

Fontoni, I. and Lozano, R. (2001) *Nonlinear Control for Underactuated Mechanical Systems*, Springer-Verlag, New York, Inc., Secaucus, NJ, USA.

Freeman, R.A. and Kokotovic, P.V. (1996) *Robust Control of Nonlinear Systems*, Birkhauser, Boston, MA.

Grasser, F., D'Arrigo, A. et al. (2002) 'JOE: a mobile, inverted pendulum', *IEEE Transactions on Industrial Electronics*, Vol. 49, No. 1, pp.107–114.

Haddad, W.M. and Chellaboina, V. (2008) *Nonlinear Dynamical Systems and Control – A Lyapunov-Based Approach*, Princeton University Press, Princeton, New Jersey.

Hitachi (2012) 'Hitachi Emiew two wheeled robot' [online] http://www.hitachi.com/New/cnews/050315a.html (accessed July 2012).

Ibañez, C.A., Frias, O.G. et al. (2005) 'Lyapunov-based controller for the inverted pendulum cart system', *Nonlinear Dynamics*, Vol. 40, No. 4, pp.367–374.

Isidori, A. (1999) *Nonlinear Control Systems*, Springer-Verlag, London.

Jankovic, M. and Kolmanovsky, I. (2000) 'Constructive Lyapunov control design for turbocharged diesel engines', *IEEE Transactions on Control Systems Technology*, Vol. 8, No. 2, pp.288–299.

Jankovic, M., Sepulchre, R. et al. (1999) 'CLF based designs with robustness to dynamic input uncertainties', *Systems & Control Letters*, Vol. 37, pp.45–54.

Kausar, Z., Stol, K. et al. (2010) 'Performance enhancement of a statically unstable two wheeled mobile robot traversing on an uneven surface', *IEEE International Conference on Robotics, Automation and Mechatronics, RAM 2010*, Singapore.

Khalil, H.K. (2002) *Nonlinear Systems*, Prentice Hall, Upper Saddle River, New Jersey.

Kim, Y., Kim, S.H. et al. (2005) 'Dynamic analysis of a nonholonomic two-wheeled inverted pendulum robot', *Journal of Intelligent and Robotic Systems: Theory and Applications*, Vol. 44, No. 1, pp.25–46.

Kristic, M., Kanellakopoulos, I. et al. (1995) *Nonlinear and Adaptive Control Design*, Wiley, New York.

Lozano, R. and Dimogianopoulos, D. (2003) 'Stabilization of a chain of integrators with nonlinear perturbations: application to the inverted pendulum', *Proceedings of the IEEE Conference on Decision and Control*, Maui, HI.

Marino, R. and Tomei, P. (1995) *Nonlinear Control Design*, Prentice-Hall, Englewood Cliffs, NJ.

Nawawi, S.W., Ahmad, M.N. et al. (2007) 'development of a two-wheeled inverted pendulum mobile robot', *Research and Development, SCOReD 2007, 5th Student Conference on*.

Pathak, K., Franch, J. et al. (2005) 'Velocity and position control of a wheeled inverted pendulum by partial feedback linearization', *IEEE Transactions on Robotics*, Vol. 21, No. 3, pp.505–513.

Robonica (2012) 'Roboni two wheeled robot' [online] http://www.slashgear.com/robonica-roboni-i-fully-programmable-two-wheel-robot-now-available-2161359/ (accessed July 2012).

Salerno, A. and Angeles, J. (2007) 'A new family of two-wheeled mobile robots: modeling and controllability', *IEEE Transactions on Robotics*, Vol. 23, No. 1, pp.169–173.

Segway (2012) 'Segway transporter' [online] http://www.segway.com/smithsoniantours/ (accessed July 2012).

Sepulchre, R., Jankovic, M. et al. (1997) *Constructive Nonlinear Control*, Springer-Verlag, London, UK.

Tsai, C-C., Huang, H-C. et al. (2010) 'Adaptive neural network control of a self-balancing two-wheeled scooter', *IEEE Transactions On Industrial Electronics*, April, Vol. 57, No. 4, pp.1420–1428.

White, W.N., Foss, M. et al. (2006) 'A direct Lyapunov approach for a class of under actuated mechanical systems', *American Control Conference*, Minneapolis, MN, USA.

White, W.N., Foss, M. et al. (2007) 'A direct Lyapunov approach for stabilization of underactuated mechanical systems', *American Control Conference*, New York, NY, USA.