Standard orthogonal polynomials-based solution of fuzzy differential equations

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Abstract: This paper proposes a new method to solve $n^{th}$ order fuzzy differential equations using collocation type of method. In the solution procedure Legendre polynomials are used in the collocation method. Three different cases have been considered for the analysis. Known example problems are solved using the proposed procedure. Obtained results are compared with the exact solution in order to illustrate the efficiency and reliability of the proposed method. Solutions are depicted in term of figures and tables.

Keywords: fuzzy number; Gaussian fuzzy number; Legendre polynomial; $n^{th}$ order; fuzzy differential equations; FDEs.


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Smita Tapaswini received her MSc in Mathematics from the National Institute of Technology, Rourkela on April 2010. She is currently working as a PhD scholar in the Department of Mathematics, National Institute of Technology, Rourkela. She is recently awarded by the Rajiv Gandhi National Fellowship (RGNF), under UGC, Government of India in the year 2010–11. Her current research interests include fuzzy set theory, fuzzy differential equations and numerical analysis.
1 Introduction

Theory of fuzzy differential equations (FDEs) has been used to model physical and engineering problems such as in fluid mechanics, viscoelasticity, biology, physics, engineering and other areas of science because this theory represents a natural way to model dynamical systems under uncertainty. Since, it is too difficult to obtain the exact solution of FDEs, so one may need a reliable and efficient numerical technique for the solution of FDEs. There exist a good number of papers dealing with FDEs and its applications in the open literatures. Some of are reviewed and cited here for better understanding of the present analysis. Chang and Zadeh (1972) first introduced the concept of a fuzzy derivative, followed by Dubois and Prade (1982) who defined and used the extension principle in their approach. Fuzzy boundary value and fuzzy initial value problems are studied by Kaleva (1987, 1990) and Seikkala (1987). Various numerical methods for solving FDEs are introduced in Abbasbandy and Allahviranloo (2002), Akin et al. (2013), Bede et al. (2007), Dahaghin and Moghadam (2010), Friedman et al. (1999), Jameel et al. (2012), Khastan and Ivaz (2008), Ma et al. (1999), Mazandarani and Kamyad (2013), Mikaeilvand and Khakrangin (2012), Mosleh and Otadi (2012) and Rodriguez-Lopez (2008).

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(Tapaswini and Chakraverty, 2014). Behera and Chakraverty (2013) obtained the solution of uncertain impulse response of imprecisely defined half order mechanical system.

Bede (2008) described the exact solutions of FDEs in his note in an excellent way. Ahmad et al. (2013) investigated analytical and numerical solutions of FDEs based on the extension principle. Buckley and Feuring (2001) applied two analytical methods for solving \( n \)th order linear differential equations with fuzzy initial conditions. In the first method, they simply fuzzify the crisp solution to obtain a fuzzy function and then checked whether it satisfies the differential equation or not, and the second method was just the reverse of the first method.

Similarly, many authors studied various other methods to solve \( n \)th order FDEs (Allahviranloo et al., 2008; Hashemi et al., 2012; Jafari et al., 2012; Jayakumar et al., 2012; Parandin, 2012; Prakash and Kalaiselvi, 2012; Yue and Guangyuan, 1998). Based on the idea of collocation method, Allahviranloo et al. (2008) investigated the numerical solution of \( n \)th order FDEs. Jafari et al. (2012) used variational iteration method (VIM) for the solution of \( n \)th order FDEs. Parandin (2012) discussed Runge-Kutta method for the numerical solution of \( n \)th order FDEs. Homotopy analysis method has been used by Hashemi et al. (2012) for the solution of system of FDEs. Yue and Guangyuan (1998) utilised time domain methods for the solutions of \( n \)th order FDEs. The Seikkala derivative and Runge-Kutta method of order five has been implemented by Jayakumar et al. (2012) for the numerical solution of FDEs. Prakash and Kalaiselvi (2012) used hybrid Euler method and hybrid predictor-corrector method for the solution of FDEs. As such we have implemented a technique based on the idea of a collocation method to solve \( n \)th order fuzzy initial value problems. In the solution procedure Legendre polynomials generated by Gram Schmidt orthogonalisation procedure are used in the collocation method.

Present paper is organised as follows: in Section 2, we give some basic preliminaries related to the present investigation. The concepts of these have been used for the solution of \( n \)th order FDE. Proposed technique is discussed in Section 3 to solve \( n \)th order FDE using collocation type of method. In Section 4, numerical example problems are solved and comparisons are given. Finally in the last section conclusions are drawn.

2 Preliminaries

In this section, we present some notations, definitions and preliminaries which are used further in this paper (Jaulin et al., 2001; Ross, 2004; Zimmermann, 2001).

2.1 Definition 2.1: fuzzy number

A fuzzy number \( \tilde{u} \) is convex normalised fuzzy set \( \tilde{u} \) of the real line \( R \) such that

\[
\{ \mu_\tilde{u}(x) : R \rightarrow [0, 1], \ \forall x \in R \}
\]

where \( \mu_\tilde{u} \) is called the membership function of the fuzzy set and it is piecewise continuous.
2.2 Definition 2.2: Gaussian fuzzy number

Let us now define an arbitrary asymmetrical Gaussian fuzzy number, \( \tilde{u} = (r, \sigma_l, \sigma_r) \). The membership function \( \mu_\tilde{u} \) of \( \tilde{u} \) will be as follows:

\[
\mu_\tilde{u}(x) = \begin{cases} 
\exp\left[-\frac{(x-r)^2}{2\sigma^2}\right] & \text{for } x \leq r \\
\exp\left[-\frac{(x-r)^2}{2\sigma^2}\right] & \text{for } x \geq r \\
\end{cases} \quad \forall x \in R
\]

where, the modal value is denoted as \( r \) and \( \sigma_l, \sigma_r \) denote the left hand and right hand spreads (fuzziness) corresponding to the Gaussian distribution. For symmetric Gaussian fuzzy number the left hand and right hand spreads are equal, i.e., \( \sigma_l = \sigma_r = \sigma \). So the symmetric Gaussian fuzzy number may be written as \( \tilde{u} = (r, \sigma, \sigma) \) and corresponding membership function may be defined as

\[
\mu_\tilde{u}(x) = \exp\left\{-\frac{\gamma(x-r)^2}{2}\right\} \quad \forall x \in R
\]

where \( \gamma = 1/2\sigma^2 \).

2.3 Definition 2.3: single parametric form of fuzzy numbers

The symmetric Gaussian fuzzy number, \( \tilde{u} = (r, \sigma, \sigma) \) in single parametric form can be represented as

\[
\tilde{u} = [\underline{u}(\alpha), \overline{u}(\alpha)] = \left[ r - \sqrt{\frac{(\log_e \alpha)}{\gamma}}, r + \sqrt{\frac{(\log_e \alpha)}{\gamma}} \right]
\]

where \( \alpha \in [0, 1] \).

It may be noted that the lower and upper bounds of the fuzzy numbers satisfy the following requirements

1. \( \underline{u}(\alpha) \) is a bounded left continuous non-decreasing function over \([0, 1]\)
2. \( \overline{u}(\alpha) \) is a bounded right continuous non-increasing function over \([0, 1]\)
3. \( \underline{u}(\alpha) \leq \overline{u}(\alpha), 0 \leq \alpha \leq 1 \).

2.4 Definition 2.4: fuzzy arithmetic

For any two arbitrary fuzzy number \( \tilde{x} = [\underline{x}(\alpha), \overline{x}(\alpha)], \tilde{y} = [\underline{y}(\alpha), \overline{y}(\alpha)] \) and scalar \( k \), the fuzzy arithmetic is defined as follows,

1. \( \tilde{x} = \tilde{y} \) if and only if \( \underline{x}(\alpha) = \underline{y}(\alpha) \) and \( \overline{x}(\alpha) = \overline{y}(\alpha) \)
2. \( \tilde{x} + \tilde{y} = [\underline{x}(\alpha) + \underline{y}(\alpha), \overline{x}(\alpha) + \overline{y}(\alpha)] \)
3. \( \tilde{x} - \tilde{y} = [\underline{x}(\alpha) - \overline{y}(\alpha), \overline{x}(\alpha) - \underline{y}(\alpha)] \)
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4 \bar{\chi} \times \bar{\psi} = \begin{bmatrix} \min \left( x(\alpha) \times y(\alpha), x(\alpha) \times \bar{y}(\alpha), \bar{x}(\alpha) \times y(\alpha), \bar{x}(\alpha) \times \bar{y}(\alpha) \right) \\ \max \left( x(\alpha) \times y(\alpha), x(\alpha) \times \bar{y}(\alpha), \bar{x}(\alpha) \times y(\alpha), \bar{x}(\alpha) \times \bar{y}(\alpha) \right) \end{bmatrix}

5 \kappa = \begin{bmatrix} [k\bar{x}(\alpha), k\bar{x}(\alpha)] \quad k < 0 \\ [k\bar{x}(\alpha), k\bar{x}(\alpha)] \quad k \geq 0 \end{bmatrix}

Now we define some standard orthogonal polynomial viz. Legendre polynomials.

2.5 Definition 2.5: Legendre polynomials (Rama Bhat and Chakraverty, 2007)

If we define the inner product of two functions \( u(t), v(t) \) in \([-1, 1]\) by

\[
< u, v > = \int_{-1}^{1} u(t)v(t)dt
\]

then we generate orthogonal polynomials from the set of functions \( f_0 = 1, f_1 = t, f_2 = \bar{r}, \ldots \) by the Gram Schmidt procedure, which are known as Legendre polynomials (weight function \( w(t) = 1 \)).

Now by Gram Schmidt procedure we will write

\[
\phi_0 = f_0 = 1,
\phi_1 = f_1 - \alpha_{10}\phi_0,
\phi_2 = f_2 - \alpha_{20}\phi_0 - \alpha_{21}\phi_1,
\phi_3 = f_3 - \alpha_{30}\phi_0 - \alpha_{31}\phi_1 - \alpha_{32}\phi_2,
\]

where \( \alpha_{10}, \alpha_{20}, \alpha_{21}, \alpha_{30}, \alpha_{31}, \alpha_{32}, \) etc. may be obtained using the orthogonality property.

So, we have

\[
\phi_1 = t,
\phi_2 = t^2 - \frac{1}{3}.
\]

The orthogonal polynomials described above are known as Legendre polynomials.

3 Proposed method

In this section, a new technique based on collocation type method (Rama Bhat and Chakraverty, 2007, Allahviranloo et al., 2008) with Legendre polynomials (Rama Bhat and Chakraverty, 2007) to solve \( n \)th order FDE has been proposed. Accordingly let us consider the \( n \)th order FDE in general form as

\[
\hat{y}^{(\alpha)}(t; \alpha) + a_{n-1}(t)\hat{y}^{(\alpha-n+1)}(t; \alpha) + \cdots + a_1(t)\hat{y}'(t; \alpha) + a_0(t)\hat{y}(t; \alpha) = \tilde{g}(t; \alpha),
\]

where \( a_i(t), 0 \leq i \leq n - 1 \) are continuous on some interval \( I \), subject to fuzzy initial conditions
\[ \tilde{y}(0) = \tilde{b}_0, \quad \tilde{y}'(0) = \tilde{b}_1, \ldots, \quad \tilde{y}^{(n-1)}(0) = \tilde{b}_{n-1}. \]

Here \( \tilde{b}_i, \ 0 \leq i \leq n-1 \) are fuzzy numbers. Here, \( \tilde{y}(t; \alpha) \) is the solution to determine.

Next, we assume an approximate solution

\[ \tilde{y}(t) = \sum_{i=0}^{n} \tilde{c}_i \phi_i(t) \]  

(3)

where \( \tilde{c}_i \) and \( \phi_i \)'s for \( i = 0, 1, 2, \ldots, n \) are unknown fuzzy constants and linearly independent Legendre polynomials respectively.

By substituting equation (3) in equation (2), one may find the residual \( \tilde{R} \) as

\[ \tilde{R}(x; \tilde{c}_0, \tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n) = \sum_{i=0}^{n} \tilde{c}_i \phi_i(t) + a_{n-1}(t) \sum_{i=0}^{n} \tilde{c}_i \phi_i^{n-1}(t) 
+ \cdots + a_1(t) \sum_{i=0}^{n} \tilde{c}_i \phi_i'(t) + a_0(t) \sum_{i=0}^{n} \tilde{c}_i \phi_i(t) - g(t). \]  

(4)

Now, forcing the residual \( \tilde{R} \) to become zero at each \( t_j \) in interval \( I \) we have

\[ \sum_{i=0}^{n} \tilde{c}_i \phi_i''(t_j) + a_{n-1}(t) \sum_{i=0}^{n} \tilde{c}_i \phi_i^{n-1}(t_j) 
+ \cdots + a_1(t) \sum_{i=0}^{n} \tilde{c}_i \phi_i'(t_j) + a_0(t) \sum_{i=0}^{n} \tilde{c}_i \phi_i(t_j) - g(t_j) = 0, \]  

(5)

for \( j = 0, 1, \ldots, n. \)

Rewriting equation (5) along with substituting the fuzzy initial condition in equation (4), we may write the following system

\[ \sum_{i=0}^{n} \tilde{c}_i \phi_i''(t_j) + a_{n-1}(t) \sum_{i=0}^{n} \tilde{c}_i \phi_i^{n-1}(t_j) 
+ \cdots + a_1(t) \sum_{i=0}^{n} \tilde{c}_i \phi_i'(t_j) + a_0(t) \sum_{i=0}^{n} \tilde{c}_i \phi_i(t_j) = g(t_j), \]

\[ \tilde{y}(0; \alpha) = \tilde{b}_0 = \sum_{i=0}^{n} \tilde{c}_i \phi_i(t_0), \]

\[ \tilde{y}'(0; \alpha) = \tilde{b}_1 = \sum_{i=0}^{n} \tilde{c}_i \phi_i'(t_0), \]

\[ \vdots \]

\[ \tilde{y}^{(n-1)}(0; \alpha) = \tilde{b}_{n-1} = \sum_{i=0}^{n} \tilde{c}_i \phi_i^{n-1}(t_0). \]  

(6)
Now, solving the above system [equation (6)] one may obtain the fuzzy coefficients, $c_i$. Then putting the values of these fuzzy coefficients in equation (3), one may obtain the approximate solution of $n^{th}$ order FDE [equation (2)].

4 Numerical examples

To illustrate the applicability of the present method we have taken three test problems. In the following paragraphs example problems are solved using the proposed method and are also compared with the exact solution obtained by the method of Bede (2008).

4.1 Example 1

Let us consider the following second order fuzzy linear differential equation with positive coefficients

$$\ddot{y}^* + \dot{y} = -t, \quad t \in [0, 1]$$

subject to the fuzzy initial conditions

$$\dot{y}(0) = [0.141421 \log \alpha, 0.141421 \log \alpha],$$

$$\ddot{y}(0) = [0.188 \log \alpha, 0.188 + 0.1 \log \alpha].$$

The exact solution is as follows

$$Y(t; \alpha) = -0.141421 \sqrt{-2 \log \alpha} (\cos(t) + \sin(t)) + 1.88 \sin(t) - t,$$

$$\dot{Y}(t; \alpha) = 0.141421 \sqrt{-2 \log \alpha} (\cos(t) + \sin(t)) + 1.88 \sin(t) - t.$$

To apply Legendre polynomials first we have to transfer the interval $t \in [0, 1]$ to $x \in [-1, 1]$.

Then the corresponding equation (7) becomes

$$4 \ddot{y} \left( \frac{x+1}{2} \right) + \dot{y} \left( \frac{x+1}{2} \right) = -\left( \frac{x+1}{2} \right), \quad x \in [-1, 1].$$

Let the solution of equation (8) be

$$\dot{y} \left( \frac{x+1}{2} \right) = \sum_{i=0}^{2} c_i \phi_i (x).$$

Using Legendre polynomial we have equation (9) as

$$\dot{y} \left( \frac{x+1}{2} \right) = c_0 + x c_1 + \left( x^2 - \frac{1}{3} \right) c_2.$$

Next, converting equation (10) in term of $t$, we get

$$\ddot{y}(t) = c_0 + (2t-1) c_1 + \left( (2t-1)^2 - \frac{1}{3} \right) c_2.$$
We find the following from equation (11) as
\[ \ddot{y}'(t) = 2\tilde{c}_1 + 4(2t-1)\tilde{c}_2, \]  
\[ \ddot{y}'(t) = 8\tilde{c}_2. \]  
Therefore, residue, \( \tilde{R} \) can be written as
\[ \tilde{R}(x, \tilde{c}_0, \tilde{c}_1, \tilde{c}_2) = \tilde{c}_0 + (2t-1)\tilde{c}_1 + \left( (2t-1)^2 + \frac{23}{3} \right) \tilde{c}_2 + t. \]  
Let us take collocating points as \( t_0 = 1/2 \) and applying initial conditions in equations (12) and (13), we get the system of equations as follows:
\[
\begin{bmatrix}
1 & 2t-1 & (2t-1)^2 & \frac{23}{3} & 0 & 0 & 0 \\
1 & -1 & \frac{2}{3} & 0 & 0 & 0 & 0 \\
0 & 2 & -4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 2t-1 & (2t-1)^2 & \frac{23}{3} \\
0 & 0 & 0 & 1 & -1 & \frac{2}{3} & -4 \\
0 & 0 & 0 & 0 & 2 & -4 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\tilde{c}_0 \\
\tilde{c}_1 \\
\tilde{c}_2 \\
\tilde{x}_0 \\
\tilde{x}_1 \\
\tilde{x}_2 \\
\end{bmatrix}
= \begin{bmatrix}
-\frac{-t}{0.188 - 0.1\sqrt{-2\log_8 \alpha}} \\
-\frac{-t}{0.188 + 0.1\sqrt{-2\log_8 \alpha}} \\
\end{bmatrix}
\]
Solving for \( \tilde{c}_0, \tilde{c}_1 \) and \( \tilde{c}_2 \) from the above system of equations we get
\[
\begin{align*}
\tilde{c}_0 &= \frac{3}{500} - \frac{23}{180} \sqrt{-2\log_8 \alpha}, \\
\tilde{c}_1 &= \frac{-19}{500} - \frac{1}{60} \sqrt{-2\log_8 \alpha}, \\
\tilde{c}_2 &= -\frac{33}{500} + \frac{1}{60} \sqrt{-2\log_8 \alpha}, \\
\tilde{x}_0 &= \frac{3}{500} + \frac{23}{180} \sqrt{-2\log_8 \alpha}, \\
\tilde{x}_1 &= \frac{-19}{500} + \frac{1}{60} \sqrt{-2\log_8 \alpha}, \\
\tilde{x}_2 &= -\frac{33}{500} - \frac{1}{60} \sqrt{-2\log_8 \alpha}.
\end{align*}
\]
Therefore,
\[
y(t; \alpha) = \left( \frac{3}{500} - \frac{23}{180} \sqrt{-2\log_8 \alpha} \right) + (2t-1) \left( \frac{-19}{500} - \frac{1}{60} \sqrt{-2\log_8 \alpha} \right) \\
+ \left( (2t-1)^2 - \frac{1}{3} \right) \left( \frac{33}{500} + \frac{1}{60} \sqrt{-2\log_8 \alpha} \right),
\]
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\[
\bar{y}(t; \alpha) = \left( \frac{3}{500} + \frac{23}{180} \sqrt{-2 \log_e \alpha} \right) + (2t - 1) \left( \frac{-19}{500} + \frac{1}{60} \sqrt{-2 \log_e \alpha} \right) \\
+ \left( (2t - 1)^2 - \frac{1}{3} \right) \left( -\frac{33}{500} - \frac{1}{60} \sqrt{-2 \log_e \alpha} \right).
\]

So from this, one may have the final solution as \( \hat{y}(t; \alpha) = \left[ y(t; \alpha), \bar{y}(t; \alpha) \right] \).

Now the results obtained by the proposed method and the exact solution obtained by the method of Bede (2008) are tabulated in Tables 1 to 4. Tables 1 and 3 give lower bound of the solution for \( t = 0 \) and 0.01 respectively. Similarly Tables 2 and 4 incorporate upper bound solution for \( t = 0 \) and 0.01. Also the absolute errors are given in the last column of Tables 1 to 4 for different values of \( t \). By looking into the results, one may conclude that the solution obtained by proposed method approximately same as that of the exact solution. Corresponding fuzzy plot is given in Figure 1.

**Table 1** Lower bound of the solution for Example 1 at \( t = 0 \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Present ( (y(0; \alpha)) )</th>
<th>Bede (2008) ( (Y(0; \alpha)) )</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-0.2145966026</td>
<td>-0.2145966026</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.1794122578</td>
<td>-0.1794122578</td>
<td>0</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.1551755654</td>
<td>-0.1551755654</td>
<td>0</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.1353728726</td>
<td>-0.1353728726</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.1177410023</td>
<td>-0.1177410023</td>
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</tr>
<tr>
<td>0.6</td>
<td>-0.1010767653</td>
<td>-0.1010767653</td>
<td>0</td>
</tr>
<tr>
<td>0.7</td>
<td>-0.08446004309</td>
<td>-0.08446004309</td>
<td>0</td>
</tr>
<tr>
<td>0.8</td>
<td>-0.06680472308</td>
<td>-0.06680472308</td>
<td>0</td>
</tr>
<tr>
<td>0.9</td>
<td>-0.0459043605</td>
<td>-0.0459043605</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 2** Upper bound of the solution for Example 1 at \( t = 0 \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Present ( (\bar{y}(0; \alpha)) )</th>
<th>Bede (2008) ( (\bar{Y}(0; \alpha)) )</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.2145966026</td>
<td>0.2145966026</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
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<td>0.1794122578</td>
<td>0</td>
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<td>0.3</td>
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<td>0.1551755654</td>
<td>0</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1353728726</td>
<td>0.1353728726</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
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<td>0.1177410023</td>
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<tr>
<td>0.9</td>
<td>0.0459043605</td>
<td>0.0459043605</td>
<td>0</td>
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<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 3  Lower bound of the solution for Example 1 at \( t = 0.01 \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Present (( y(0.01;\alpha) ))</th>
<th>Bede (2008) (( Y(0.01;\alpha) ))</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-0.2148746622</td>
<td>-0.2148520011</td>
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<td>-0.1793175779</td>
<td>2.324e-5</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.154863376</td>
<td>-0.1548397344</td>
<td>2.364e-5</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.1348639765</td>
<td>-0.1348400082</td>
<td>2.396e-5</td>
</tr>
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<td>0.5</td>
<td>-0.1170569629</td>
<td>-0.1170327036</td>
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</tr>
<tr>
<td>0.6</td>
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<td>-0.1002026603</td>
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</tr>
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</tr>
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<td>-0.04450674382</td>
<td>-0.04448129926</td>
<td>2.544e-5</td>
</tr>
<tr>
<td>1</td>
<td>0.0018536</td>
<td>0.001879802001</td>
<td>2.620e-5</td>
</tr>
</tbody>
</table>

Table 4  Upper bound of the solution for Example 1 at \( t = 0.01 \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Present (( \tilde{y}(0.01;\alpha) ))</th>
<th>Bede (2008) (( \tilde{Y}(0;\alpha) ))</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.2185818622</td>
<td>0.2186116051</td>
<td>2.974e-5</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1830480196</td>
<td>0.1830771819</td>
<td>2.916e-5</td>
</tr>
<tr>
<td>0.3</td>
<td>0.158570576</td>
<td>0.1585993384</td>
<td>2.876e-5</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1385711765</td>
<td>0.1385996122</td>
<td>2.843e-5</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1207641629</td>
<td>0.1207923077</td>
<td>2.814e-5</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1039343945</td>
<td>0.1039622643</td>
<td>2.787e-5</td>
</tr>
<tr>
<td>0.7</td>
<td>0.08715261285</td>
<td>0.08718020848</td>
<td>2.759e-5</td>
</tr>
<tr>
<td>0.8</td>
<td>0.06932191667</td>
<td>0.06934922097</td>
<td>2.730e-5</td>
</tr>
<tr>
<td>0.9</td>
<td>0.04821394382</td>
<td>0.04824090326</td>
<td>2.695e-5</td>
</tr>
<tr>
<td>1</td>
<td>0.0018536</td>
<td>0.001879802001</td>
<td>2.620e-5</td>
</tr>
</tbody>
</table>

Figure 1  Fuzzy solution of Example 1 using the proposed method (see online version for colours)
4.2 Example 2

Now, consider the following second order fuzzy linear differential equation

\[ \ddot{y} - \frac{2}{t^2} \dot{y} = \frac{2}{t}, \quad t \in [1, 2] \]  

(15)

subject to the fuzzy initial conditions as

\[ \ddot{y}(1) = \dot{y}'(1) = \left[ -0.1\sqrt{-2 \log_\alpha \alpha}, 0.1 \sqrt{-2 \log_\alpha \alpha} \right]. \]

The exact fuzzy solution we have obtained as follows

\[ \ddot{Y}(t; \alpha) = -\frac{1}{210t} \begin{bmatrix} -70 - 140t^3 + 210t^2 + t^{3/2} \sqrt{-2 \log_\alpha \alpha} \\ 21 \cos \left( \frac{\sqrt{7}}{2} \log_\alpha t \right) \\ + 3\sqrt{7} \sin \left( \frac{\sqrt{7}}{2} \log_\alpha t \right) \end{bmatrix}, \]

\[ \ddot{Y}(t; \alpha) = \frac{1}{210t} \begin{bmatrix} 70 + 140t^3 - 210t^2 + t^{3/2} \sqrt{-2 \log_\alpha \alpha} \\ 21 \cos \left( \frac{\sqrt{7}}{2} \log_\alpha t \right) \\ + 3\sqrt{7} \sin \left( \frac{\sqrt{7}}{2} \log_\alpha t \right) \end{bmatrix}. \]

By following the proposed method the original equation (15) may again be converted to system of equations as discussed above. Consider the collocating point as \( t_0 = \frac{3}{2} \).

\[ \begin{bmatrix} 0 & 0 & 8 & -\frac{8}{4t^2} & \frac{8(2t-3)}{4t^2} & \frac{8((2t-3)^2 - (1/3))}{4t^2} \\ 1 & -1 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 2 & -4 & 0 & 0 & 0 \\ \frac{8}{4t^2} & \frac{8(2t-3)}{4t^2} & \frac{8((2t-3)^2 - (1/3))}{4t^2} & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{bmatrix} \]
Therefore,
\[ y(t; \alpha) = \left( \frac{1}{4} - \frac{13}{100} \sqrt{-2 \log_\alpha} \right) + (2t - 3) \left( \frac{3}{8} - \frac{1}{50} \sqrt{-2 \log_\alpha} \right) \\
+ \left( (2t - 3)^2 - \frac{1}{3} \right) \left( \frac{3}{16} + \frac{3}{200} \sqrt{-2 \log_\alpha} \right), \]
\[ \bar{y}(t; \alpha) = \left( \frac{1}{4} + \frac{13}{100} \sqrt{-2 \log_\alpha} \right) + (2t - 3) \left( \frac{3}{8} + \frac{1}{50} \sqrt{-2 \log_\alpha} \right) \\
+ \left( (2t - 3)^2 - \frac{1}{3} \right) \left( \frac{3}{16} - \frac{3}{200} \sqrt{-2 \log_\alpha} \right). \]

Again, the results obtained by proposed method and the exact solution are given in Tables 5 to 8, to get the bounds of the solutions. Also the absolute errors are given in Tables 5 to 8 for various values of \( t \) and from this we conclude that the solution from present method is approximately equal to the exact solution by the method of Bede (2008). Fuzzy plot for Example 2 are depicted in Figure 2.

**Table 5**  Lower bound of the solution for Example 2 at \( t = 1 \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Present (( y(t; \alpha) ))</th>
<th>Bede (2008) (( Y(t; \alpha) ))</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-0.2145966026</td>
<td>-0.2145966026</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.1794122578</td>
<td>-0.1794122578</td>
<td>0</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.1551755654</td>
<td>-0.1551755654</td>
<td>0</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.1353728726</td>
<td>-0.1353728726</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.1177410023</td>
<td>-0.1177410023</td>
<td>0</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.1010767653</td>
<td>-0.1010767653</td>
<td>0</td>
</tr>
<tr>
<td>0.7</td>
<td>-0.08446004309</td>
<td>-0.08446004309</td>
<td>0</td>
</tr>
<tr>
<td>0.8</td>
<td>-0.06680472308</td>
<td>-0.06680472308</td>
<td>0</td>
</tr>
<tr>
<td>0.9</td>
<td>-0.0459043605</td>
<td>-0.0459043605</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 6**  Upper bound of the solution for Example 2 at \( t = 1 \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Present (( \bar{y}(t; \alpha) ))</th>
<th>Bede (2008) (( \bar{Y}(t; \alpha) ))</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.2145966026</td>
<td>0.2145966026</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1794122578</td>
<td>0.1794122578</td>
<td>0</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1551755654</td>
<td>0.1551755654</td>
<td>0</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1353728726</td>
<td>0.1353728726</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1177410023</td>
<td>0.1177410023</td>
<td>0</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1010767653</td>
<td>0.1010767653</td>
<td>0</td>
</tr>
<tr>
<td>0.7</td>
<td>0.08446004309</td>
<td>0.08446004309</td>
<td>0</td>
</tr>
<tr>
<td>0.8</td>
<td>0.06680472308</td>
<td>0.06680472308</td>
<td>0</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0459043605</td>
<td>0.0459043605</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 7  Lower bound of the solution for Example 2 at $t = 1.01$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Present ($y(1.01; \alpha)$)</th>
<th>Bede (2008) ($Y(1.01; \alpha)$)</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-0.2166546929</td>
<td>-0.2166215106</td>
<td>3.3182e-5</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.1811206156</td>
<td>-0.181088829</td>
<td>3.1787e-5</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.1566430105</td>
<td>-0.1566121852</td>
<td>3.0825e-5</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.136643479</td>
<td>-0.1366134392</td>
<td>3.0040e-5</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.1188363478</td>
<td>-0.1188070075</td>
<td>2.9340e-5</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.1020064683</td>
<td>-0.101977789</td>
<td>2.8679e-5</td>
</tr>
<tr>
<td>0.7</td>
<td>-0.08522457592</td>
<td>-0.0851965557</td>
<td>2.8020e-5</td>
</tr>
<tr>
<td>0.8</td>
<td>-0.06739376203</td>
<td>-0.06736644214</td>
<td>2.7320e-5</td>
</tr>
<tr>
<td>0.9</td>
<td>-0.04628564985</td>
<td>-0.04625915901</td>
<td>2.6491e-5</td>
</tr>
<tr>
<td>1</td>
<td>0.000075</td>
<td>0.000099669967</td>
<td>2.4670e-5</td>
</tr>
</tbody>
</table>

Table 8  Upper bound of the solution for Example 2 at $t = 1.01$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Present ($\bar{Y}(1.01; \alpha)$)</th>
<th>Bede (2008) ($\bar{Y}(1.01; \alpha)$)</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.2168046929</td>
<td>0.2168208505</td>
<td>1.6158e-5</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1812706156</td>
<td>0.1812881689</td>
<td>1.7553e-5</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1567930105</td>
<td>0.1568115252</td>
<td>1.8515e-5</td>
</tr>
<tr>
<td>0.4</td>
<td>0.136793479</td>
<td>0.1368127791</td>
<td>1.9300e-5</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1189863478</td>
<td>0.1190063474</td>
<td>2.0000e-5</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1021564683</td>
<td>0.1021771289</td>
<td>2.0661e-5</td>
</tr>
<tr>
<td>0.7</td>
<td>0.08537457592</td>
<td>0.08539589564</td>
<td>2.1320e-5</td>
</tr>
<tr>
<td>0.8</td>
<td>0.06754376203</td>
<td>0.06756578208</td>
<td>2.2020e-5</td>
</tr>
<tr>
<td>0.9</td>
<td>0.04643564985</td>
<td>0.04645849894</td>
<td>2.2849e-5</td>
</tr>
<tr>
<td>1</td>
<td>0.000075</td>
<td>0.000099669967</td>
<td>2.4670e-5</td>
</tr>
</tbody>
</table>

Figure 2  Fuzzy solution of Example 2 using the proposed method (see online version for colours)
4.3 Example 3

Next, take the following fuzzy linear differential equation

\[ \dddot{y} - 4\ddot{y} + 4y = 0, \quad t \in [0, 1] \]  \hspace{1cm} (16)

subject to the fuzzy initial conditions as

\[ \ddot{y}(0) = \left[3 - 0.1\sqrt{-2\log\alpha}, 3 + 0.1\sqrt{-2\log\alpha}\right], \]

\[ \dddot{y}'(0) = \left[6 - 0.1\sqrt{-2\log\alpha}, 6 + 0.1\sqrt{-2\log\alpha}\right]. \]

The exact fuzzy solution may be obtained as

\[ Y(t; \alpha) = 3e^{2t} - \left(\frac{1}{10}\right)e^{2t}(1 + 3t)\sqrt{-2\log\alpha}, \]

\[ \bar{Y}(t; \alpha) = 3e^{2t} + \left(\frac{1}{10}\right)e^{2t}(1 + 3t)\sqrt{-2\log\alpha}. \]

By following the proposed method the original equation (16) is converted to set of system of equations and collocating point is taken at \( t_0 = 1/2 \). As such we have the following matrix equation as before:

\[
\begin{bmatrix}
4 & 4(2t - 1) & \frac{20}{3} + 4(2t - 1)^2 & 0 & -8 & -16(2t - 1)
\end{bmatrix}
\begin{bmatrix}
\xi_0 \\
\xi_1 \\
\xi_2 \\
\xi_3 \\
\xi_4 \\
\xi_5
\end{bmatrix}
= \begin{bmatrix}
0 \\
3 - 0.1\sqrt{-2\log\alpha} \\
3 + 0.1\sqrt{-2\log\alpha} \\
0 \\
6 - 0.1\sqrt{-2\log\alpha} \\
6 + 0.1\sqrt{-2\log\alpha}
\end{bmatrix}.
\]
Therefore,

\[ y(t; \alpha) = (6 - 0.1023809524 \sqrt{-2 \log_\alpha \alpha}) + (2t - 1) \left( 3 + 0.0214285714 \sqrt{-2 \log_\alpha \alpha} \right) + \left( (2t - 1)^2 - \frac{1}{3} \right) \left( 0.0357142857 \sqrt{-2 \log_\alpha \alpha} \right), \]

\[ \bar{y}(t; \alpha) = (6 + 0.1023809524 \sqrt{-2 \log_\alpha \alpha}) + (2t - 1) \left( 3 - 0.0214285714 \sqrt{-2 \log_\alpha \alpha} \right) + \left( (2t - 1)^2 - \frac{1}{3} \right) \left( -0.0357142857 \sqrt{-2 \log_\alpha \alpha} \right). \]

Solution obtained by proposed method is again compared with the exact solutions and corresponding results are tabulated in Tables 9 and 10. From the tables, we observe that the present solution is very close to the exact solution. Fuzzy plot for this example are also depicted in Figure 3.

**Table 9** Lower bound of the solution for Example 3 at \( t = 0.01 \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Present ( (y(0.01; \alpha)) )</th>
<th>Bede (2008) ( (\bar{y}(0.01; \alpha)) )</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2.830514481</td>
<td>2.843946296</td>
<td>0.0134</td>
</tr>
<tr>
<td>0.2</td>
<td>2.868139949</td>
<td>2.879468573</td>
<td>0.0113</td>
</tr>
<tr>
<td>0.3</td>
<td>2.894058206</td>
<td>2.90393805</td>
<td>0.0099</td>
</tr>
<tr>
<td>0.4</td>
<td>2.915234829</td>
<td>2.923930941</td>
<td>0.0087</td>
</tr>
<tr>
<td>0.5</td>
<td>2.934900015</td>
<td>2.941732159</td>
<td>0.0076</td>
</tr>
<tr>
<td>0.6</td>
<td>2.951910433</td>
<td>2.95855645</td>
<td>0.0066</td>
</tr>
<tr>
<td>0.7</td>
<td>2.969680039</td>
<td>2.97532769</td>
<td>0.0057</td>
</tr>
<tr>
<td>0.8</td>
<td>2.988560302</td>
<td>2.993157662</td>
<td>0.0046</td>
</tr>
<tr>
<td>0.9</td>
<td>3.010910751</td>
<td>3.014258765</td>
<td>0.0033</td>
</tr>
<tr>
<td>1</td>
<td>3.06</td>
<td>3.06060402</td>
<td>6.0402e-4</td>
</tr>
</tbody>
</table>

**Table 10** Upper bound of the solution for Example 3 at \( t = 0.01 \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Present ( (\bar{y}(0.01; \alpha)) )</th>
<th>Bede (2008) ( (\bar{y}(0.01; \alpha)) )</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>3.289485519</td>
<td>3.277261744</td>
<td>0.0122</td>
</tr>
<tr>
<td>0.2</td>
<td>3.251860051</td>
<td>3.241739467</td>
<td>0.0101</td>
</tr>
<tr>
<td>0.3</td>
<td>3.225941794</td>
<td>3.21726999</td>
<td>0.0087</td>
</tr>
<tr>
<td>0.4</td>
<td>3.204765171</td>
<td>3.1972771</td>
<td>0.0075</td>
</tr>
<tr>
<td>0.5</td>
<td>3.185909985</td>
<td>3.179475882</td>
<td>0.0064</td>
</tr>
<tr>
<td>0.6</td>
<td>3.168089567</td>
<td>3.162651591</td>
<td>0.0054</td>
</tr>
<tr>
<td>0.7</td>
<td>3.150319961</td>
<td>3.145875271</td>
<td>0.0044</td>
</tr>
<tr>
<td>0.8</td>
<td>3.131436998</td>
<td>3.128050378</td>
<td>0.0034</td>
</tr>
<tr>
<td>0.9</td>
<td>3.109089249</td>
<td>3.106949275</td>
<td>0.0021</td>
</tr>
<tr>
<td>1</td>
<td>3.06</td>
<td>3.06060402</td>
<td>6.0402e-4</td>
</tr>
</tbody>
</table>
4.4 Example 4

Finally, the differential equation of vibration of a spring mass system with \( m = 1 \) slug in Figure 4 (Buckley and Feuring, 2001) is considered as an application problem. The spring constant is \( k = 4 \) lb/ft with no damping force and the forcing function is \( 100 \cos \omega t \) for \( \omega > 0 \). Corresponding differential equation of motion is

\[
\ddot{y} + 4\dot{y} = 100 \cos \omega t, \quad t \in [0, 1]
\]  

subject to the fuzzy initial conditions as

\[
\dot{y}(0) = \left[ -0.1 \sqrt{-2 \log_e \alpha}, 0.1 \sqrt{-2 \log_e \alpha} \right],
\]

\[
\dot{y}(0) = \left[ -0.1 \sqrt{-2 \log_e \alpha}, 0.1 \sqrt{-2 \log_e \alpha} \right].
\]

We may obtain the exact solution as

\[
\bar{y}(t; \alpha) = -\frac{1}{20} \sqrt{-2 \log_e \alpha} \sin 2t - \frac{100 \cos \omega t}{-4 + \omega^2} - \frac{1}{10} \left( \cos 2t \left( -4 \sqrt{-2 \log_e \alpha} + \omega^2 \sqrt{-2 \log_e \alpha} \right) - 1,000 \right),
\]

\[
\bar{y}(t; \alpha) = \frac{1}{20} \sqrt{-2 \log_e \alpha} \sin 2t - \frac{100 \cos \omega t}{-4 + \omega^2} + \frac{1}{10} \left( \cos 2t \left( -4 \sqrt{-2 \log_e \alpha} + \omega^2 \sqrt{-2 \log_e \alpha} \right) + 1,000 \right).
\]
By following the similar procedure, proposed method the original equation (17) is converted to system of equations, where the collocating point has been taken as $t_0 = 1/4$.

\[
\begin{bmatrix}
1 & 2t - 1 & (2t - 1)^2 + \frac{23}{3} & 0 & 0 & 0 \\
1 & -1 & \frac{2}{3} & 0 & 0 & 0 \\
0 & 2 & -4 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 2t - 1 & (2t - 1)^2 + \frac{23}{3} \\
0 & 0 & 0 & 1 & -1 & \frac{2}{3} \\
0 & 0 & 0 & 0 & 2 & -4 \\
\end{bmatrix}
= 
\begin{bmatrix}
\xi_0 \\
\xi_1 \\
\xi_2 \\
\xi_0 \\
\xi_1 \\
\xi_2 \\
\end{bmatrix}
\begin{bmatrix}
100 \cos \omega t \\
-0.1 \sqrt{-2 \log_\alpha \alpha} \\
-0.1 \sqrt{-2 \log_\alpha \alpha} \\
100 \cos \epsilon t \\
0.1 \sqrt{-2 \log_\alpha \alpha} \\
0.1 \sqrt{-2 \log_\alpha \alpha} \\
\end{bmatrix}
\]
Solution of the above system are obtained as

\[
y(t; \alpha) = \left( \frac{1,600}{99} \cos \omega t - \frac{257}{1,980} \sqrt{-2 \log_e \alpha} \right) \\
+ (2t - 1) \left( \frac{800}{33} \cos \omega t - \frac{13}{660} \sqrt{-2 \log_e \alpha} \right) \\
+ (2t - 1)^2 \left( \frac{400}{33} \cos \omega t + \frac{1}{66} \sqrt{-2 \log_e \alpha} \right),
\]

\[
\bar{y}(t; \alpha) = \left( \frac{1,600}{99} \cos \omega t + \frac{257}{1,980} \sqrt{-2 \log_e \alpha} \right) \\
+ (2t - 1) \left( \frac{800}{33} \cos \omega t + \frac{13}{660} \sqrt{-2 \log_e \alpha} \right) \\
+ (2t - 1)^2 \left( \frac{400}{33} \cos \omega t - \frac{1}{66} \sqrt{-2 \log_e \alpha} \right).
\]

### Table 11
Lower bound of the solution for Example 4 at \( t = 0.001 \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Present ( (y(0.001; \alpha)) )</th>
<th>Bede (2008) ( (\bar{y}(0.001; \alpha)) )</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-0.203794443</td>
<td>-0.214760769</td>
<td>0.0110</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.170373225</td>
<td>-0.179541311</td>
<td>0.0092</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.147351060</td>
<td>-0.155280430</td>
<td>0.0079</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.128540702</td>
<td>-0.135457974</td>
<td>0.0069</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.111792385</td>
<td>-0.117808507</td>
<td>0.0060</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.095963211</td>
<td>-0.101127639</td>
<td>0.0052</td>
</tr>
<tr>
<td>0.7</td>
<td>-0.080179171</td>
<td>-0.084494334</td>
<td>0.0043</td>
</tr>
<tr>
<td>0.8</td>
<td>-0.063408579</td>
<td>-0.066821394</td>
<td>0.0034</td>
</tr>
<tr>
<td>0.9</td>
<td>-0.043555557</td>
<td>-0.045900173</td>
<td>0.0023</td>
</tr>
<tr>
<td>1</td>
<td>0.000048484</td>
<td>0.000049999</td>
<td>1.5150e-6</td>
</tr>
</tbody>
</table>

### Table 12
Upper bound of the solution for Example 4 at \( t = 0.001 \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Present ( (\bar{y}(0.001; \alpha)) )</th>
<th>Bede (2008) ( (\bar{y}(0.001; \alpha)) )</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.203891413</td>
<td>0.214860769</td>
<td>0.0110</td>
</tr>
<tr>
<td>0.2</td>
<td>0.170470195</td>
<td>0.179641311</td>
<td>0.0092</td>
</tr>
<tr>
<td>0.3</td>
<td>0.14744803</td>
<td>0.15538043</td>
<td>0.0079</td>
</tr>
<tr>
<td>0.4</td>
<td>0.128637672</td>
<td>0.135557974</td>
<td>0.0069</td>
</tr>
<tr>
<td>0.5</td>
<td>0.111889354</td>
<td>0.117908507</td>
<td>0.0060</td>
</tr>
<tr>
<td>0.6</td>
<td>0.096060181</td>
<td>0.101227639</td>
<td>0.0052</td>
</tr>
<tr>
<td>0.7</td>
<td>0.080276141</td>
<td>0.084594334</td>
<td>0.0043</td>
</tr>
<tr>
<td>0.8</td>
<td>0.063505548</td>
<td>0.066921394</td>
<td>0.0034</td>
</tr>
<tr>
<td>0.9</td>
<td>0.043652526</td>
<td>0.0460001</td>
<td>0.0023</td>
</tr>
<tr>
<td>1</td>
<td>0.000048484</td>
<td>0.000049999</td>
<td>1.5150e-6</td>
</tr>
</tbody>
</table>
Again the solution obtained by proposed method is compared with the exact solutions. It is interesting to note that present solution is almost same as that of the exact solution obtained by the method of Bede (2008). Corresponding results are given in Tables 11 and 12 and plots for this example are defined in Figure 5.

5 Conclusions

In this paper, Legendre polynomials are used in the collocation method to obtain the numerical solution of $n^{th}$ order FDEs. Gaussian convex normalised fuzzy initial conditions are considered for the analysis. Obtained solutions are depicted in term of plots and tables. Finally, the results of the present method are compared with the exact solution and are found to be in good agreement.

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References


