Stability analysis of an eco-epidemiological SIN model with impulsive control strategy for integrated pest management considering stage-structure in predator

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Abstract: Pest outbreak is a major threat to agricultural resources and livestock. Integrated pest management is widely being used nowadays to control the pest population. Natural enemies are immensely beneficial to control the outbreak of pests. To achieve the same, in this paper, microbial and biological pest control techniques are applied simultaneously by impulsively releasing natural enemies and infected pests. Therefore a SIN (prey-predator) model considering infection in prey with two classes (susceptible-infected) and stage structure in predator is investigated for the cause of integrated pest management. Prey acts as pest and predator plays the role of a natural enemy. Firstly, local and global stability of pest extinction periodic solution is carried out, then condition for the permanence of system is derived using Stroboscopic map, comparison analysis technique and Floquet theory of impulsive differential equations. Further, it is observed that there exists a threshold value of the impulsive period which plays an important role in the dynamics of the system. Finally, for validating the established results, numerical simulation is done using MATLAB.

Keywords: biological control; impulsive control strategy; pest management; stage-structure.

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1 Introduction

Eradication of agricultural pests is a matter of great concern over the past few decades. Many times their outbreak has resulted in production loss because of the destruction of crops and economic impoverishment due to spending on measures to avert these losses. Therefore, with the advancement in agricultural technology, farmers are acquiring the best pest control techniques. One such widely used technique is chemical control consisting of spraying pesticides. Also, biological control using specific living organisms as natural enemies of pests is implemented on a large scale nowadays. Other techniques include physical control by killing and removing pests with hand using manpower and remote sensing. Big achievements have been made by eminent researchers in this regard. Although, Hong et al. (2007) and Mcewen (1979) observed that pesticides are proved to be very effective to eradicate pests. But they are contributing a lot to environmental pollution, identified as a paramount health hazard to mankind and also harmful for certain beneficial pests such as pollinators as described in Kalmakoff and Longworth (1980). Several pest species have become resistant to pesticides due to long term use. Cherry et al. (1999) also studied that due to high cost, small scale farmers are finding it hard to use chemical pesticides.

Therefore, biological control is the best alternative. It is executed in two ways. Freedman (1976) explained that the first way includes some specific natural enemies and these act as predators for the targeted pests. Second is the microbial control that involves spreading of some infectious diseases in pests using viruses. These are bacteria, fungi, nematodes and protozoa. Again, there are two ways to insert these insect pathogens in the targeted pest community. In the first method, to create an epidemic in the pest population, a marginal amount of pathogens is introduced in the pest population. In the second method, pathogens are used as biopesticides. In this approach, the pathogen is applied when the targeted pests reach at an economically significant level and the pathogen cannot survive for a long time in the environment as explained in Burges and Hussey (1971) and Falcon (1976). Therefore, these organisms are capable of creating an epidemic in the pest population by interfering with their biological process. One such example is given in Lacey et al. (2015) that entomopathogenic bacteria *Bacillus thuringiensis* acts as a strong microbial control agent against many species of Lepidopteran pests (Cotton Bollworm, Pink Bollworm). Moreover, biological pest control is considered as a boon for both open crop fields and greenhouses. Lanteren and Woets (1988) stated that more than fifty percent of the world's greenhouse area is covered by the Netherlands and the UK. Biological control had been a great success in these countries as the parasitoid *Encassia formosa* is widely used to control tomato pest *Trialeurodes vaporariorum*.

Mary and Robert (1981) explained that integrated pest management is to suppress the pest population below the acceptable range which is called economic injury level (EIL) in order to avoid major economic and yield loss. Because complete eradication of pests is very expensive, so integrated pest management (IPM) is emerging as a topic of broad interest for the past few years. Many researchers are working in this area and they have provided different strategies to hinder the growth of the targeted pests using a combination of chemical and biological control. Nandi et al. (2015) developed an ecological model consisting of predator-prey interaction with two-stage infection in prey for pest management using ordinary differential equations. The authors have analysed the dynamics of the system at five different equilibria. They found that there was rapid increase in the pest population below some critical value of the carrying capacity in the absence of natural enemy and infection.

Further, the dynamics of pest control models using biological and chemical control techniques is studied effectively with the help of impulsive differential equations as these techniques involve the instantaneous implication of viruses or natural predators of specified pests. Impulsive differential equations have a plethora of applications in modelling in ecology, population dynamics and other applied sciences described in Lakshmikantham et al. (1989), Bainov and Simenov (1993) and Dishlieva (2012). These act as a good mathematical tool to represent several real-life phenomena that undergo short term perturbations, see Liu and Chen (2003), Dong et al. (2005), Jiao et al. (2009), Tan et al. (2012) and Kalra and Kaur (2019). Liu et al. (2005) have studied the dynamics of the prey-dependent consumption model with linear impulsive control strategy. The authors have established that the pest population can be suppressed by taking an impulsive period greater than the specified threshold value to prevent an outbreak. Similarly, valuable results in terms of threshold impulsive period and the release amount of infected pests and natural enemies are obtained in Jiao et al. (2009), Su et al. (2008), Shi et al. (2009); Shi and Chen (2010), Zhao et al. (2012) and Huang and Wang (2013) to check pest population. It is also found that nonlinear impulsive control measures can also be used to control the pest population where periodic release amount of natural enemies and pests depend on their amount already present in the field. Tian et al. (2019) studied nonlinear impulsive control actions which are based on the density of the natural predator. They opined that these measures could develop complex switching pattern which warns about the possibility of an epidemic of the prey population because of ecological challenge. In order to make these impulsive pest control models more effective and realistic, many scholars have also incorporated the concept of delay factor (time delay and gestation delay). Kumar et al. (2019) analysed plant-pest-natural enemy model assuming gestation delay time in natural enemies as well as pest population. They proved boundedness of the system and carried out the bifurcation analysis. Kumari et al. (2020) studied integrated pest management approach with time delay which significantly suppresses pest population and prevents pest resistance to yield. They opined that these control measures are proved to be more effective in reducing pest population if these are applied in combination. This approach also led to positive economic and environment outcomes. Recently, the researchers have paid more attention in the implementation of microbial pest control by dividing the pest population into two or three parts as susceptible-infected (SI), susceptible-exposed-infected (SEI) and susceptible-exposed-infected -natural enemy (SEIN) models. Wang and Song (2010) have studied an impulsive SEI model for pest management considering nonlinear incidence rate and established threshold impulsive period which was the key parameter for the permanence of the system. Extending this work, Mathur and Dhar (2018) analysed an eco-epidemiological SEIN model considering impulsive control and have observed that natural enemies play an important role to calculate the threshold value of the impulsive period.

Furthermore, a good biological understanding of different life stages of pests and natural enemies must be there for the effectiveness of biological pest control. It is easy to identify different life stages (such as eggs, larva and moult) of insects. Many researchers have studied stage-structuring in prey-predator dynamics considering immature and mature population of one or both the species involved. One such model is proposed by Ma et al. (2010). The authors have taken two stages of predator, immature larvae and mature adults. The mortality rate of both immature and mature predator population is taken as same. Motivated by this, Jatav and Dhar (2014) considered a stage-structured plant-pest-natural enemy (food chain) model for impulsive pest control strategy. Again, Bhanu et al. (2020) extended the work done in Mathur and Dhar (2018) by analysing stage- structure in susceptible pest population. The authors examined that impulsive control method has acquired special significance in the extinction and permanence of pests. The results of the study brought out that the annihilation of susceptible pests (immature or mature) and exposed preys completely leans on the pulse releasing amount and impulsive period. The study further pointed out that biological control methods which include releasing of predators or infected pests are very effective to suppress the pest population. Kumari et al. (2018) proposed and examined a plant-pest-natural enemy model with hybrid impulsive control strategy. They have considered Holling type II functional response for plant-pest interaction which incorporates the time taken by pests to process the food. They have established threshold value of impulsive period for the extinction of pest population.

In this paper, therefore, a stage-structured predator-prey model is taken into consideration by acknowledging infection in prey for IPM. Prey acts as pest and predator plays the role of a natural enemy. Also, the functional response of the prey population to predator plays a significant role in predator-prey interactions. This refers to the intake rate of the predator as a function of prey density. It can be prey dependent or predator dependent. Most of the work done in this field subsumed Holling type I and II functional responses for interactions between prey and predator population. However, in this paper, the functional response of susceptible pest population to the predator is taken as Holling type IV. It incorporated the situation of group defence by prey species that makes the model more realistic. Thus, there is decrease in predation rate because the ability of prey species to defend themselves get enhanced in a group resulting decrease in the threshold value of the impulsive period for the permanence of the system.

2 Mathematical model

In this modelling process, stage structuring in the predator population is considered by categorising predator into two stages, immature larvae and mature adults. The mortality rate of both immature and mature predator population is taken as same. Therefore, the following predator-prey model is proposed and examined in this paper by taking an immature and mature class of predator and infection in prey.

$$\begin{cases} \left.\begin{aligned} \frac{dx_s(t)}{dt} &= \alpha x_s\left(1-\frac{x_s}{\beta}\right) - \frac{\beta_i x_s x_i}{1+\gamma_4 x_s} - \frac{\alpha_n x_s y_{ea}}{1+\gamma_2 x_s+\gamma_3 x_s^2}, \\ \frac{dx_i(t)}{dt} &= \frac{\beta_i x_s x_i}{1+\gamma_4 x_s} - \delta_1 x_i, \\ \frac{dy_{em}(t)}{dt} &= \frac{\gamma_1 \alpha_n x_s y_{ea}}{1+\gamma_2 x_s+\gamma_3 x_s^2} - \mu_{em} y_{em} - \delta_2 y_{em}, \\ \frac{dy_{ea}(t)}{dt} &= \mu_{em} y_{em} - \delta_2 y_{ea}, \end{aligned}\right\} t \neq n\tau, \\ \left.\begin{aligned} \Delta x_s(t) &= 0, \\ \Delta x_i(t) &= e_1, \\ \Delta y_{em}(t) &= e_2, \\ \Delta y_{ea}(t) &= e_3, \end{aligned}\right\} t = n\tau, \quad n \in Z_+. \end{cases} \tag{1}$$

where $x_s(t),\ x_i(t),\ y_{em}(t),\ y_{ea}(t)$ be the population densities of susceptible prey, infected prey, immature and mature natural enemies population respectively at time t with initial conditions $x_s(0) > 0,\ x_i(0) > 0,\ y_{em}(0) > 0$ and $y_{ea}(0) > 0$. The model is formulated under some assumptions as follows:

a The logistic growth of prey population is taken in the absence of infection.

b The infected prey population is neither able to reproduce nor recover. Also they do not contribute towards carrying capacity of the total prey population.

c Mature predator only catch susceptible pest and immature predator is not capable of predation. So, their growth mainly depends on mature predator.

d There are four different kinds of Holling type functional responses are available depending on the situation. In this paper, the crowding effect of the susceptible pest population is incorporated. Therefore, Holling II type incidence rate is considered for transmission from susceptible to infected pest population.

e Functional response of prey population to predator is taken as Holling type IV.

f For the integrated pest control, infected pests, immature and mature natural enemies are released periodically at time $t = n\tau$ with intensities e_1, e_2, e_3 respectively. $\Delta x_s(t) = x_s(t^+) - x_s(t),\ \Delta x_i(t) = x_i(t^+) - x_i(t),\ \Delta y_{em}(t) = y_{em}(t^+) - y_{em}(t),\ \Delta y_{ea}(t) = y_{ea}(t^+) - y_{ea}(t)$, where τ is the impulsive period.

The different parameters used in system (1) are defined as follows:

1 $\alpha > 0$ is the internal growth rate of susceptible pests and $\beta > 0$ is the carrying capacity

2 $\alpha_n > 0$ measures the efficiency of the prey to avoid predator's attack

3 $\gamma_2 > 0,\ \gamma_3 > 0,\ \gamma_4 > 0$ are the half saturation constants from Holling type IV and II functional responses

4 μ_{em} is the conversion rate from immature to mature predator

5 β_i is the transmission rate from susceptible to infected pest and δ_1 is the death rate of infected prey

6 δ_2 is the death rate of immature and mature natural enemies

7 γ_1 represents the fraction of prey available to immature predator.

3 Preliminaries

Let $R_+ = [0,\ \infty),\ R_+^4 = \{x\ \in\ R^4 : x \geq 0\},\ \Omega = intR_+^4$. The map defined by the right hand of the system (1) is given as $g = (g_1,\ g_2,\ g_3,\ g_4)^T$. Let $S_0 = \{V : R_+ \times R_+^4 \mapsto R_+$, continuous on $(n\tau,\ (n+1)\,\tau] \times R_+^4$ and $\lim_{(t,\ y)\to(n\tau,\ x),\ t>n\tau} S\,(t,\ x) = S\,(n\tau^+,\ x)$ exits$\}$.

Definition 3.1: $S \in S_0$, then for $(t,x) \in (n\tau,(n+1)\tau] \times R_+^4$, the upper right derivative of $S(t,x)$ with respect to the impulsive differential system (1) is defined as

$$D^+S(t,x) = \limsup_{h\to 0^+} \frac{1}{h}[S(t+h, x+hf(t,x)) - S(t,x)]. \tag{2}$$

Definition 3.2: Consider that $P(t) = (x_s(t), x_i(t), y_{em}(t), y_{ea}(t))^T$ be the solution of equation (1). It is piece-wise continuous function from R_+ to R_+^4 because solution changes its behaviour only at moments of impulse. Therefore, $P(t)$ is continuous in the interval $(n\tau,(n+1)\tau), n \in Z_+$ and $\lim_{t\to n\tau^+}(P(t)) = P(n\tau^+)$ exists also $\lim_{t\to n\tau^-}(P(t)) = P(n\tau))$.

The required system (1) is said to be permanent if $\exists Q \geq q > 0$ such that $q \leq x_s(t), x_i(t), y_{em}(t), y_{ea}(t) \leq Q$ for sufficiently large t and $P(0^+) > 0$.

Our main aim here is to suppress the pests in a targeted region beneath a tolerable limit so that it does not cause major production loss. To achieve the same, we need the following lemma:

Lemma 1: Consider the following impulsive system:

$$\begin{cases} \psi'(t) = -c\psi(t), & t \neq n\tau, \\ \psi(t^+) = \psi(t) + d, & t = n\tau, \\ n \in Z_+. \end{cases} \tag{3}$$

It has periodic solution $\overline{\psi}(t)$ and for any solution $\psi(t)$ of equation (3)

$$\mid \psi(t) - \overline{\psi}(t) \mid \to 0 \ as\ t \to \infty \quad where\ \ \bar{\psi}(t) = \frac{d\exp(-c(t-n\tau))}{1-\exp -c\tau}.$$

Thus $\psi\bar{(t)}$ is globally stable.

4 Boundedness and global stability

4.1 Upper bound of all the variables

Here, in this section, firstly, upper bound for all the variables of system (1) are obtained in the coming lemma:

Lemma 2: For sufficiently large t, $\exists Q_0 > 0$ such that $x_s(t) \leq Q_0, x_i(t) \leq Q_0, y_{em}(t) \leq Q_0, y_{ea}(t) \leq Q_0$. That is there is an upper bound for every solution of equation (1).

Proof: Consider $X(t) = (x_s(t), x_i(t), y_{em}(t), y_{ea}(t))$ as any solution of equation (1).
Let $W(t,\ X(t)) = x_s(t) + x_i(t) + y_{em}(t) + y_{ea}(t)$ for $t \neq n\tau$,

$$\begin{aligned}
D^+W(t) + \theta W(t) = {} & \alpha x_n(t) - \frac{\alpha x_n(t)^2}{\beta} - \beta_i x_n(t) x_i(t) - \frac{\alpha_n x_n(t) y_{ea}(t)}{1 + \gamma_2 x_n(t)} \\
& + \beta_i x_n(t) x_i(t) - \delta_1 x_i(t) + \frac{\gamma_1 \alpha_n x_n(t) y_{ea(t)}}{1 + \gamma_2 x_n(t)} - \delta_2 y_{em}(t) \\
& + \mu_{em} y_{em}(t) + \theta(x_n(t) + x_i(t) + y_{em}(t) + y_{ea}(t)) \\
& - \mu_{em} y_{em}(t) \\
= {} & \alpha x_n(t) - \frac{\alpha x_n(t)^2}{\beta} - (\delta_2 - \theta)(y_{em}(t) + y_{ea}(t)) \\
& - (1 - \gamma_1) \frac{\alpha_n x_n(t) y_{ea}(t)}{1 + \gamma_2 x_n(t)} - (\delta_1 - \theta) x_i(t) \\
\leq {} & (\alpha + \theta) x_n(t) - \frac{\alpha x_n(t)^2}{\beta} \quad (\gamma \leq 1) \\
\leq {} & \frac{\beta(\alpha + \theta)^2}{4\alpha} = L_0
\end{aligned}$$

$$W(t^+) = W(t) + e_1 + e_2 + e_3, for\ t = n\tau.$$

Therefore by Theorem 1.4.1 of Lakshmikantham et al. (1989),

$$\begin{aligned}
W(t) \leq {} & W(0) \exp\left(\int_0^t (-\theta)\right) ds + (e_1 + e_2 + e_3) \sum_{0<n\tau<t} \exp \int_{n\tau}^t (-\theta) ds \\
& + \int_0^t \left(L_0 \exp \int_s^t (-\theta d\sigma) \right) ds \\
& \leq W(0) \exp(-\theta t) + (e_1 + e_2 + e_3) \sum_{0<n\tau<t} \exp(-\theta(t - n\tau)) \\
& + \frac{L_0}{\theta}(1 - \exp(-\theta t)) \\
& \leq W(0) \exp(-\theta t) + \frac{L_0}{\theta}(1 - \exp(-\theta t)) \\
& + \frac{(e_1 + e_2 + e_3)(\exp(-\theta(t - \tau)))}{1 - \exp(-\theta\tau)} \\
& + \frac{(e_1 + e_2 + e_3)(\exp(\theta t))}{\exp(\theta\tau) - 1}
\end{aligned}$$

$$\rightarrow \frac{L_0}{\theta} + \frac{(e_1+e_2+e_3)(\exp(\theta\tau))}{\exp(\theta\tau)-1} = Q_0 \quad as \quad t \rightarrow \infty$$

Thus, W(t) is uniformly bounded. Hence, $\exists$ the constant Q_0 such that $x_s(t) \leq Q_0,\ x_i(t) \leq Q_0,\ y_{em}(t) \leq Q_0,\ y_{ea}(t) \leq Q_0$. This completes the proof. $\square$

Lemma 3: If $V(t)$ be any solution of system (1) with $V(0) \geq 0$ then $V(t) \geq 0$ for all $t \geq 0$. Also $V(t) > 0$ for all $t \geq 0$ if $V(0) > 0$.

After using microbial and natural pest control, when pest population becomes extinct, then $x_s(t) = 0$, the impulsive system (1) reduces to

$$\begin{cases} \left.\begin{aligned} \frac{dx_i(t)}{dt} &= -\delta_1 x_i(t), \\ \frac{dy_{em}(t)}{dt} &= -\mu_{em} y_{em}(t) - \delta_2 y_{em}(t), \\ \frac{dy_{ea}(t)}{dt} &= \mu_{em} y_{em}(t) - \delta_2 y_{ea}(t), \end{aligned}\right\} t \neq n\tau, \\ \left.\begin{aligned} &\Delta x_i(t) = e_1, \\ &\Delta y_{em}(t) = e_2, \\ &\Delta y_{yea}(t) = e_3, \end{aligned}\right\} t = n\tau, \quad n \in\ Z_+. \end{cases} \tag{4}$$

From first and third equations of system (4) and using Lemma 1, we get globally asymptotically stable periodic solution $\bar{x_i}(t)$ as:

$$\bar{x_i}(t) = \frac{e_1 \exp((-\delta_1)(t-n\tau))}{1-\exp(-\delta_1\tau)}; \quad \bar{x_i}(0^+) = \frac{e_1}{1-\exp(-\delta_1 t)}. \tag{5}$$

Similarly, applying Lemma 1 on second and fourth equations of system (4), we have

$$\begin{aligned} \bar{y}_{em}(t) &= \frac{e_2 \exp(-(\mu_{em}+\delta_2)(t-n\tau))}{1-\exp((-\mu_{em}+\delta_2)\tau)}; \\ \bar{y}_{em}(0^+) &= \frac{e_2}{1-\exp((-\mu_{em}+\delta_2)\tau)}. \end{aligned} \tag{6}$$

Now substituting the value of $\bar{y}_{em}(t)$ in third equation of system (4), we get the following subsystem

$$\begin{cases} \frac{dy_{ea}(t)}{dt} = \mu_{em}\bar{y}_{em}(t) - \delta_2 y_{ea}(t), & t \neq n\tau, \\ \Delta y_{ea}(t) = e_3, & t = n\tau, \\ n \in Z_+. \end{cases} \tag{7}$$

Integrating first equation of system (7) on $t \in (n\tau, (n+1)\tau)$,

$$\begin{aligned} y_{ea}(t) = &\ \frac{e_2(\exp(-\delta_2(t-n\tau))) - \exp(-(\mu_{em}+\delta_2)(t-n\tau))}{1-\exp(-(\mu_{em}+\delta_2)\tau)} \\ &+ y_{ea}(n\tau_+)\exp(-\delta_2(t-n\tau)), \end{aligned}$$

where $n\tau < t \leq (n+1)\tau$. After solution is effected by impulse at time $(n+1)\tau$, the stroboscopic map is given as

$$\begin{aligned} y_{ea}(t) &= \frac{e_2(\exp(-\delta\tau) - \exp(-(\mu_{em}+\delta_2)\tau)}{1-\exp(-(\mu_{em}+\delta_2)\tau)} + y_{ea}(n\tau_+)\exp(-\delta_2\tau) + e_3 \\ &= h(y_{ea}(n\tau_+)) \ ; \ n\tau < t < (n+1)\tau. \end{aligned} \tag{8}$$

The above map (8) has a unique fixed point y^*_{ea}. Therefore,

$$\begin{aligned} h(y^*_{ea}) = y^*_{ea} \Rightarrow y^*_{ea} = &\ \frac{e_2(1-\exp(-\mu_{em}\tau))\exp(-\delta_2\tau)}{(1-\exp-(\delta_2+\mu_{em})\tau)(1-\exp(-\delta_2\tau))} \\ &+ \frac{e_3}{1-\exp(-\delta_2\tau)}. \end{aligned}$$

As from equation (8), $h(y_{ea})$ is an increasing function, therefore $0 < y_{ea} < y^*_{ea} \Rightarrow y_{ea} < h(y_{ea}) < y^*_{ea}$ and $y_{ea} > y^*_{ea} \Rightarrow y^*_{ea} < h(y_{ea}) < y_{ea}$. Thus by Cull (1981), y^*_{ea} is globally stable. Hence, the corresponding periodic solution of system (7) is

$$\bar{y}_{ea}(t) = \frac{-e_2\exp((-\delta_2+\mu_{em})(t-n\tau))}{1-\exp(-(\delta_2+\mu_{em})\tau)} + \frac{(e_2+e_3)\exp(-\delta_2(t-n\tau))}{1-\exp(-\delta_2\tau)}, \tag{9}$$

where

$$\begin{aligned} \bar{y}_{ea}(0^+) &= y^*_{ea} \\ &= \frac{-e_2}{1-\exp(-\delta_2+\mu_{em})\tau} + \frac{e_2+e_3}{1-\exp(-\delta_2\tau)}; \ t \in (n\tau,(n+1)\tau), \end{aligned} \tag{10}$$

which is globally asymptotically stable.

Theorem 4: There exists a threshold value (τ_{max}) of the impulsive period such that if $\tau \leq \tau_{max}$, then the susceptible pest eradication solution $(0, \bar{x}_i(t),\ \bar{y}_{em}(t),\ \bar{y}_{ea}(t))$ is locally asymptotically stable and if $\tau > \tau_{max}$, it is unstable where

$$\tau_{max} = \frac{1}{\alpha}\left[\frac{\beta_i e_1}{\delta_1} - \frac{\alpha_n e_2}{\delta_2+\mu_{em}} + \frac{\alpha_n(e_2+e_3)}{\delta_2}\right].$$

Proof: Here, we use small perturbation method to prove the local stability of the required solution. Let $\zeta_1(t), \zeta_2(t), \zeta_3(t), \zeta_4(t)$ be the small perturbations in the periodic solution $(0, \bar{x}_i(t), \bar{y}_{em}(t), \bar{y}_{ea}(t))$ respectively. Then

$$\begin{aligned} &x_s(t) = \zeta_1(t), x_i(t) = \bar{x}_i + \zeta_2(t), y_{em}(t) = \bar{y}_{em}(t) + \zeta_3(t), \\ &y_{ea}(t) = \bar{y}_{ea}(t) + \zeta_4(t). \end{aligned}$$

Putting these values in system (1), it reduces to

$$\begin{cases} \left.\begin{aligned} \frac{d\zeta_1(t)}{dt} &= \alpha\zeta_1(t) - \beta_i\bar{x}_i(t)\zeta_1(t) - \alpha_n\bar{y}_{ea}(t)\zeta_1(t), \\ \frac{d\zeta_2(t)}{dt} &= \beta_i\bar{x}_i(t)\zeta_1(t) - \delta_1\zeta_2(t), \\ \frac{d\zeta_3(t)}{dt} &= \gamma_1\alpha_n\bar{y}_{ea}(t)\zeta_1(t) - \mu_{em}\zeta_3(t) - \delta_2\zeta_3(t), \\ \frac{d\zeta_4(t)}{dt} &= \mu_{em}\zeta_3(t) - \delta_2\zeta_4(t), \end{aligned}\right\} t \neq n\tau, \\ \left.\begin{aligned} \zeta_1(t^+) &= \zeta_1(t), \\ \zeta_2(t^+) &= \zeta_2(t), \\ \zeta_3(t^+) &= \zeta_3(t), \\ \zeta_4(t^+) &= \zeta_4(t), \end{aligned}\right\} t = n\tau, \quad n \in \ Z_+. \end{cases} \tag{11}$$

The above system represents system of linear differential equations, which can be written in matrix form. Hence for $t \neq n\tau$, the coefficient matrix is given as

$$B = \begin{bmatrix} \alpha - \beta_i\bar{x}_i(t) - \alpha_n\bar{y}_{ea}(t) & 0 & 0 & 0 \\ \beta_i\bar{x}_i(t) & -\delta_1 & 0 & 0 \\ \gamma_1\alpha_n\bar{y}_{ea}(t) & 0 & -(\mu_{em} + \delta_2) & 0 \\ 0 & 0 & \mu_{em} & -\delta_2 \end{bmatrix},$$

and for $t = n\tau$,

$$\begin{bmatrix} \zeta_1(t^+) \\ \zeta_2(t^+) \\ \zeta_3(t^+) \\ \zeta_4(t^+) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \\ \zeta_3(t) \\ \zeta_4(t) \end{bmatrix}.$$

Let $\Phi(t)$ be the fundamental solution of system (11), then

$$\frac{d\Phi(t)}{dt} = B\Phi(t), \tag{12}$$

$$\Phi(\tau) = \Phi(0)\exp(\int_0^T B\ dt),$$

with $\Phi(0) = I$, the identity matrix. Solving, we have

$$\Phi(\tau) = \begin{bmatrix} e^{(\int_0^\tau \alpha - \beta_i\bar{x}_i(t) - \alpha_n\bar{y}_{ea}(t)dt)} & 0 & 0 & 0 \\ e^{(\int_0^\tau \beta_i\bar{x}_i(t)dt)} & e^{(\int_0^\tau -\delta_1 dt)} & 0 & 0 \\ e^{(\int_0^\tau \gamma_1\alpha_n\bar{y}_{ea}(t))} & 0 & e^{(\int_0^\tau -(\mu_{em}+\delta_2)dt)} & 0 \\ 0 & 0 & e^{(\int_0^\tau \mu_{em}dt)} & e^{(\int_0^\tau -\delta_2 dt)} \end{bmatrix},$$

which is upper triangular matrix. Now according to Floquet theory of impulsive differential equations [Theorem 3.1 and 3.5 of Bainov and Simenov (1993)], if absolute values of all the eigenvalues of Monodromy matrix M are less than one, then the required solution is globally stable where

$$M = [\Phi(0)]^{-1}\Phi(\tau).$$

Since M is an upper triangular matrix, therefore, eigenvalues of M are

$$
\begin{aligned}
\lambda_1 &= \exp\left(\int_0^{\tau} \alpha - \beta_i \bar{x}_i(t) - \alpha_n \bar{y}_{ea}(t)dt\right),\\
\lambda_2 &= \exp\left(\int_0^{\tau} -\delta_1 dt\right),\\
\lambda_3 &= \exp\left(\int_0^{\tau} -(\mu_{em} + \delta_2)dt\right),\\
\lambda_4 &= \exp\left(\int_0^{\tau} -\delta_2 dt\right).
\end{aligned} \tag{13}
$$

Now, it is obvious from equation (13), that $\mid \lambda_2 \mid < 1$, $\mid \lambda_3 \mid < 1$, $\mid \lambda_4 \mid < 1$ and $\mid \lambda_1 \mid < 1$ if $\tau \leq \tau_{max}$. Hence the required result. □

4.2 Global stability

Theorem 5: There is a threshold value $(\check{\tau})$ of the impulsive period such that if $\tau < \check{\tau}$, then the susceptible pest extinction solution $(0, \bar{x}_i(t), \bar{y}_{em}(t), \bar{y}_{ea}(t))$ is globally asymptotically stable, where

$$
\check{\tau} = \frac{1}{\alpha}\left[\frac{\beta_i e_1}{\delta_1(1+\gamma_4\beta)} - \left(\frac{1}{1+\gamma_2\beta+\gamma_3\beta^2}\right)\left(\frac{\alpha_n e_2}{\delta_2+\mu_{em}} + \frac{\alpha_n(e_2+e_3)}{\delta_2}\right)\right].
$$

Proof: Let $(x_s(t), x_i(t), y_{em}(t), y_{ea}(t))$ be an arbitrary solution of equation (1). Given that $\tau < \check{\tau}$, so, it is possible to find sufficiently small $\grave{\varepsilon} > 0$such that

$$
\int_0^{\tau}\left(\alpha - \frac{\beta_i(\bar{x}_i(t)-\grave{\varepsilon})}{1+\gamma_4\beta} - \frac{\alpha_n(\bar{y}_{ea}(t)-\grave{\varepsilon})}{1+\gamma_2\beta+\gamma_3\beta^2}\right)dt = \varrho_1 < 0. \tag{14}
$$

From equation (1),

$$
\begin{cases}
\frac{dx_i(t)}{dt} \geq -\delta_1 x_i(t), & t \neq n\tau\\
\Delta x_i(t) = e_1, & t = n\tau,\\
n \in Z_+.
\end{cases} \tag{15}
$$

Consider the corresponding comparison impulsive system of equation (15) as

$$
\begin{cases}
\frac{dw_i(t)}{dt} = (-\delta_1 w_i), & t \neq n\tau\\
\Delta w_i(t) = e_1, & t = n\tau.
\end{cases} \tag{16}
$$

Applying Lemma 1, system equation (16) has periodic solution

$$
\bar{w_i}(t) = \frac{e_1\exp((-\delta_1)(t-n\tau))}{1-\exp(-\delta_1\tau)},\ t \in (n\tau, (n+1)\tau],
$$

which is globally asymptotically stable. Therefore by Theorem 1.4.1 of Lakshmikantham et al. (1989), $x_i(t) \geq w_i(t) \to \bar{w}_i(t)$ and $\bar{w}_i(t) = \bar{x}_i(t)$. Hence, $\exists$ a positive integer κ such that $x_i(t) > \bar{x}_i(t) - \grave{\varepsilon}\ \forall\ t \geq \kappa\tau$. Again from equation (1)

$$
\begin{cases}
\frac{dy_{em}(t)}{dt} \geq -(\mu_{em}+\delta_2)y_{em}(t), & t \neq n\tau,\\
\Delta y_{em}(t) = e_2, & t = n\tau.
\end{cases} \tag{17}
$$

Consider the following comparison system of equation (17)

$$\begin{cases} \frac{dw_{em}(t)}{dt} = -(\mu_{em} + \delta_2)w_{em}(t), & t \neq n\tau, \\ \Delta w_{em}(t) = e_2, & t = n\tau. \end{cases} \tag{18}$$

Similarly, by using Lemma 1 and comparison analysis technique of impulsive differential equations, it is obtained that $y_{em}(t) \geq w_{em}(t) \to \bar{w}_{em}(t)$ and $\bar{w}_{em}(t) = \bar{y}_{em}(t)$. Hence, $\exists$ a positive integer κ_1 such that $y_{em}(t) > \bar{y}_{em}(t) - \grave{\varepsilon}\ \forall\ t \geq \kappa_1\tau$. Now, from fourth equation of equation (1), we have,

$$\begin{cases} \frac{dy_{ea}(t)}{dt} \geq \mu_{em}(\bar{y}_{em}(t) - \grave{\varepsilon}) - \delta_2)y_{ea}(t), & t \neq n\tau, \\ \Delta y_{ea}(t) = e_3, & t = n\tau. \end{cases} \tag{19}$$

Consider its corresponding comparison impulsive system

$$\begin{cases} \frac{dw_{ea}(t)}{dt} = \mu_{em}(\bar{y}_{em}(t) - \grave{\varepsilon}) - \delta_2)w_{ea}(t), & t \neq n\tau, \\ \Delta w_{ea}(t) = e_3, & t = n\tau. \end{cases} \tag{20}$$

So, by applying Lemma 1, we get periodic solution of system (20)

$$\begin{aligned} \bar{w}_{ea}(t) = &\ \frac{-e_2 \exp((-\delta_2 + \mu_{em})(t - n\tau))}{1 - \exp(-(\delta_2 + \mu_{em})\tau)} \\ &+ \frac{(e_2 + e_3)\exp(-\delta_2(t - n\tau))}{1 - \exp(-\delta_2\tau)} - \frac{\mu_{em}\grave{\varepsilon}}{\delta_2}. \end{aligned}$$

Applying comparison theorem of IDE, $y_{ea}(t) \geq w_{ea}(t) \to \bar{w}_{ea}(t)$. Hence, $\exists$ a positive integer κ_2 such that $y_{ea}(t) > \bar{y}_{ea}(t) - \grave{\varepsilon}\ \forall\ t \geq \kappa_2\tau$ $(\kappa_2 > \kappa_1 > \kappa)$. Therefore, for $t \geq \kappa_2\tau$, first equation of (1) gives

$$\frac{dx_s(t)}{dt} \leq \left[\alpha - \frac{\beta_i(x_i - \grave{\varepsilon})}{1 + \gamma_4\beta} - \frac{\alpha_n(y_{ea} - \grave{\varepsilon})}{1 + \gamma_2\beta + \gamma_3\beta^2}\right] x_s. \tag{21}$$

Integration of equation (21) on $(\kappa_2\tau, (\kappa_2 + 1)\tau]$ gives

$$x_s((\kappa_2 + q)\tau) \leq x_s(\kappa_2\tau)\exp(q\varrho_1) \to 0 \text{ as } t \to \infty\ (\because \varrho_1 < 0). \tag{22}$$

This implies, there exists a positive integer $\kappa_3 > \kappa_2$ and sufficiently small $\grave{\varepsilon}_1 > 0$ such that $x_s(t) < \grave{\varepsilon}_1$ for $t \geq \kappa_3$ and $\grave{\varepsilon}_1 < \frac{\delta_1}{\beta_i}$. Using maximum value of $x_s(t)$ in the second equation of system (1), we get

$$\begin{cases} \frac{dx_i(t)}{dt} \leq (\beta_i\grave{\varepsilon}_1 - \delta_1)x_i, & t \neq n\tau, \\ \Delta x_i(t) = e_1, & t = n\tau. \end{cases}$$

Analysing again its comparison impulsive system

$$\begin{cases} \frac{du_i(t)}{dt} = (\beta_i\grave{\varepsilon}_1 - \delta_1)x_i, & t \neq n\tau, \\ \Delta u_i(t) = e_1, & t = n\tau. \end{cases} \tag{23}$$

Applying Lemma 1, system (23) has periodic solution

$$\bar{u_i}(t) = \frac{e_1 \exp((\beta_i \grave{\varepsilon}_1 - \delta_1)(t - n\tau))}{1 - \exp(\beta_i \grave{\varepsilon}_1 - \delta_1 \tau)}, t \in (n\tau, (n+1)\tau],$$

which is globally asymptotically stable. Therefore by Theorem 1.4.1 of Lakshmikantham et al. (1989), $x_1(t) \leq u_i(t) \to \bar{u}_i(t)$. Hence, $\exists$ a positive integer κ_4 such that

$$x_i(t) < \bar{u}_i(t) + \grave{\varepsilon} \ \forall \ t \geq \kappa_4 \tau. \tag{24}$$

From third equation of system (1)

$$\begin{cases} \frac{dy_{em}(t)}{dt} \leq (\gamma_1 \alpha_n \grave{\varepsilon}_1 Q_0) - (\mu_{em} + \delta_2) y_{em}(t), & t \neq n\tau, \\ \Delta y_{em}(t) = e_2, & t = n\tau. \end{cases}$$

Applying the same argument, $\exists$ a positive integer κ_5 such that

$$y_{em}(t) \leq \bar{u}_{em}(t) + \grave{\varepsilon} \forall t \geq \kappa_5 \tau, \tag{25}$$

where

$$\bar{u}_{em}(t) = \frac{\gamma_1 \alpha_n \grave{\varepsilon}_1 Q_0}{\mu_{em} + \delta_2} + \frac{e_2 exp(-(\mu_{em} + \delta_2)(t - n\tau))}{1 - exp(-(\mu_{em} + \delta_2)\tau)}.$$

Now, from fourth equation of system (1), we have

$$\begin{cases} \frac{dy_{ea}(t)}{dt} \leq \mu_{em}(\bar{u}_{em}(t) + \grave{\varepsilon}) - \delta_2) y_{ea}(t), & t \neq n\tau, \\ \Delta y_{ea}(t) = e_3, & t = n\tau. \end{cases} \tag{26}$$

Similarly, as above $\exists$ a positive integer κ_6 such that

$$y_{ea}(t) \leq \bar{u}_{ea}(t) + \grave{\varepsilon}_2 \forall \ t \geq \kappa_6 \tau, \tag{27}$$

where

$$\begin{aligned} \bar{u}_{ea}(t) = & \ \frac{-e_2 \exp((-\delta_2 + \mu_{em})(t - n\tau))}{1 - \exp(-(\delta_2 + \mu_{em})\tau)} + \frac{(e_2 + e_3) \exp(-\delta_2(t - n\tau))}{1 - \exp(-\delta_2 \tau)} \\ & + \frac{\mu_{em}}{\delta_2} \left(\frac{\gamma_1 \alpha_n \grave{\varepsilon}_1 Q_0}{\mu_{em} + \delta_2} + \grave{\varepsilon} \right). \end{aligned} \tag{28}$$

As $\grave{\varepsilon} > 0, \grave{\varepsilon}_1 > 0$ and $\grave{\varepsilon}_2 > 0$ are sufficiently small, therefore $\bar{u}_{em}(t) \to \bar{y}_{em}(t)$ and $\bar{u}_{ea}(t) \to \bar{y}_{ea}(t)$ as $t \to \infty$ $(\grave{\varepsilon}_1 \to 0)$. Hence it is established that $x_s(t) \to 0, \ x_i(t) \to \bar{x}_i(t), \ y_{em}(t) \to \bar{y}_{em}(t),$ and$y_{ea} \to \bar{y}_{ea}(t)$ as $t \to \infty$. This completes the proof. □

5 Permanence

The required condition for the system to be permanent is established as follows:

Theorem 6: The system (1) is permanent if $\tau > \tau_{max}$.

Proof: Upper bound of $x_s(t),\ x_i(t),\ y_{em}(t),\ y_{ea}(t)$ of the system is already been obtained in Lemma 2. Also in the above section, it is proved that

$$\begin{aligned} &x_i(t) > \bar{x}_i(t) - \grave{\varepsilon} = q_1\ \forall\ t \geq \kappa_4\tau,\\ &y_{em}(t) > \bar{y}_{em}(t) - \grave{\varepsilon} = q_2\ \forall\ t \geq \kappa_5\tau,\\ &y_{ea}(t) > \bar{y}_{ea}(t) - \grave{\varepsilon} = q_3\ \forall\ t \geq \kappa_6\tau. \end{aligned} \tag{29}$$

Thus, for permanence of the system (1), there must exists a constant $q_4 < min\left(\beta, \frac{\delta_1}{\beta_i}\right)$ such that $x_s(t) \geq q_4$ for sufficiently large t. This is done in two steps as follows

Step 1: To start with, assume that $x_s(t) \geq q_4$ is not true $\forall\ t$. Thus $\exists$ a positive integer l_1 such that $x_s < q_4\ \forall\ t \geq l_1\tau$. Considering this assumption, from system (1), we have

$$\begin{cases} \frac{dx_i(t)}{dt} \leq -(\delta_1 - \beta_i q_4), & t \neq n\tau,\\ \Delta x_i(t) = e_1, & t = n\tau. \end{cases}$$

Consider the following impulsive system

$$\begin{cases} \frac{d\breve{u}_i(t)}{dt} = -(\delta_1 - \beta_i q_4)\breve{u}_i, & t \neq n\tau,\\ \Delta \breve{u}_i(t) = e_1, & t = n\tau. \end{cases} \tag{30}$$

Applying Lemma 1, equation (30) has periodic solution

$$\overline{\breve{u}}_i(t) = \frac{e_1 \exp(-(\delta_1 - \beta_i q_4)(t - n\tau))}{1 - \exp(-(\delta_1 - \beta_i q_4)\tau)},\ t \in (n\tau, (n+1)\tau].$$

which is globally asymptotically stable. Therefore by Theorem 1.4.1 of Lakshmikantham et al. (1989), $x_i(t) \leq u_i(t) \to \breve{u}_i(t)$. Hence, $\exists$ a positive integer l_2 such that

$$x_i(t) \leq \overline{\breve{u}}_i(t) + \grave{\varepsilon}_3\ \forall\ t \geq l_2\tau. \tag{31}$$

From the third equation of the system (1)

$$\begin{cases} \frac{dy_{em}(t)}{dt} \leq (\gamma_1\alpha_n q_4 Q_0) - (\mu_{em} + \delta_2)y_{em}(t), & t \neq n\tau,\\ \Delta y_{em}(t) = e_2, & t = n\tau. \end{cases}$$

Now, consider the following impulsive system

$$\begin{cases} \frac{dv_{em}(t)}{dt} = (\gamma_1\alpha_n q_4 Q_0) - (\mu_{em} + \delta_2)y_{em}(t), & t \neq n\tau,\\ \Delta v_{em}(t) = e_2, & t = n\tau. \end{cases} \tag{32}$$

Applying the same argument, $\exists$ a positive integer l_2 such that

$$y_{em}(t) \leq \bar{v}_{em}(t) + \grave{\varepsilon}_3 \;\forall\; t \geq \; l_2\tau, \tag{33}$$

where

$$\bar{v}_{em}(t) = \frac{\gamma_1\alpha_n q_4 Q_0}{\mu_{em}+\delta_2} + \frac{e_2 exp(-(\mu_{em}+\delta_2)(t-n\tau))}{1-exp(-(\mu_{em}+\delta_2)\tau)}, \tag{34}$$

$$\bar{v}_{em}(0^+) = \frac{\gamma_1\alpha_n q_4 Q_0}{\mu_{em}+\delta_2} + \frac{e_2}{1-exp(-(\mu_{em}+\delta_2)\tau)}. \tag{35}$$

Now, from fourth equation of system (1), we have

$$\begin{cases} \frac{dy_{ea}(t)}{dt} \leq \mu_{em}(\bar{v}_{em}(t) + \grave{\varepsilon}_3) - \delta_2)y_{ea}(t), & t \neq n\tau, \\ \Delta y_{ea}(t) = e_3, & t = n\tau. \end{cases}$$

Considering again its comparison system as below

$$\begin{cases} \frac{dv_{ea}(t)}{dt} = \mu_{em}(\bar{v}_{em}(t) + \grave{\varepsilon}_3) - \delta_2)v_{ea}(t), & t \neq n\tau, \\ \Delta v_{ea}(t) = e_3, & t = n\tau. \end{cases} \tag{36}$$

Applying Lemma 1 and comparison theorem, $\exists$ a positive integer l_3 such that

$$y_{ea}(t) \leq \bar{v}_{ea}(t) + \grave{\varepsilon}_3 \forall\; t \geq l_3\tau, \tag{37}$$

where

$$\begin{aligned} \bar{v}_{ea}(t) = \; & \frac{-e_2\exp(-(\delta_2+\mu_{em})(t-n\tau))}{1-\exp(-(\delta_2+\mu_{em})\tau)} + \frac{(e_2+e_3)\exp(-\delta_2(t-n\tau))}{1-\exp(-\delta_2\tau)} \\ & + \frac{\mu_{em}}{\delta_2}\left(\frac{\gamma_1\alpha_n \grave{q}_4 Q_0}{\mu_{em}+\delta_2} + \grave{\varepsilon}_3\right). \end{aligned} \tag{38}$$

Therefore, for $t \geq l_2\tau$, first equation of system (1) gives

$$\frac{dx_s(t)}{dt} \geq \left[\alpha - \frac{\alpha q_4}{\beta}\beta_i(\bar{\breve{u}} - \grave{\varepsilon}_3) - \alpha_n(\bar{v}_{ea} + \grave{\varepsilon}_3)\right]x_s(t)$$

Integration of above equation on $(l_2\tau, (l_2+1)\tau]$ gives

$$\begin{aligned} & x_s((l_2+1)\tau) \\ & \quad \geq x_s(l_2\tau)\text{exp}\left(\int_{l_2\tau}^{(l_2+1)\tau}\left[\alpha - \tfrac{\alpha q_4}{\beta} - \beta_i(\bar{\breve{u}} + \grave{\varepsilon}_3) - \alpha_n(\bar{v}_{ea} + \grave{\varepsilon}_3)\right]dt\right) \\ & \quad \geq x_s(l_2\tau)\exp(\varrho_2). \end{aligned}$$

where

$$\varrho_2 = \int_{l_2\tau}^{(l_2+1)\tau}\left[\alpha - \frac{\alpha q_4}{\beta} - \beta_i(\bar{\breve{u}} + \grave{\varepsilon}_3) - \alpha_n(\bar{v}_{ea} + \grave{\varepsilon}_3)\right]dt.$$

Because $\tau > \tau_{max}$, so it is possible to find $q_4 > 0$ and $\grave{\varepsilon}_3 > 0$ such that $\varrho_2 > 0$. This implies

$$x_s[(l_2 + l)\tau] \geq x_s(l_2\tau)\exp(l\varrho_2) \to \infty$$

as $l \to \infty$ This is in contradiction to our assumption that $x_s < q_4 \ \forall \ t \geq l_1\tau, \ (l_2 > l_1)$. Hence $\exists \ \mathring{t} > l_1\tau$ such that $x_s(\mathring{t}) \geq q_4$.

Step 2: There is nothing to prove if $x_s(t) \geq q_4 \ \ \forall t > \mathring{t}$. But if this is not true, let $\mathring{t_1} = \inf\{t \mid \ x_s(t) < q_4; t > \mathring{t}\}$. Thus $x_s(t) \geq q_4 \forall t \in [\mathring{t}, \ \mathring{t_1}], \ \mathring{t_1} \in (\breve{n}_1\tau, (\breve{n_1} + 1)\tau]$. $x_s(\mathring{t_1}) = q_4$, because of continuity of $x_s(t)$. Let $\tau^* = (\breve{n}_2 + \breve{n}_3)\tau$ where $\breve{n}_2 = \breve{n}_{21} + \breve{n}_{22} + \breve{n}_{23}$ and $\breve{n}_{21}, \ \breve{n}_{22}, \ \breve{n}_{23}, \ \breve{n}_3$ satisfying the following conditions:

$$\begin{aligned}
&(\breve{n}_{21})\tau > -\left(\frac{1}{\delta_1 - \beta_i q_4}\right)\ln\frac{\grave{\varepsilon}_3}{Q_0 + e_1}, \\
&(\breve{n}_{22})\tau > -\left(\frac{1}{\mu_{em} + \delta_2}\right)\ln\frac{\grave{\varepsilon}_3}{Q_0 + e_2}, \\
&(\breve{n}_{23})\tau > -\left(\frac{1}{\mu_{em} + \delta_2}\right)\ln\frac{\grave{\varepsilon}_3}{Q_0 + e_3}, \\
&\exp(\breve{n}_3\varrho_2 - \upsilon(\breve{n}_2 + 1)\tau) > 1, \upsilon = \left(\frac{\alpha q_4}{\beta} + \beta_i Q_0 + \alpha_n Q_0\right).
\end{aligned} \tag{39}$$

Now, we will prove that $\exists \ \mathring{t_2} \in ((\breve{n}_1 + 1)\tau, (\breve{n}_1 + 1)\tau + \tau^*]$ such that $x_s(\mathring{t_2}) \geq q_4$. Suppose this is not true, then $x_s(t) < q_4 \forall t \in ((\breve{n}_1 + 1)\tau, (\breve{n}_1 + 1)\tau + \tau^*]$. If system (30) is considered with $\breve{u}_i((\breve{n}_1 + 1)\tau^+) = x_s((\breve{n}_1 + 1)\tau^+)$, then using Lemma 1 for $t \in ((\breve{n}_1 + 1)\tau, (\breve{n_1} + 1)\tau + \tau^*]$, we have

$$\begin{aligned}
\breve{u}_i(t) = &\left[\breve{u}_i(\breve{n}_1 + 1)\tau^+) - \frac{e_1}{1 - \exp(-(\delta_1 - \beta_i q_4)\tau)}\right] \\
&\exp(-(\delta_1 - \beta_i q_4)(t - (n+1)\tau)) + \overline{\breve{u}}_i(t)
\end{aligned}$$

This implies

$$\begin{aligned}
\mid \breve{u}_i(t) - \overline{\breve{u}}_i(t) \mid &\leq (Q_0 + e_1)\exp(-(\delta_1 - \beta_i q_4)(t - n\tau)) \\
&\leq \grave{\varepsilon}_3.
\end{aligned}$$

which depicts that $x_i(t) \leq \breve{u}_i(t) < \overline{\breve{u}}_i(t) + \grave{\varepsilon}_3, \ (\breve{n}_1 + \breve{n}_{21} + 1)\tau \leq t \leq (\breve{n}_1 + 1)\tau + \tau^*$. Now consider the system (32) with $v_{em}((\breve{n}_1 + \breve{n}_{21} + 1)\tau^+) = y_{em}((\breve{n}_1 + \breve{n}_{21} + 1)\tau^+)$, then using Lemma 1, we have,

$$\begin{aligned}
&v_{em}(t)\left[v_{em}((\breve{n}_1 + \breve{n}_{21} + 1)\tau^+) - \overline{v}_{em}(0^+)\right] \\
&\exp(-(\mu_{em} + \delta_2)(t - (\breve{n}_1 + \breve{n}_{21} + 1)\tau))\overline{v}_{em}(t). \\
&|v_{em}(t) - \overline{v}_{em}(t)| \leq \ (Q_0 + e_2)\exp(-(\mu_{em} + \delta_2)(t - (\breve{n}_1 + \breve{n}_{21} + 1)\tau)) \\
&\leq \ \grave{\varepsilon}_3 \forall (\breve{n}_1 + \breve{n}_{21} + \breve{n}_{22} + 1) \leq t \leq (\breve{n}_1 + 1)\tau + \tau^*
\end{aligned}$$

which concludes that $y_{em}(t) \leq \overline{v}_{em}(t) + \grave{\varepsilon}_3$. Finally, consider equation (36) with $v_{ea}((\breve{n}_1 + \breve{n_{21}} + \breve{n_{22}} + 1)\tau^+) = y_{ea}((\breve{n}_1 + \breve{n_{21}} + \breve{n_{22}} + 1)\tau^+) \geq 0$, then using Lemma 1, we have,

$$\begin{aligned} v_{ea}(t) = \ & (v_{ea}(\breve{n}_1 + \breve{n_{21}} + \breve{n_{22}} + 1)\tau^+ - \overline{v}_{ea}(0^+)) \\ & \exp(-(\mu_{em} + \delta_2)(t - (\breve{n}_1 + \breve{n_{21}} + \breve{n_{22}} + 1)\tau) + \overline{v}_{ea}(t) \\ & \Rightarrow |\, v_{ea}(t) - \overline{v}_{ea}(t) \,| \\ & \leq (Q_0 + e_3)\exp(-(\mu_{em} + \delta_2)(t - (\breve{n}_1 + \breve{n_{21}} + \breve{n_{22}} + 1)\tau)) \\ & \leq \grave{\varepsilon}_3 \forall (\breve{n}_1 + \breve{n}_2 + 1)\tau \leq t \leq (\breve{n}_1 + 1)\tau + \tau^* \end{aligned}$$

Therefore, $y_{ea}(t) \leq \overline{v}_{ea}(t) + \grave{\varepsilon}_3$. Hence

$$\frac{dx_s(t)}{dt} \geq \left[\alpha - \frac{\alpha q_4}{\beta}\beta_i(\overline{\breve{u}} - \grave{\varepsilon}_3) - \alpha_n(\overline{v}_{ea} + \grave{\varepsilon}_3)\right] x_s(t).$$

Integrating on $[(\breve{n}_1 + \breve{n}_2 + 1)\tau, (\breve{n}_1 + \breve{n}_2 + \breve{n}_3 + 1)\tau]$, we get

$$x_s((\breve{n}_1 + \breve{n}_2 + \breve{n}_3 + 1)\tau) \geq x_s((\breve{n}_1 + \breve{n}_2 + 1)\tau)\exp(\varrho_2 \breve{n}_3). \tag{40}$$

Further, for $t \in [\mathring{t}_1, (\breve{n}_1 + 1)\tau]$, two possibilities are there.
Case 1: If $x_s(t) < q_4 \forall\ t \in [\mathring{t}_1, (\breve{n}_1 + 1)\tau]$, then from above assumption $x_s(t) < q_4\ \forall t \in [\mathring{t}_1, (\breve{n}_1 + 1)\tau + \tau^*]$. This implies

$$\frac{dx_s(t)}{dt} \geq \left(-\frac{\alpha q_4}{\beta} - \beta_i Q_0 - \alpha_n Q_0\right) x_s(t). \tag{41}$$

Integrating equation (41) in $[\mathring{t}_1, (\breve{n}_1 + \breve{n}_2 + 1)\tau]$, we have

$$x_s((\breve{n}_1 + \breve{n}_2 + 1)\tau) \geq x_s(\mathring{t}_1)\exp(-\upsilon(\breve{n}_2 + 1)\tau). \tag{42}$$

Using equation (40) in equation (42)

$$x_s((\breve{n}_1 + \breve{n}_2 + \breve{n}_3 + 1)\tau) \geq x_s(\mathring{t}_1)\exp(\varrho_2 \breve{n}_3)\exp(-\upsilon(\breve{n}_2 + 1)\tau) > q_4.$$

But this contradicts our assumption. Therefore, $x_s(t) \geq q_4$ in $[\mathring{t}_1, (\breve{n}_1 + \breve{n}_2 + \breve{n}_3 + 1)\tau]$ for some t. Let $\mathring{t}_3 = \inf\{t \mid x_s(t) \geq q_4;\ t > \mathring{t}_2\}$. Due to continuity of $x_s(t)$, $x_s(\mathring{t}_3) = q_4$. Now integration of equation (41) on the interval $[\mathring{t}_2, \mathring{t}_3]$ gives

$$\begin{aligned} x_s(t) &\geq x_s(\mathring{t}_2)\exp((-\upsilon)(t - \mathring{t}_2) \\ &\geq q_4 \exp((-\upsilon)(t - \mathring{t}_2) \\ &\geq q_4 \exp(\upsilon(\breve{n}_2 + \breve{n}_3 + 1)\tau) = \overline{q_4}. \end{aligned}$$

Since $x_s(\mathring{t}_3) \geq \overline{q}_4$, so similar process can be continued for $t > \mathring{t}_3$. Hence $x_s(t) \geq \overline{q}_4 \forall t > t_1$.

Case 2: if $\exists\ \mathring{t}_4 \in [\mathring{t}_1, (\breve{n}_1 + 1)\tau]$ such that $x_s(\mathring{t}_4) \geq q_4$, then let $\mathring{t}_5 = \inf\{t \mid x_s(t) \geq q_4; t > \mathring{t}_2\}$. Therefore, $x_s(t) < q_4$ for $t \in [\mathring{t}_2, \mathring{t}_5]$ and $x_s(\mathring{t}_5) = q_4$. Now, integration of equation (41) on the interval $[\mathring{t}_2, \mathring{t}_5]$ gives

$$x_s(t) \geq x_s(\mathring{t}_2)\exp((-\upsilon)(t - \mathring{t}_2)) \geq q_4 \exp(\upsilon\tau) = \overline{q}_4.$$

Because $x_s(\overset{\circ}{t_5}) \geq \overline{q}_4$, so, similar argument can be followed for $t > \overset{\circ}{t_5}$. Hence, it is concluded that $x_s(t) \geq \overline{q}_4 \ \forall t > \overset{\circ}{t}$.

Step 3: Let $a = \min\{q_1, q_2, q_3, \overline{q}_4\}, \Theta = \{R_+^3 : a \leq x_i(t), x_s(t), y_{em}(t), y_{ea}(t) \leq Q_0\}$. Thus, from above steps and Lemma 2, it is proved that each solution of system (1) will always remain in region Θ. Therefore, by Definition 3.1, system (1) is permanent. □

6 Numerical analysis and discussion

In this paper, a prey-predator model, with stage structure in predator and infection in prey, is constituted and investigated to control the outbreak of pest population. Susceptible prey is considered as a pest and predator acts as the natural enemy. Here, the main aim is to substantiate the theoretical findings and to numerically investigate that how the period of impulsive perturbations and releasing amounts of infected pests and natural enemies population is beneficial for integrated pest management. For this, a set of parametric values of the system (1) in biologically feasible range are chosen per week as given in Table 1.

Table 1 Parametric values chosen for numerical simulation of SIN model (1)

Parameter	*Representation*	*Its value (per week)*
α	Reproduction rate of susceptible pest	1.7
β	Carrying capacity	3
β_i	Contact rate of susceptible pest per unit time infected pest	2.6
α_n	Rate of predation by mature natural enemy	0.3
γ_2	half saturation constant by Holling IV	0.1
δ_1	Death rate of infected pest	0.5
γ_1	Conversion rate of pest to immature natural enemy	0.7
μ_{em}	Conversion rate of immature to mature natural enemy	0.4
δ_2	Death rate of mature and immature natural enemy	0.3
e_1	Impulsive releasing amount of infected pests	0.5
e_2	Impulsive releasing amount of immature natural enemy	2
e_1	Impulsive releasing amount of mature natural enemy	4
γ_3	half saturation constant	0.2
γ_4	half saturation constant	0.1

It is analysed that in the absence of impulsive release of infected pests and natural enemies, stable limit cycles exist for susceptible pest and infected pest population while immature and mature predator is driven towards extinction as shown in Figure 1. The global stability of pest-free equilibrium point is established and then it is derived that the system (1) is permanent. For ensuring the same, the threshold value of the impulsive period is calculated that depends on amount of infected pests and natural enemies.

The initial values of population densities of susceptible prey, infected prey, immature and mature predator are $x_s(0) = 0.5, x_i(0) = 0.8, y_{em}(0) = 0.8, y_{ea}(0) = 4$. Therefore, by using all these numerical values, it is observed that if there is no impulsive release of infected pests, immature and mature natural enemies, that is $e_1 = e_2 = e_3 = 0$, then there exist stable limit cycles for susceptible and infected pest population. But immature and mature predators become extinct as shown in Figure 1. Further from Theorems 5 and 6, we get $\check{\tau} = 1.836$ and $\tau_{max} = 4.55$. Therefore, it is obtained that susceptible pest free solution is globally stable if $\tau > \check{\tau}$ as depicted in Figure 2. Also, phase portrait of susceptible pest verses infected pest in Figure 2(e) shows that stable limit cycle moves towards chaotic behaviour. But complete extinction of pests is not encouraged biologically. Thus, Theorem 6 implies that if $\tau > \tau_{max}$ system (1) is permanent as shown in Figure 3 and exhibit chaotic behaviour [see Figures 3(e) and 3(f)].

Figure 1 Stable limit cycles of susceptible pests and infectious pests when $e_1 = e_2 = 2 = e_3 = 0$ and $x_s(0) = 0.5,\ x_i(0) = 0.8,\ y_{em}(0) = 0.8,\ y_{ea}(0) = 4$

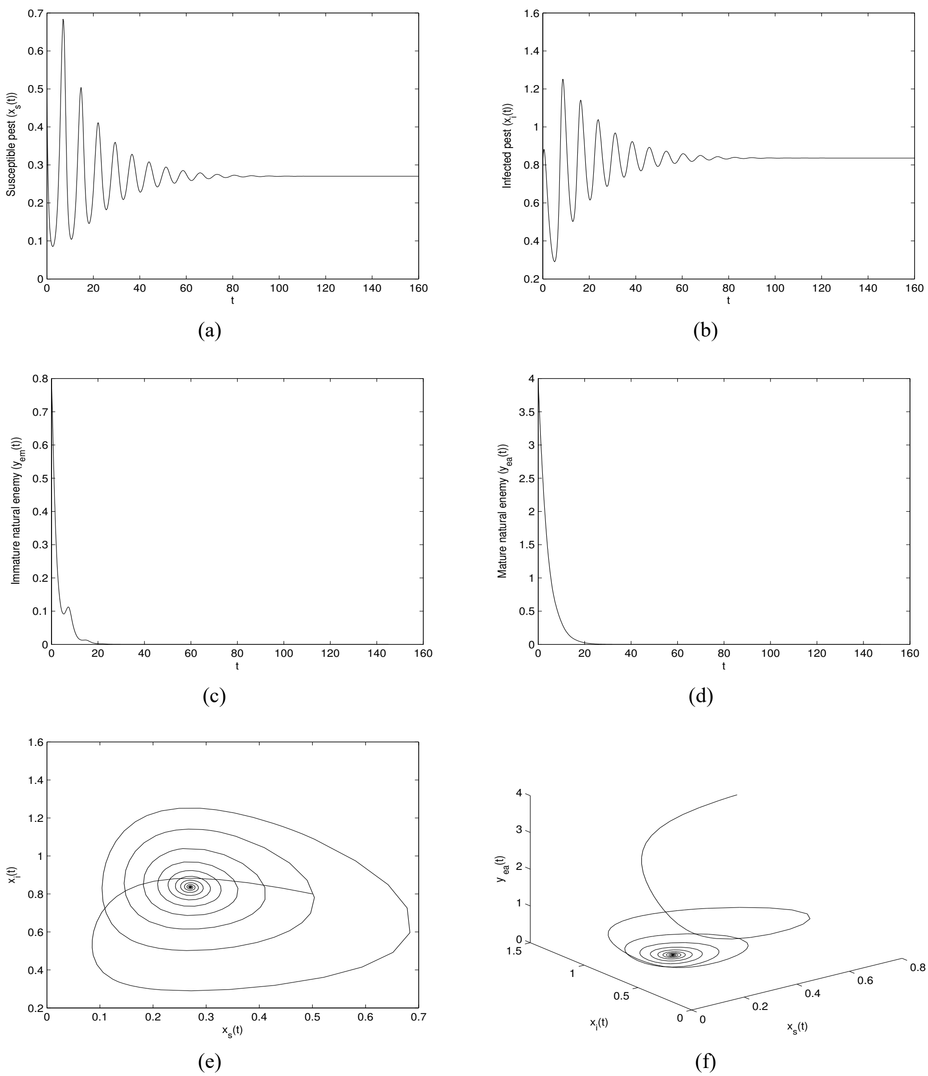
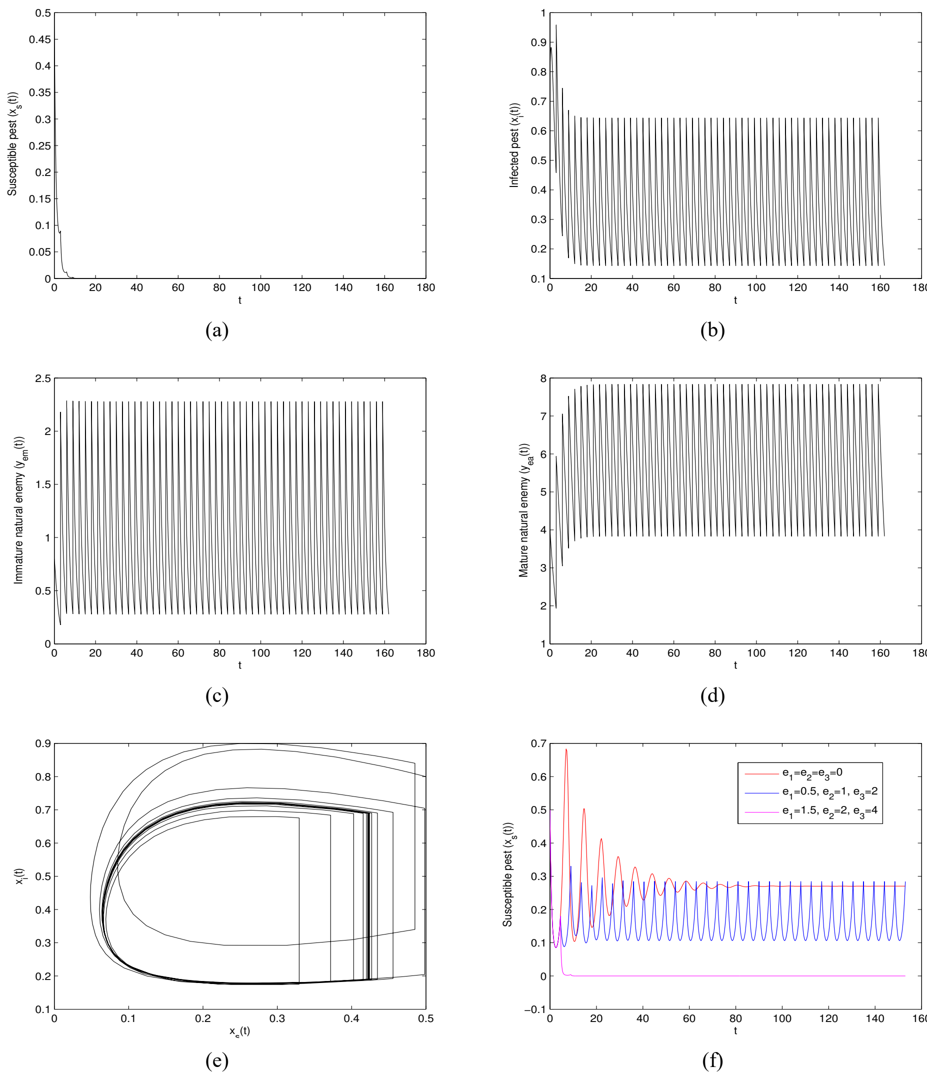
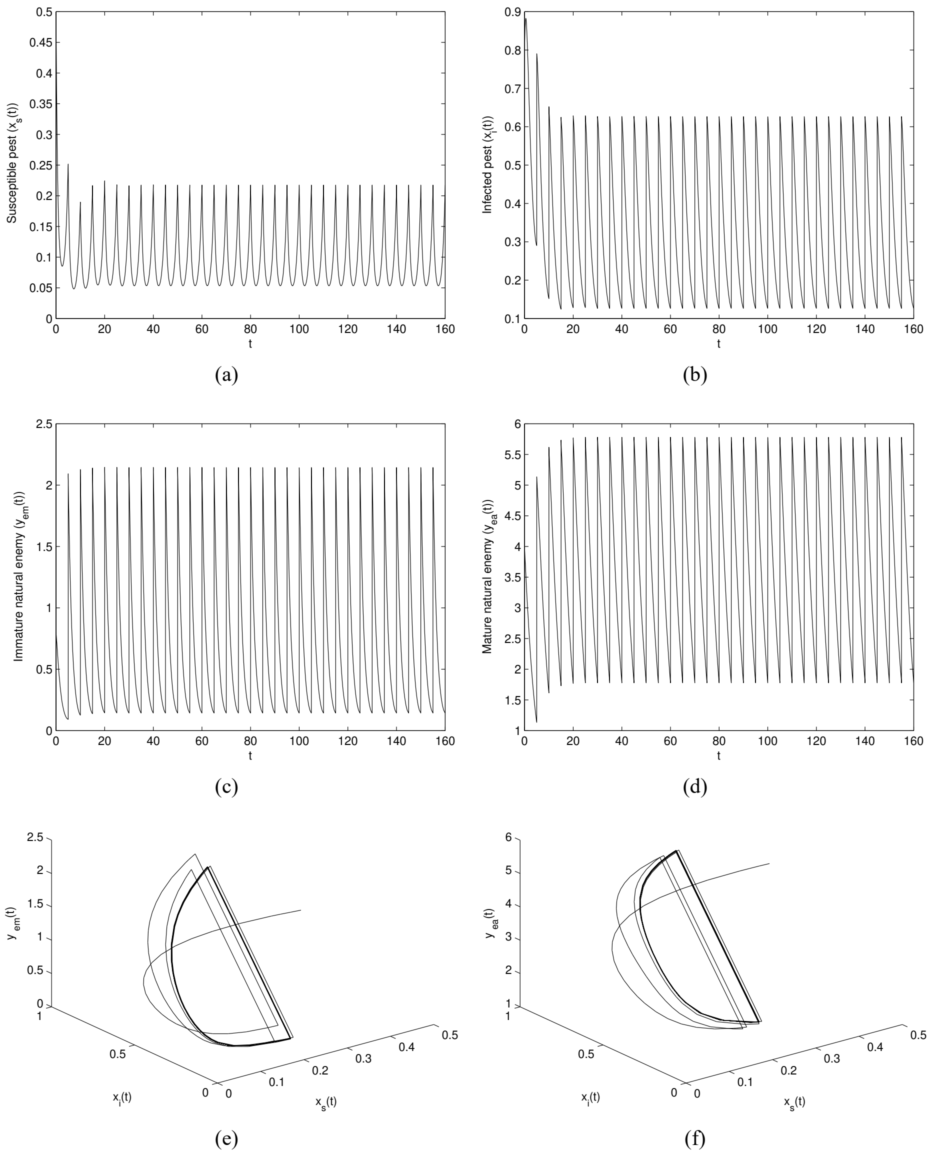


Figure 2 Global stability of pest extinction periodic solution $(0, x_i(t), y_{em}(t), y_{ea}(t))$ of system (1) at $\tau < \check{\tau}(= 4.7)$ with $e_1 = 0.5, e_2 = 2, e_3 = 4$ (see online version for colours)



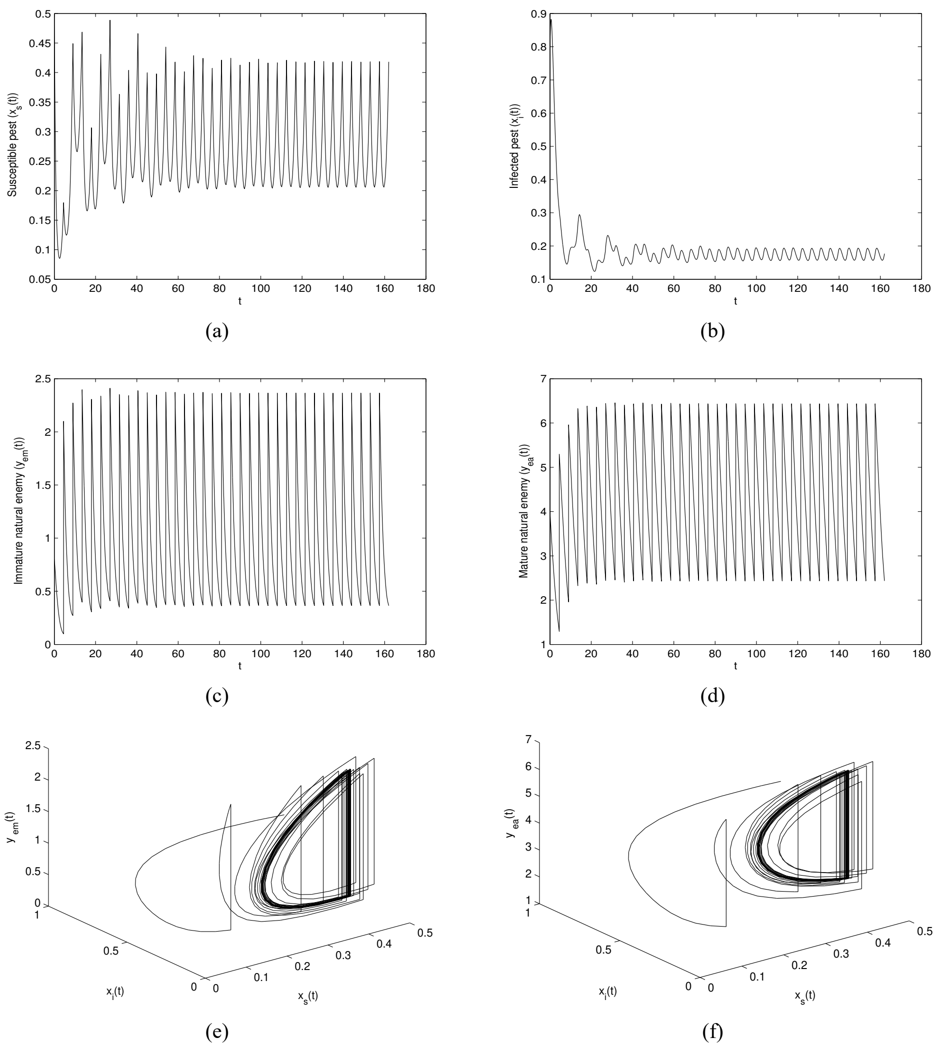
Apart from this, it is also analysed that if $e_1 = 0$, that is only natural enemies are released then $\tau_{max} = 3.02511$. As shown in Figure 4, the system is again permanent and shows chaotic behaviour but threshold value of impulsive period decreased which means that natural enemies are to be released at a fast pace. But this is not always feasible specially when natural enemies are not native species and are being reared. Similarly, if $e_2 = 0$ then $\tau_{max} = 3.88$. Permanence and chaotic behaviour of the system is shown graphically in Figure 5. If $e_3 = 0$, then $\tau_{max} = 2.20168$ and permanence of the system (1) is shown in Figure 6. In all these situations, permanence of the system is achieved but at relatively low values of τ_{max} than the situation when we released both infected pests and natural enemies (immature and mature) simultaneously. Also numerical simulation is performed to examine the effect of impulsive releasing of natural enemies and infected pest population on the extinction of susceptible pest population. It is observed that susceptible pest population moves towards extinction as impulsive release is increased as shown in Figure 2(f). Thus it is concluded that combination of microbial and natural control is very effective for pest control.

Figure 3 Permanence of the system (1) at $\tau > \tau_{max}(= 4.55)$ with $x_s(0^+ = 0.5)$, $x_i(0^+) = 0.8$, $y_{em}(0^+) = 0.8$, $y_{ea}(0^+) = 4$ with $e_1 = 0.5$, $e_2 = 2$, $e_3 = 4$



(a) (b) (c) (d) (e) (f)

Figure 4 Permanence of the system (1) at $\tau > \tau_{max}(= 3.02511)$ with $x_s(0^+) = 0.5$, $x_i(0^+) = 0.8$, $y_{em}(0^+) = 0.8$, $y_{ea}(0^+) = 4$ and $e_1 = 0$, $e_2 = 2$, $e_3 = 4$



6.1 Comparison with other results

A similar SIN model is considered by Shi and Chen (2010) for integrated pest management. However, the researchers ignored the concept of stage structuring and mutual interference between natural enemies and did not verify the results numerically. But here extensive numerical simulation is performed to depict that how stable limit cycles shift towards chaotic behaviour with the impulsive release of natural enemies and infected pests. The results obtained in this paper also support those obtained by Mathur and Dhar (2018). But the model (1) incorporates the concept of stage-structure in natural enemies that makes it more realistic. Additionally, crowding effect of susceptible pest population and mutual interference between natural enemies is considered with the help of Holling II and Holling IV functional responses. As a result the threshold value of the impulsive period for complete eradication of susceptible pests $\check{\tau}$ is decreased while the threshold value for the coexistence of pest and natural enemies population τ_{max} is increased. Thus infected pests and natural enemies are to be release dafter longer period which enhance the effectiveness of pest control model (1) in terms of economic reduction.

Figure 5 Permanence of the system (1) at $\tau > \tau_{max} (= 3.88)$ with $x_s(0^+) = 0.5$, $x_i(0^+) = 0.8$, $y_{em}(0^+) = 0.8$, $y_{ea}(0^+) = 4$ and $e_1 = 0.5$, $e_2 = 0$, $e_3 = 4$

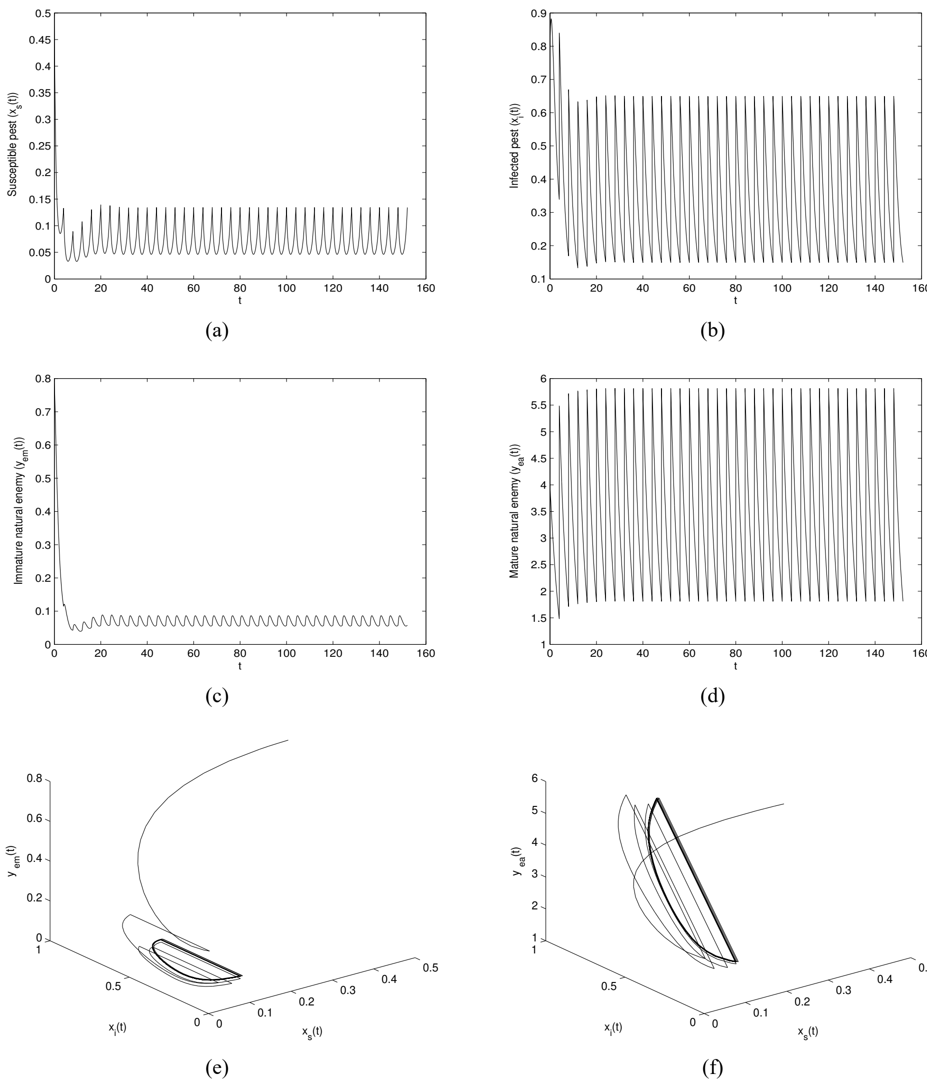
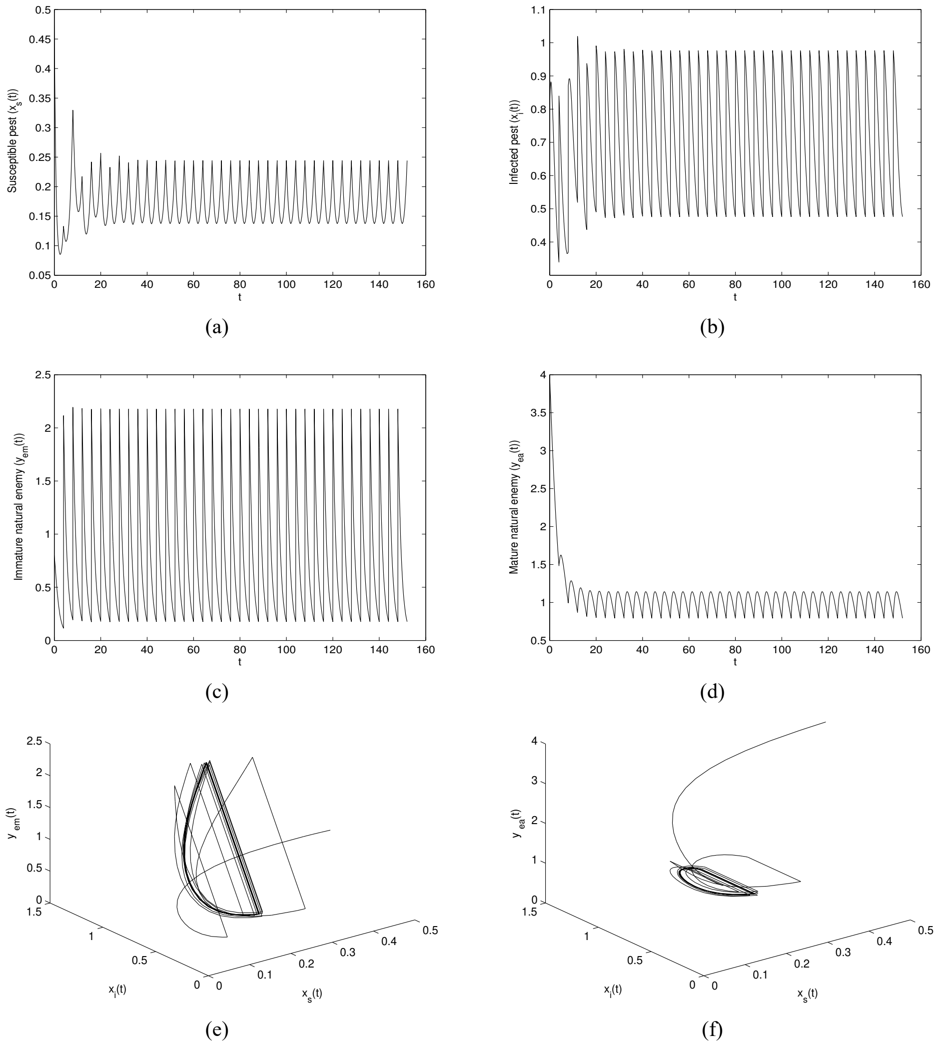


Figure 6 Permanence of the system (1) at $\tau > \tau_{max}(= 2.20168)$ with $x_s(0^+) = 0.5$, $x_i(0^+) = 0.8$, $y_{em}(0^+) = 0.8$, $y_{ea}(0^+) = 4$ and $e_1 = 0.5$, $e_2 = 2$, $e_3 = 0$



7 Conclusions

The war between pests and humans is going on from several decades. From time to time, different pest control techniques are acquired by mankind. Working on the same path, here, we investigated a stage structure predator-prey model for the purpose of integrated pest management. It is found that instead of using pesticides, microbial control agents along with natural enemies are more efficient in pest control. In Theorem 3, the threshold value of impulsive period (τ_{max}) is obtained and it is established that susceptible pests can coexist with infected pests and natural enemies if $\tau > \tau_{max}$. Also, the effect of releasing the number of infected pests and natural enemies is discussed and found that greater releasing amount supports pest eradication.

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