Nash type games in competitive facilities location

Athanasia Karakitsiou*
Department of Business Administration, 
Technological Educational Institute of Central Macedonia, 
62100, Serres, Greece 
Email: karakitsiou@teiser.gr 
*Corresponding author

Athanasios Migdalas
Industrial Logistics, ETS Institute, 
Luleå University of Technology, 
971 87 Luleå, Sweden and 
Department of Civil Engineering, 
Aristotle University of Thessalonike, 
54124, Thessalonike, Greece 
Email: athmig@ltu.se 
Email: samig@auth.gr

Abstract: In a non-cooperative game, a Nash equilibrium corresponds to a set of strategies, which ensure that none of the players are better off by unilaterally changing his strategy. This work focuses on the application of Nash Equilibrium to the competitive facility location. Its aim is to provide an up to date review of the literature concerning the discrete version of the problem.

Keywords: Nash games; competitive location; equilibrium.


Biographical notes: Athanasia Karakitsiou received her PhD in Production Management and Engineering at the Technical University of Crete, Greece. Her research interests are related to optimisation, supply chain management and game theory. She has published research papers at the national and international journals, conference proceedings as well as chapters of books.

Athanasios Migdalas obtained his PhD on Optimisation and Operations Research from the Faculty of Engineering of the Linköping University, Sweden. He has eight books published by international houses and edited several special issues of international scientific journals. He has published more than 70 research papers and chapters in international scientific journals and scientific collections. His primary research interests are in mathematical programming, in the design and implementation of high performance computational algorithms and in their applications to decision problems that arise in large scale networks, in supply chain management and logistics.
1 Introduction

Location problems form a wide class of mathematical operations research models, which is interesting both practically and from the point of view of the combinatorial optimisation theory. In the basic model, there is a predefined cost for opening a facility and also connecting a customer to a facility, the goal is to minimise the total cost (Daskin, 1995). Competitive location models additionally incorporate the fact that location decisions have been or will be made by independent decision-makers who will subsequently compete with each other for market share, profit maximisation etc. Additionally, the assignment of customers to be served by these facilities and also how these facilities are connected with each other constitute interesting decisions considered within the problem. It is widely accepted that the competitive location analysis was initiated by Hotelling (1929).

The nature of the competition is one of the primary factors determining the class of the competitive facility location (CFL) problems. The two main classes are simultaneous location problems and sequential location problems. The competing firms simultaneously (or independently, that is, in ignorance of each other’s decision) decide on their facility locations in the former case, whereas there exists a priority among the competing firms with regard to the timing of the actions in the latter (Dobson and Karmarkar, 1987; Hakimi, 1983).

Sequential location problems are usually modelled as hierarchical (or multi-level) programming models (see for example, Hakimi, 1983; Alekseeva et al., 2015; Beresnev, 2009, 2013, 2014; Berman and Krass, 2002; Beresnev and Mel’nikov, 2011). The research effort on simultaneous competitive models aims at developing insights concerning the existence of a set of locations and pricing or production quantities that will ensure a Nash equilibrium (Nash, 1951). Discrete versions of such problems arise when it is assumed that there are a finite number of candidate locations and the markets consist of point demands. The monograph (Karakitsiou, 2015) surveys and provides insights regarding modelling, complexity and algorithmic approaches to discrete competitive location problems.

The majority of the relevant literature is dealing with linear markets (see for example, Hotelling, 1929, Osborne and Pitchic, 1987) mainly due to the complexity associated with the solution of location problems in other spaces such as network or discrete spaces (Plastria, 2001; Eiselt et al., 1993).

In this work we focus on discrete simultaneous location problems. Our intention is to provide an up to date review of modelling and optimisation approaches.

2 Location-pricing equilibria

In location literature, firms typically use either a mill pricing policy or a delivered pricing policy. The seller sets the same price to all customers in the market and the buyer pays the transportation cost in the first case, while the seller charges a specific price in each market and takes care of the transportation cost in the second case.

The location-pricing problems assume that competing firms are involved in a Bertrand type game in their effort to maximise profits. The equilibrium analysis of such problems aims to investigate the existence of equilibrium with simultaneous decisions on price and location. The notion of sub-game Nash equilibrium is used for this purpose, which
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captures the idea that when firms select their location, they all anticipate the consequences of their choice on price competition.

More precisely, firms are supposed to choose location and price, one at a time in a two-stage game, with the aim of maximising their own profits. In the first stage firms simultaneously make their location decision. Given the outcome of the first stage, firms then simultaneously determine their price in the second stage.

However, when a firm sets a mill pricing, a price equilibrium at the second stage of the game rarely exists. As a result the majority of the research work devoted to discrete or network location problems have been using parametric prices (Serra and ReVelle, 1999; García et al., 2004). On the other hand the existence of a price equilibrium is possible when a delivered pricing mechanism is used (Lederer and Thisse, 1990; Dorta-González et al., 2005; Pelegrín-Pelegrín et al., 2011; Pelegrín et al., 2007). If for simplicity, a duopoly is assumed, then for any fixed sets of locations, $x^1, x^2$, the equilibrium prices $(\hat{p}^1, \hat{p}^2)$ determined by the second stage of the problem are:

\[
\begin{align*}
\hat{p}^1(x^1, x^2) &= \begin{cases} p^2(x^2) & \text{if } p^1(x^1) < p^2(x^2) \\ p^1(x^1) & \text{otherwise} \end{cases} \\
p^2(x^1, x^2) &= \begin{cases} p^1(x^1) & \text{if } p^2(x^2) < p^1(x^1) \\ p^2(x^2) & \text{otherwise,} \end{cases}
\end{align*}
\]

The pair $(\bar{x}^1, \bar{x}^2)$, is a location equilibrium if

\[
\begin{align*}
s(\bar{x}^1, \bar{x}^2) &\leq s(x^1, x^2), & \forall x^1 \\
s(\bar{x}^1, \bar{x}^2) &\leq s(x^1, x^2), & \forall x^2
\end{align*}
\]

For any fixed location set, $s(x^1, x^2)$ is the total cost incurred to supply the demand $q$ of the customers if each customer pays the minimum delivered price. This total cost (often called social cost) is given by

\[
s(x^1, x^2) = \min\{p^1(x^1), p^2(x^2)\}q
\]

It is pointed out by Hamilton (1989) and Gupta (1994) that the socially optimal location does not necessarily guarantee the existence of an equilibrium in the first stage of the game, if demand is elastic with respect to price or if the marginal cost of production is not constant. In such a case an equilibrium is obtained by an alternate procedure where each firm determines its optimal location considering the location of its rivals as given. Alternatively, the location-price problem is treated in García et al. (2011) as a location game of an entering firm where the reaction of its rivals is to change their price instead of their locations. In other words, the entering firm first selects its locations and sequentially it gets involved in a Bertrand game with the existing firms aiming at maximising its profit. Under reasonable assumptions the authors in García et al. (2011) show that there is a price equilibrium to the Bertrand game in which each firm sets the optimal price in its monopolised markets and sets a price equal to the minimum delivered cost in other markets, i.e.,
Consequently, if all competing firms set the equilibrium prices, the location-price problem reduces to the following location problem

\[
\max \sum_{h \in H^*} \sum_{\ell \in L_h} q_{h\ell} \left( p_{h\ell}^*(t) - c_{h\ell} \right) y_{h\ell}
\]

s.t.

\[
\sum_{\ell \in L} x_{\ell} y_{h\ell} \leq 1, \quad h \in H^*
\]

\[
y_{h\ell} \leq x_{\ell}, \quad h \in H^*, \quad \ell \in L_h
\]

\[
\sum_{\ell \in L} x_{\ell} = s
\]

\[
x_{\ell}, y_{h\ell} \in \{0,1\},
\]

where \(L\) is the set of potential location of the entering firm. \(L_h\) is the set of the location candidates at which the entering firm can price below its competitors, i.e., \(L_h = \{\ell \in L : c_{h\ell} < c_{h\ell}^{\text{com}}\}\). \(H^*\) is the set of markets in which the entering firm make a positive profit, \(H^* = \{h \in H : L_h \neq \emptyset\}\), \(x_{\ell}\) and \(y_{h\ell}\) are location and allocation variables respectively with \(x_{\ell} = 1\) if a facility is located at \(\ell, 0\) otherwise, and \(y_{h\ell} = 1\) if market \(h\) is served from location \(\ell, 0\) otherwise.

Lederer and Thisse (1990) consider a duopoly with inelastic demand and constant marginal cost. The firms compete through their decisions concerning location, production technology and delivered prices. The problem is formulated as a two-stage game. In the first stage each firm \(i\) chooses a location \(x_i\) and a production technology \(\tau_i\);

The marginal production cost at \(x_i\) is then given by

\[
m_i(s_i) = \sum_{j=1}^{J} \min_{h \in H, j} \left\{ p_{jh} + t_j \left[ d(y_{jh}, x_i) \right] \right\} q_{ji},
\]

where \(j = 1, \ldots, J\) are the inputs used by firm \(i\) in order to produce the product. \(H_j\) represents the possible sources of input \(j\) in \(N\), denoted by \(y_{jh}, h = 1, \ldots, H_j\). \(p_{jh}\) denotes the price of input \(j\), at \(y_{jh}\) which is assumed to be given. While, the cost of transporting one unit of input \(j\) from \(y_{jh}\) to \(x_i\), \(t_j(d(y_{jh}, x_i))\), is assumed to be increasing and concave with distance \(d(y_{jh}, x_i)\). Finally, \(q_{ji}\) is the amount of input \(j\) used by firm \(i\) to produce one unit of the product.

Customers are located at \(z_h, h = 1, \ldots, H\), distinct markets on a network \(N = (V, E)\). At each market \(z_h\) customers have fixed demand of a total amount \(D_h\). The unit transportation cost between firm’s \(i\), \(i = \{1, 2\}\), location \(x\) and market \(z_h\) is denoted by \(t_i = t_i[d(x, z_h)]\) and is assumed to be positive, concave and increasing with respect to distance.

For a given location-technology choice \(s_i\) the most efficient mix of inputs to produce output \(q_i > 0\) is the one that minimises the firm’s \(i\) total cost which is given by

\[
TC_i(q_i, s_i) = F_i(s_i) + C_i(q_i, s_i),
\]
where \( F_i \) and \( C_i = m_i(s_i)q_i \) denote the fixed and variable production cost respectively. In the second stage, the firms simultaneously set delivery price schedules. For the solution of the problem, Lederer and Thisse (1990), first, they proved that for any location-technology choices \((s_1, s_2)\) there exists a unique pair of delivered prices \((p^*_1, p^*_2)\) satisfying,

\[
p^*_i(z_h) = p^*_2(z_h) = \max \left\{ m_i(s_1) + t_1 \left[ d(x_1, z_h) \right], m_2(s_2) + t_2 \left[ d(x_2, z_h) \right] \right\}
\]

for all \(H\) markets, which constitutes a unique Nash equilibrium in price (stage two). Then, this equilibrium is used to search for the equilibrium in location and technology at the first stage of the game. The study of the equilibrium of the first stage is facilitated by the use of the so called social cost function

\[
SK(s_1, s_2) = \sum_{h=1}^{H} \min \left\{ m_i(s_1) + t_1 \left[ d(x_1, z_h) \right] \right\} D_h + F_1(s_1) + F_2(s_2).
\]

The social cost is defined as the total delivered cost if each customer were served with the lowest marginal delivered cost. Lederer and Thisse (1990) show that a location equilibrium exists and is the global optimiser of the social cost. As in Hakimi (1983), they also proved that if the firm transport costs are strictly concave, then the set of locational choices of the firm is reduced to the vertices of the network.

Dorta-González et al. (2005) obtained the same result in the case of oligopoly i.e. when \(1 \leq i \leq r\). They proved the existence of a subgame perfect Nash equilibrium at the vertices of the network when transportation costs are concave with respect to distance.

In Pelegrín-Pelegrín et al. (2011) the case of multi-facility duopoly is considered and the following integer linear programming formulation aiming at minimising the social cost is proposed to find a location equilibrium in discrete location space.

\[
\min \sum_{h=1}^{H} \sum_{\ell \in L_2} D_h p^*_{ih} y_{ih} + \sum_{h=1}^{H} \sum_{\ell \in L_2} D_h p^*_{2h} y_{2h}
\]

s.t.

\[
\sum_{\ell \in L_1} y_{ih} + \sum_{\ell \in L_2} y_{2h} = 1, h \in H
\]

\[
y^*_{ih} \leq x^*_i, h \in H, \ell \in L_1
\]

\[
y^*_{2h} \leq x^*_2, h \in H, \ell \in L_2
\]

\[
\sum_{\ell \in L_1} x^*_i = r
\]

\[
\sum_{\ell \in L_2} x^*_2 = s
\]

\[
y^*_{ih} \in \{0, 1\}, h \in H, i \in [1, 2],
\]

where \(L_i\) is the set of potential locations for firm \(i\). \(p^*_{ih}\) is the marginal delivered cost of firm \(i\) from location \(\ell\) to market \(h\). \(x_i\) is a vector with components \(x^*_i\) with 0-1 values,
where \( x_i' = 1 \) indicates that facility location \( \ell \) is chosen by firm \( i \). For any fixed location set \( x_1 \) and \( x_2 \), \( y_{ih}\) is a 0-1 variable indicating whether firm \( i \) serves market \( h \) from facility located at \( \ell \) or not, and \( r \) and \( s \) is the number of facilities located by firm 1 and 2 respectively. Once the above problem is solved, the profit of each firm is obtained by taking into account that profit of a firm plus social cost equals total delivered cost of its rival (Pelegrín-Pelegrín et al., 2011). A genetic like algorithm is proposed in Pelegrín et al. (2007) for the solution of the problem.

### 3 Location-quantity equilibria

The studies belonging to this case describe a non-cooperative game where competing firms decide on location and quantity supplied to market with goal to maximise their own profit. Because the choice of location is usually prior to decision, the location-quantity game is, in most of the cases, formulated as a two stage game where in the first stage firms simultaneously decide their location and in the second stage give the chosen sets of location they decide the quantity to be offered at the market. Because of firms compete in quantities rather in price the second stage is a Cournot game.

Labbé and Hakimi (1991) analyse a duopolistic game with linear demand and zero fixed cost in a network \( N = (V, E) \) connecting spatially separated markets. A market is located at each vertex \( v_k \in V \), and at each market a product is sold at price \( p_k \) with 0/otherwise\

\[
P_k = \begin{cases} 
  a_k - \beta_k q_k & \text{if } 0 \leq q_k \leq q_k^* / \beta_k \\
  0 & \text{otherwise}
\end{cases}
\]

with \( a_k \geq 0 \) and \( \beta_k \geq 0 \). Each firm locates at its facility points \( x_1 \) and \( x_2 \) respectively and transport the product to the market \( v_k \) with unit transportation cost \( T(d(x_i, v_k)) \) which is assumed to be increasing with distance. The marginal cost of production \( C(x_i) \) is assumed to be concave and independent from the quantity produced. Consequently, \( c_k(x_i) = C(x_i) + T(d(x_i, v_k)) \) is the unit delivered cost of firm \( i \) at market \( v_k \). Labbé and Hakimi (1991) solve the game using backward induction. First, the second stage of the game is solved for a fixed pair of location. They proved that there is a unique production pair,

\[
q_{1k}^* = \arg \max_{q_{1k}} \left\{ \left( a_k - \beta_k (q_{1k} + q_{2k}^*) - c_k (x_i) \right) q_{1k} \right\}
\]

and

\[
q_{2k}^* = \arg \max_{q_{2k}} \left\{ \left( a_k - \beta_k (q_{1k}^* + q_{2k}) - c_k (x_i) \right) q_{2k} \right\}
\]

which constitutes a Cournot-Nash equilibrium of the second stage of the game. Next, given the equilibrium quantities, the sub-game perfect Nash equilibrium is obtained by determining the equilibrium locations, i.e., by solving the first stage of the game. They state that if the unit delivered cost is sufficiently small so that is always profitable to offer some quantity at each market then the there always exists a pair of equilibrium locations that consists of a pair of vertices for the first stage of the game. Sarkar et al. (1997) extended the work in Labbé and Hakimi (1991) by considering the case of oligopoly. They find similar condition for the existences of an equilibrium.
The game was further extended by Rhim et al. (2003) by including capacity decisions as well. Their settings is a three stage location game in which identical firms, first, decide their location, second, their capacity and finally the production quantity for each market. Rhim et al. (2003) demonstrated the existence of a sub-game Nash equilibrium by showing that the stage one game can be reduced into a single stage decision related to the so-called congestion games (Rosenthal, 1973). At equilibrium, the opened facilities select the markets they serve based on the demand, and the costs incurred by capacity acquisition, production and transportation. The assumption of identical firm was relaxed by Sáiz and Hendrix (2008), who assumed that the $n$ competitors have their own cost structure. Sáiz and Hendrix (2008) presented the necessary optimality conditions for the Nash equilibria and developed algorithms to find them.

Konur and Geunes (2011) adopt the common two stage approach to study the effect of traffic congestion on competitive firms’ equilibrium facility location and supply quantity decision. They study a set of $k$ competitive firms considering the location of facilities at $m$ possible locations in order to serve customer markets at $n$ locations. Each firm incurs transportation, traffic congestion, and fixed facility location costs as a result of their location and distribution volume decisions. The transportation cost is linear in the quantity flow from facility $i \in I = \{1, 2, \ldots, m\}$ to market $j \in J = \{1, 2, \ldots, n\}$, with unit transportation cost $c_{ij}$.

The congestion cost is a non decreasing convex function of the total quantity flow on the link $(i, j)$ of the form $g_{ij}(q_{ij}) = \alpha_{ij}q_{ij}$, where $q_{ij}$ is the total quantity shipped from location $i$ to market $j$, $q_{ij} = \sum_{r \in R} q_{ijr}$ and $q_{ijr}$ is the quantity shipped from the facility of firm $r \in R = \{1, 2, \ldots, k\}$ at location $i$ to market $j$. $\alpha_{ij}$ denotes the traffic congestion cost factor.

Konur and Geunes (2011) conclude that firms will choose identical facility locations in equilibrium and that increased congestion hampers efficient location of facilities since it forces firms to use either congested links or links that are not close to the market. These results are further extended to the case of no-identical firms in Konur and Geunes (2012).

The model proposed by Nagurney et al. (2002) combines both concepts Nash equilibrium and the spatial price equilibrium (SPE) in order to develop the supply chain network equilibrium (SCNE) model, for investigating the economic behaviours of the players in a decentralised supply chain with the market competition. Their model can find the equilibrium shipment and price patterns between manufacturers and retailers, which will become necessary inputs in modelling the CFL problem on the supply chains. Indeed, the authors in Meng et al. (2009), building upon the SCNE model, proposed a generic mathematical program which can simultaneously determine facility location of the entering firm and the production levels of these facilities.

References


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