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## Defining the almost-entropic regions by algebraic inequalities

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**Abstract:** We study the definability of the almost-entropic regions by finite lists of algebraic inequalities. First, we study linear information inequalities and polyhedrality, we present a proof of a theorem of Matus, which claims that the almost-entropic regions are not polyhedral. Then, we study polynomial inequalities and semialgebraicity, we show that the semialgebraicity of the almost-entropic regions is something that depends on the essentially conditionality of a certain class of conditional information inequalities. Those results suggest that the almost-entropic regions are not semialgebraic. We conjecture that those regions are not decidable.

**Keywords:** entropic regions; entropic vectors; entropy; information inequalities; Shannon inequalities.

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Linear information inequalities are the linear homogeneous inequalities satisfied by Shannon entropy. An important example is the *basic inequalities* claiming that all the Shannon information measures are non-negative. A second important example is the class of Shannon inequalities, which are the linear inequalities that can be expressed as positive linear combinations of the basic inequalities.

Linear information inequalities play an important role in the analysis of communication problems. Unfortunately, it is not easy to decide if a given linear expression involving Shannon entropies is non-negative. Actually, it is not known if the set of linear information

inequalities is a decidable set. We want to study the set of linear information inequalities and the sets of polymatroidal vectors defined by them, which are the so-called *almost-entropic regions*. We focus on the case of four random variables.

Let  $X_1, X_2, X_3, X_4$  be four random variables, given  $i, j, k \in \{1, 2, 3, 4\}$ , we use the symbol  $I(i; j)$  to denote the *mutual information* of  $X_i$  and  $X_j$ , and the symbol  $I(i; j | k)$  to denote the mutual information of  $X_i$  and  $X_j$  conditioned by  $X_k$  (for definitions see Yeung, 2002). We use the symbol  $\mathcal{I}$  to denote the vector

$$-I(1; 2) + I(1; 2 | 3) + I(1; 2 | 4) + I(3; 4)$$

which is the linear expression encoding the famous Ingleton's inequality (Ingleton, 1971). Consider the following question

**Question 1:** *Let  $\lambda$  be a large positive integer, does the inequality*

$$\mathcal{I} + \lambda(I(1; 3 | 2) + I(2; 3 | 1) + I(1; 4 | 2) + I(2; 4 | 1)) \geq 0$$

*hold for all four-tuples of random variables?*

The above question seems to be a hard one, and we cannot give a conclusive answer to it. Last example shows, among other things, that it is hard to recognise the linear inequalities satisfied by Shannon entropy.

*Entropic vectors* (also called entropic functions or entropic polymatroids) are the vectors that can be formed by collecting together the join entropies of a given tuple of random variables. Given  $i_1, \dots, i_k \in \{1, 2, 3, 4\}$ , we use the symbol  $h_{i_1 \dots i_k}$  to denote the join entropy of the tuple  $X_{i_1}, \dots, X_{i_k}$ . The entropic vector associated to the given four-tuple of random variables is the 15-dimensional vector

$$(h_1, h_2, h_3, h_4, h_{12}, h_{13}, h_{14}, h_{23}, h_{24}, h_{34}, h_{123}, h_{124}, h_{134}, h_{234}, h_{1234})$$

Almost-entropic vectors are the vectors that can be approximated by a sequence of entropic polymatroids, and they are the elements of the *almost-entropic regions*. Notice that entropic and almost-entropic vectors satisfy the same linear inequalities. Thus, given  $n \geq 1$ , the almost-entropic region of dimension  $2^n - 1$  (of order  $n$ ) is precisely the set that is defined by the set of linear information inequalities over  $n$  random variables.

We do not know of the existence of a general method which can be used to determine if a given linear expression involving Shannon information measures is non-negative, we also do not know of the existence of an algorithm that can be employed to decide if a given polymatroidal function is almost-entropic. The algorithmic solvability of those two problems is related to the existence and computability of good definitions for the almost-entropic regions. We study the definability of the almost-entropic regions by means of algebraic inequalities. Notice that if the almost-entropic region of order  $n$  can be defined by a finite list of algebraic inequalities, then one can use such a list to check whether a given vector of dimension  $2^n - 1$  is an almost-entropic vector. Thus, it can be said that those algebraic definitions are algorithmically useful. An example of such a useful definition is a polyhedral definition. It is known that if one could compute polyhedral definitions of the almost-entropic regions of any dimension, then he could use linear programming to solve the problem of recognising the infinite set of *solvable communication networks*

(Chan and Grant, 2014). It can also be proved that if one could compute semialgebraic definitions of those regions, then he could solve the problem of computing the capacity of any information network given as input. Those two problems, related to the analysis of communication networks, are the most important algorithmic problems in network coding, and it is worth to remark that it is unknown if they can be solved by algorithmic means.

The non-polyhedrality of the almost-entropic regions was established by Matúš (2007), who proved that the almost-entropic regions of order larger than four are not polyhedral; we provide an elementary exposition of this proof. The core idea of this proof, related to the analysis of two-dimensional sections, can be applied to cope with the following question: Are the almost-entropic regions semialgebraic? We investigate the definability of the almost-entropic regions by finite lists of polynomial inequalities, that is, we investigate the semialgebraicity of those regions. We provide some evidence showing that the almost-entropic regions are not semialgebraic.

We know that the almost-entropic regions are not polyhedral. We try to prove, in this work, that the almost-entropic regions are not semialgebraic. Those two undefinability phenomena could be corollaries of a more general conjecture: The almost-entropic regions cannot be defined in an algorithmically useful way. How can one prove that the almost-entropic regions do not admit algorithmically useful definitions? Any possible type of algorithmically useful definition must imply decidability. Thus, we think that the best way of coping with the last question is by trying to prove that those regions are not decidable. Here, a remark is in order. Notice that the almost-entropic regions are subsets of higher dimensional euclidean spaces, whose interior is non-empty, and it implies that all those regions contain uncomputable elements. Then, it must be clarified what is the meaning of the almost-entropic regions being decidable. We say that the almost-entropic region of order  $n$ , denoted with the symbol  $\bar{\Gamma}_n^*$ , is decidable, if and only if the set  $\bar{\Gamma}_n^* \cap \mathbb{Z}^{2^n - 1}$  is a Turing decidable set of integer-valued vectors of dimension  $2^n - 1$ . We conjecture that the almost-entropic regions of order larger than 4 are not decidable in the above sense.

**Organisation of the work and contributions.** This work is organised into four sections. In Section 1, we introduce the basic definitions and concepts. In Section 2, we present a proof of Matúš' theorem. In Sections 3 and 4, we investigate the semialgebraicity of the almost-entropic regions, and we provide strong evidence suggesting that those sets are not semialgebraic.

## 1 Background: linear information inequalities and entropic functions

Let  $\vec{X} = (X_1, \dots, X_n)$  be a  $n$ -tuple of random variables, it determines a  $n$ -order *entropic function*  $h_{\vec{X}} : (\mathcal{P}([n]) - \emptyset) \rightarrow \mathbb{R}$ , which is defined by:

$$h_{\vec{X}}(I) = H(X_I)$$

where  $X_I$  denotes the tuple  $(X_i)_{i \in I}$  and  $H(X_I)$  its *Shannon entropy*. Thus, we have that the entropic function (vector)  $h_{\vec{X}}$  is constructed by collecting together the entropies of the  $2^n - 1$  non-empty subtuples of the tuple  $\vec{X}$ .

We use the symbol  $\Gamma_n^*$  to denote the set of all the entropic vectors of order  $n$ , which is the so-called *entropic region* of order  $n$ . The set  $\Gamma_n^*$  spans a solid convex cone of dimension  $2^n - 1$  (Yeung, 2002). We use the symbol  $\overline{\Gamma}_n^*$  to denote this cone, which we call *The almost-entropic region of order  $n$* .

Notice that any element of  $\Gamma_n^*$  can be expressed as a linear combination  $\sum_{I \subseteq [n]} e_I h_I$ , where given  $I \subseteq [n]$ , the symbol  $e_I$  denotes the vector of  $\mathbb{R}^{2^n - 1}$  defined by the condition:

$$\text{For all } h_{\vec{X}} \in \Gamma_n^* \text{ we have that } \langle h_{\vec{X}}, e_I \rangle = H(X_I)$$

We get from the above definitions the following equality

$$(\overline{\Gamma}_n^*)^\circ = (\Gamma_n^*)^\circ$$

where given  $A \subseteq \mathbb{R}^n$ , the symbol  $A^\circ$  denotes its *polar cone* (also called the dual cone of  $A$ ). Recall that the polar of  $A$  is the cone

$$\{v \in \mathbb{R}^n : \forall w \in A (\langle v, w \rangle \geq 0)\}$$

That is, the set  $(\Gamma_n^*)^\circ$ , the polar of the entropic region contains all the linear inequalities over  $n$  random variables satisfied by Shannon entropy. The elements of  $(\overline{\Gamma}_n^*)^\circ$  are called *linear information inequalities* over  $n$  random variables. Pippenger (1986) argued that those inequalities encode the fundamental laws of *information theory*, which determine the ultimate limits of information transmission and data compression. Thus, if one subscribes Pippenger's claim, he is obligated to consider that the theory of linear information inequalities is a fundamental fragment of information theory. Then, it is natural to ask, Is this restricted theory decidable? Is it finitely axiomatisable?

It has been showed (argued) that solving many important problems in network coding and secret sharing (Csirmaz, 1997; Gomez, Mejía and Montoya, 2014a, 2014b), database theory (Gottlob et al., 2012), graph guessing (Baber et al., 2013) and index coding (Jafar and Sun, 2013) are something that strongly depend on our ability for computing finite checkable definitions of the entropic regions of all orders. The definability of the almost-entropic regions is entailed by the definability of their polars. Thus, computing finite checkable definitions of the polars of the entropic regions is an important problem. Actually, it has been claimed that it is the most important theoretical problem in network coding (Bassoli et al., 2015), and it has also been claimed that it is one of the most important open problems in information theory (Gottlob et al., 2012).

### 1.1 A suitable basis for $\mathbb{R}^{15}$

In this section we begin our study of the almost-entropic region of order four, which we denote with the symbol  $\overline{\Gamma}_4^*$ .

Recall that, a typical element of  $\overline{\Gamma}_4^*$  is a 15-dimensional vector, say

$$(h_1, h_2, h_3, h_4, h_{12}, h_{13}, h_{14}, h_{23}, h_{24}, h_{34}, h_{123}, h_{124}, h_{134}, h_{234}, h_{1234})$$

where given  $\{i_1, \dots, i_k\} \subseteq \{1, 2, 3, 4\}$ , the entry  $h_{i_1, \dots, i_k}$  is equal to  $\langle h, e_{\{i_1, \dots, i_k\}} \rangle$ . Thus, we will be working in the 15-dimensional euclidean space, and then it is a good idea to fix a suitable basis. Given  $i, j, k, l \in \{1, 2, 3, 4\}$ , and given  $h \in \overline{\Gamma}_4^*$ , we set

- $I_h(i : j) = h_i + h_j - h_{ij}$
- $I_h(i : j | k) = h_{ik} + h_{jk} - h_{ijk} - h_k$
- $H_h(i | j, k, l) = h_{ijkl} - h_{jkl}$

Let  $\vec{X} = (X_1, \dots, X_4)$ , and suppose that  $h_{\vec{X}}$  is the entropic vector that is associated with the tuple  $\vec{X}$ , notice that

- $I_{h_{\vec{X}}}(i : j)$  is equal to the mutual information of  $X_i$  and  $X_j$ , and it is also equal to  $\langle h_{\vec{X}}, e_{\{i\}} + e_{\{j\}} - e_{\{i,j\}} \rangle$ . We use the symbol  $I(i : j)$  to denote the vector  $e_{\{i\}} + e_{\{j\}} - e_{\{i,j\}}$ .
- $I_{h_{\vec{X}}}(i : j | k)$  is the mutual information of  $X_i$  and  $X_j$  given  $X_k$ , and it is also equal to  $\langle h_{\vec{X}}, e_{\{i,k\}} + e_{\{j,k\}} - e_{\{i,j,k\}} - e_{\{k\}} \rangle$ . We use the symbol  $I(i : j | k)$  to denote the vector  $e_{\{i,k\}} + e_{\{j,k\}} - e_{\{i,j,k\}} - e_{\{k\}}$ , and we use the symbol  $H(i | j, k, l)$  to denote the vector  $e_{\{i,j,k,l\}} - e_{\{j,k,l\}}$ .

Matúš and Studený (1995) introduced the *natural basis* of  $\mathbb{R}^{15}$ , which seems to be appropriate to describe the almost-entropic region  $\overline{\Gamma}_4^*$  (see Csirmaz and Matúš, 2016), and which is constituted by the following vectors:

$$\begin{aligned}
 v_1 &= -I(1 : 2) + I(1 : 2 | 3) + I(1 : 2 | 4) + I(3 : 4). \\
 v_2 &= I(1 : 2 | 3). \\
 v_3 &= I(1 : 2 | 4). \\
 v_4 &= I(1 : 3 | 2). \\
 v_5 &= I(1 : 4 | 2). \\
 v_6 &= I(2 : 3 | 1). \\
 v_7 &= I(2 : 4 | 1). \\
 v_8 &= I(3 : 4 | 1). \\
 v_9 &= I(3 : 4 | 2). \\
 v_{10} &= I(3 : 4). \\
 v_{11} &= I(1 : 2 | 3, 4) = e_{\{1,3,4\}} + e_{\{2,3,4\}} - e_{\{1,2,3,4\}} - e_{\{3,4\}}. \\
 v_{12} &= H(1 | 2, 3, 4). \\
 v_{13} &= H(2 | 1, 3, 4). \\
 v_{14} &= H(3 | 1, 2, 4). \\
 v_{15} &= H(4 | 1, 2, 3).
 \end{aligned}$$

Notice that  $v_1$  is the vector encoding the famous *Ingleton inequality* (Ingleton, 1971). Ingleton inequality was the first ever discovered *linear rank inequality*, which is not a linear information inequality, that is, Given a *linear rank function*  $h$ , we have that  $\langle h, v_1 \rangle \geq 0$ , while there exists an entropic function  $h$  such that  $\langle h, v_1 \rangle < 0$ . We will use the symbol  $\mathcal{I}$  to denote the vector  $v_1$ . Notice also that for all  $i \geq 2$ , the vector  $v_i$  encodes a basic inequality asserting that a certain Shannon information measure is non-negative, that is, for all  $i \geq 2$  and for any almost-entropic vector  $h$ , it happens that  $\langle h, v_i \rangle \geq 0$ .

## 2 Linear inequalities and polyhedrality

In this section, we study the (non)polyhedrality of the almost-entropic regions. First, some definitions.

**Definition 1:**  $A \subset \mathbb{R}^n$  is polyhedral, if and only if the set  $A$  can be defined by a finite list of linear inequalities, and we say that such a finite list of linear inequalities is a polyhedral definition of  $A$ .

Remark that polyhedral definitions are easy to check. If one knows a polyhedral definition of  $A \subset \mathbb{R}^n$ , he can use this definition to recognise the elements of  $A$ . Moreover, if he knows a polyhedral definition of  $A$ , he can use it, and linear programming, to compute the maximum (minimum) of any linear functional defined over  $A$ .

We say that  $A \subset \mathbb{R}^n$  is a *convex cone* if and only if it is closed under positive linear combinations, and we say that the convex cone  $A$  is a *closed convex cone*, if and only if it is a closed set in the euclidean topology of  $\mathbb{R}^n$ . It is known that the almost-entropic regions are closed convex cones (Yeung, 2002)

We present, in this section, an elementary exposition of the proof of a theorem of Matúš (2007), which claims that for all  $n \geq 4$ , the cone  $\overline{\Gamma}_n^*$  is not polyhedral. Our exposition is based on the following facts:

- 1 Given a closed convex cone  $A \subseteq \mathbb{R}^n$ , the cone  $A$  is polyhedral, if and only if  $A^\circ$  is polyhedral. Thus, if one wants to prove  $\overline{\Gamma}_4^*$  is not polyhedral, he can focus on proving that  $(\overline{\Gamma}_4^*)^\circ$  is not polyhedral.
- 2 Given a closed, convex, polyhedral cone  $A \subseteq \mathbb{R}^n$ , all the projections of  $A$  are polyhedral. It is easy to check that for all  $n \geq 4$ , the set  $\overline{\Gamma}_4^*$  is a projection of the set  $\overline{\Gamma}_n^*$ . Thus, if one proves that  $(\overline{\Gamma}_4^*)^\circ$  is not polyhedral, he can get as a corollary that for all  $n \geq 4$ , the set  $\overline{\Gamma}_n^*$  is not polyhedral.

Given the above facts, we focus on proving that  $(\overline{\Gamma}_4^*)^\circ$  is not polyhedral.

**Notation 2:** *The first quadrant of  $\mathbb{R}^2$ , denoted by  $FQ$ , is the set  $\{(a, b) : a, b > 0\}$ . Given  $A \subset FQ$ , the symbol  $\pi_1(A)$  denotes the projection of  $A$  over the  $X$ -axis.*

**Definition 3:** Given a vector  $v \in \mathbb{R}^n$ , the ray determined by  $v$  is the set  $\{\lambda v : \lambda \geq 0\}$

Sometimes, it is fairly easy to prove that a certain set is not polyhedral, consider the following lemma.

**Lemma 4:** *Let  $A \subset \mathbb{R}^2$ , suppose that  $\pi_1(A \cap FQ)$  is an unbounded set, and suppose that  $A$  contains the line  $\{(0, y) : y \geq 0\}$ . Then, if for all  $\lambda, \beta > 0$ , the set  $A$  does not contain the ray determined by  $\lambda e_1 + \beta e_2$ , we have that  $A$  is not polyhedral.*

The proof of the lemma is straightforward and we omit it. Moreover, it suggests that proofs of non-polyhedrality could become easy when one has to cope with two-dimensional sets. Recall that  $(\overline{\Gamma}_4^*)^\circ$  is a 15-dimensional set. Klee (1959) proved that a set  $A \subset \mathbb{R}^n$  is polyhedral, if and only if all its  $(n - 1)$ -dimensional sections are polyhedral, it means that the set  $A \subset \mathbb{R}^n$  is polyhedral, if and only if, for all  $(n - 1)$ -dimensional affine subspace of  $\mathbb{R}^n$ , say  $V$ , the set  $A \cap V$  is polyhedral. Next lemma is an easy corollary of Klee's theorem.

**Lemma 5:** *Let  $A \subseteq \mathbb{R}^n$ , the set  $A$  is polyhedral, if and only if all the two-dimensional sections of  $A$  are polyhedral.*

Thus, if  $(\overline{\Gamma}_4^*)^\circ$  is not polyhedral, there must exist  $P$  a two-dimensional plane contained in  $\mathbb{R}^{15}$ , such that  $(\overline{\Gamma}_4^*)^\circ \cap P$  is not polyhedral. Then, If one can determine the right plane, he succeeds, given that it only remains to be proved that the two-dimensional set

$(\overline{\Gamma}_4^*)^\circ \cap P$  is not polyhedral, and, according to the above facts, most of the time this remaining task is easy.

In next section, we define a two-dimensional section of  $(\overline{\Gamma}_4^*)$ , and we apply Lemma 4 to verify that such a two-dimensional section is not polyhedral

### 2.1 A two-dimensional view of Matúš' theorem

Recall that

- $v_1 = \mathcal{I} = -I(1; 2) + I(1; 2 | 3) + I(1; 2 | 4) + I(3; 4)$ .
- $v_2 = I(1; 2 | 3)$ .
- $v_4 = I(1; 3 | 2)$ .
- $v_6 = I(2; 3 | 1)$ .

Matúš proved that the sequence

$$\left\{ s\mathcal{I} + \left( \frac{s(s+1)}{2} \right) (v_2 + v_4) + v_6 \right\}_{s \geq 0}$$

is contained in  $(\overline{\Gamma}_4^*)^\circ$ . We observe that this sequence is contained in a two-dimensional section of  $(\overline{\Gamma}_4^*)^\circ$ .

Let  $v \in \mathbb{R}^{15}$  and let  $W \subset \mathbb{R}^{15}$ . From now on, we use the symbol  $Tr_v(W)$  to denote the translation of  $W$  by the vector  $v$ . We use the symbol  $\mathcal{J}_{\frac{1}{2}}$  to denote the vector  $v_2 + v_4$ .

Let  $Q$  be the plane spanned by the vectors  $\mathcal{I}$  and  $\mathcal{J}_{\frac{1}{2}}$  (i.e.  $Q$  is the plane  $\langle \{\mathcal{I}, \mathcal{J}_{\frac{1}{2}}\} \rangle$ ), and let  $P = Tr_{v_6}(Q)$ . Notice that Matúš' sequence is contained in  $(\overline{\Gamma}_4^*)^\circ \cap P$ .

In order to fix a system of rectangular coordinates for  $P$ , we only have to choose a point in  $P$ , (which plays the role of the origin) together with two independent vectors. The natural choice corresponds to designate the point  $v_6$  as the origin, and the vectors  $\mathcal{I}$  and  $\mathcal{J}_{\frac{1}{2}}$  as the coordinate axis. If we work in this system of coordinates, Matúš' sequence becomes equal to the sequence  $\left\{ \left( s, \frac{s(s+1)}{2} \right) \right\}_{s \geq 0}$ .

Let  $FQ(P)$  be the first quadrant of the plane  $P$  (as determined by our coordinate system). We observe that

- 1 Matúš' sequence is contained in  $(\overline{\Gamma}_4^*)^\circ \cap FQ(P)$ , and hence  $\pi_1 \left( (\overline{\Gamma}_4^*)^\circ \cap FQ(P) \right)$  is an unbounded set.
- 2  $(\overline{\Gamma}_4^*)^\circ \cap FQ$  contains the line  $\left\{ t \cdot \mathcal{J}_{\frac{1}{2}} : t \geq 0 \right\}$ . The containment follows from the fact that  $\mathcal{J}_{\frac{1}{2}}$  is a Shannon inequality, and for all  $t \geq 0$ , the vector  $v_6 + t \cdot \mathcal{J}_{\frac{1}{2}}$  is a positive combination of Shannon inequalities.

The above two facts indicate that we can apply Lemma 4 on the two-dimensional set  $(\overline{\Gamma}_4^*)^\circ \cap P$ .

**Lemma 6:** *For all  $\lambda, \beta > 0$ , the vector  $\lambda\mathcal{I} + \beta\mathcal{J}_{\frac{1}{2}}$  is not an information inequality.*

*Proof:* The set of linear information inequalities is closed under positive scalar multiples, then it is enough to prove that for all  $\beta > 0$ , the vector  $\mathcal{I} + \beta \mathcal{J}_{\frac{1}{2}}$  is not an information inequality. To this end, we use a parameterised family of probability distributions, which we define below.

Given  $\delta \in (0, \frac{1}{4}]$ , distribution  $D_\delta$  is given by the following table

| 1 | 2 | 3 | 4 | Pr                     |
|---|---|---|---|------------------------|
| 1 | 1 | 1 | 1 | $\frac{1}{4}$          |
| 1 | 1 | 0 | 1 | $\frac{1}{4} - \delta$ |
| 0 | 0 | 1 | 0 | $\frac{1}{4} - \delta$ |
| 0 | 0 | 0 | 0 | $\frac{1}{4}$          |
| 1 | 0 | 1 | 1 | $\delta$               |
| 1 | 0 | 0 | 0 | $\delta$               |

Given  $\delta$ , we use the symbol  $h_{D_\delta}$  to denote the entropic vector generated by the distribution  $D_\delta$ . Kaced and Romashchenko (2013) proved that there exists  $c < 0$ , such that when  $\delta$  tends to zero, the equalities

$$\begin{aligned} \langle h_{D_\delta}, \mathcal{I} \rangle &= c \cdot \delta^2 + \Theta(\delta^3) \\ \langle h_{D_\delta}, I(1:3 | 2) \rangle, \langle h_{D_\delta}, I(2:3 | 1) \rangle &= \Theta(\delta^3) \end{aligned}$$

hold. Then, given  $\beta > 0$ , there exists  $\delta > 0$  such that  $\langle h_{D_\delta}, \mathcal{I} + \beta \mathcal{J}_{\frac{1}{2}} \rangle < 0$ . Therefore, we have that  $\mathcal{I} + \beta \mathcal{J}_{\frac{1}{2}}$  is not an information inequality, and it is true for all  $\beta > 0$ .  $\square$

Now, we can get as a corollary Matúš' theorem

**Theorem 7:** *Given  $m \geq 4$ , the set  $\overline{\Gamma_m^*}$  is not polyhedral.*

### 3 Polynomial inequalities and semialgebraicity

In this section, we study the semialgebraicity of the almost-entropic regions. For the basics of semialgebraic sets, we refer the reader to Bochnak, Coste and Roy (1998).

Sets defined by a finite list of polynomial inequalities are called *basic semialgebraic*, and the finite union of those sets are called *semialgebraic sets*. Semialgebraic sets behave almost as well as polyhedral sets: It is possible to effectively solve optimisation problems that are defined by semialgebraic constraints and semialgebraic functions (Lasserre, 2009). Thus, it would be good news if we could prove that the almost-entropic regions are semialgebraic. Actually, we conjecture that those regions are not semialgebraic.

Notice that polynomial inequalities could be very much more expressive than linear inequalities. Consider the following example. Let  $\mathcal{C}$  be the set

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

This set, the two-dimensional unit disk, can be defined by a single polynomial inequality, but it should be clear that it cannot be defined by a finite list of linear inequalities. Thus, in despite of their non-polyhedrality, the almost-entropic regions could be semialgebraic.

Chan and Grant (2008) provided some weak evidence concerning this fact, they discovered a quadratic information inequality which cannot be entailed by any finite set of linear inequalities, and which is almost as strong as the whole infinite sequence of linear inequalities that we used in the proof of Theorem 7 (the so called Matúš' sequence). Chan and Grant (2008) asked the following

**Problem 8** (CG problem): *Is it possible to define the almost-entropic regions using finite lists of polynomial inequalities?*

Thus, The CG problem is the question about the semialgebraicity of the almost-entropic regions.

### 3.1 The logical connection

In this section, we study the well-known relations between semialgebraicity and first-order definability (Tarski, 1951). For the basics of first order logic and model theory, we refer the reader to Hodges (1997).

We have the following important theorem (Tarski, 1951).

**Theorem 9** (Tarski-Seidenberg): *Semialgebraic sets are exactly the sets that can be defined in first-order logic over the real closed field.*

We can use the logical connection (given by the Tarski-Seidenberg theorem) to establish some basic facts.

**Theorem 10:** *Let  $A \subset \mathbb{R}^n$  be a closed convex cone.*

- 1  *$A$  is semialgebraic, if and only if,  $A^\circ$  is semialgebraic.*
- 2 *If  $A$  is semialgebraic, and  $P$  is a plane with rational parameters, then  $A \cap P$  is semialgebraic.*
- 3 *Semialgebraic sets are closed under projections.*

*Proof:* Suppose that  $A \subset \mathbb{R}^n$  is semialgebraic, then  $A$  is first-order definable over the real closed field  $(\mathbb{R}, +, \times, 0, 1)$ . Thus, there exists a first-order formula over the alphabet  $\{+, \times, 0, 1\}$ , say  $\psi(x_1, \dots, x_n)$ , such that

$$A = \{(a_1, \dots, a_n) \in \mathbb{R}^n : (\mathbb{R}, +, \times, 0, 1) \models \psi[a_1, \dots, a_n]\}$$

Let  $\varphi(x_1, \dots, x_n)$  be the formula

$$\forall y_1, \dots, y_n (\psi(y_1, \dots, y_n) \implies \exists z ((x_1 y_1) + \dots + (x_n y_n) = z z))$$

It is easy to check that  $\varphi(x_1, \dots, x_n)$  defines the set  $A^\circ$ . Then,  $A^\circ$  is semialgebraic.

Suppose that  $A^\circ$  is semialgebraic. We have that  $(A^\circ)^\circ = A$ , given that  $A$  is a closed convex cone. Then, it follows that the set  $A$  must be semialgebraic.

Now suppose that  $A$  is semialgebraic, and let  $P$  be a plane with rational parameters. There exists a first-order formula  $\alpha(x_1, \dots, x_n)$  defining  $P$ . Let  $\psi(x_1, \dots, x_n)$  be a first-order formula defining the semialgebraic set  $A$ , we have that the formula

$$\alpha(x_1, \dots, x_n) \wedge \psi(x_1, \dots, x_n)$$

defines the set  $A \cap P$ .

Finally, we observe that the class of semialgebraic sets is closed under projections, and this fact is equivalent to the theorem of Tarski and Seidenberg.  $\square$

Let us prove, as a warm up, that the entropic regions of order larger than 2 are not semialgebraic.

**Theorem 11:** *If  $n \geq 3$ , the entropic region  $\Gamma_n^*$  is not semialgebraic.*

*Proof:* We prove that  $\Gamma_3^*$  is not semialgebraic. It is known (see Yeung, 2002) that  $(a, a, a, 2a, 2a, 2a, 2a) \in \Gamma_3^*$ , if and only if there exist three pairwise independent uniformly distributed random variables, say  $X_1, X_2, X_3$ , such that

$$h_{X_1 X_2 X_3} = (a, a, a, 2a, 2a, 2a, 2a)$$

Recall that if  $X$  is a random variable which is uniformly distributed over the set  $\Omega$ , then its Shannon entropy is equal to  $\log(|\Omega|)$ .

It means that  $(a, a, a, 2a, 2a, 2a, 2a) \in \Gamma_3^*$  if and only if there exists  $n \in \mathbb{N}$  such that  $a = \log(n)$ . Suppose that  $\psi(x_1, \dots, x_7)$  is a first-order formula defining the set  $\Gamma_3^*$ . Consider the formula

$$\begin{aligned} \varphi(x_1, \dots, x_7) = & \psi(x_1, \dots, x_7) \wedge (x_1 = x_2 = x_3) \\ & \wedge (x_4 = x_5 = x_6 = x_7 = (1 + 1)x_1) \end{aligned}$$

Notice that  $Def_\varphi$  is equal to the set

$$\{(a, a, a, 2a, \dots, 2a) \in \mathbb{R}^7 : \exists n \in \mathbb{N}^+ (a = \log(n))\}$$

which is not semialgebraic because it has infinitely many connected components (Bochnak, Coste and Roy, 1998). Thus, we get a contradiction if we suppose that  $\Gamma_3^*$  is semialgebraic. Therefore, we can conclude that for all  $n \geq 3$ , the set  $\Gamma_n^*$  is not semialgebraic.  $\square$

### 3.2 *Disproving semialgebraicity: a two-dimensional view*

Theorem 11 shows that the entropic regions are not semialgebraic. It also shows that it is possible to establish the non-semialgebraicity of some sets. We used, in the proof, a key fact: The set  $\Gamma_3^*$  is not closed, and the intersection of this set with its boundary has a complex structure which is not semialgebraic. We cannot use the same idea with the set  $\overline{\Gamma}_4^*$ , given that it is a closed set<sup>1</sup>.

We begin with the following observation: Given  $A \subset \mathbb{R}^n$ , if there exists a two-dimensional plane  $P$  such that  $A \cap P$  is not semialgebraic, then the set  $A$  is not semialgebraic.

We know that the converse of the above fact is not true, consider the following elementary example:

Let  $S$  be the set  $\{(t, t^2, t^3) : t \in \mathbb{N}\}$ . The set  $S$  is not semialgebraic, given that it has infinitely many connected components. On the other hand, we have that the intersection of  $S$  with a two-dimensional subspace of  $\mathbb{R}^3$  is always a finite set. Finite sets are

semialgebraic, and it means that  $S$  is an example of a non-semialgebraic set such that all its two-dimensional sections are semialgebraic.

It is important to remark that all the known examples of non-semialgebraic sets, whose two-dimensional sections are semialgebraic, are very similar in nature to the above example. We have the following conjecture

**Conjecture 12:** *Let  $A \subset \mathbb{R}^n$  be a convex set, the set  $A$  is semialgebraic, if and only if all its two-dimensional sections are semialgebraic.*

It could happen that our conjecture is false, and that  $(\overline{\Gamma_4^*})^\circ$  is not semialgebraic, but that all its two-dimensional sections are semialgebraic. It is, in some sense, the worst possible scenario. Nevertheless, we are confident in our conjecture, and we think that the easiest way of proving that  $(\overline{\Gamma_4^*})^\circ$  is not semialgebraic consists in showing that there exists a two-dimensional section that is not semialgebraic.

We have to ask: How can one prove that a two-dimensional set is not semialgebraic? We will use the logical connection to establish some important facts concerning semialgebraic and(or) first-order definable sets.

First, three technical propositions:

**Proposition 13:** *Given a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , if  $f$  is an exponential growing function, then it is not first-order definable over the real closed field.*

*Proof:* Given that  $(\mathbb{R}, +, \times, 0, 1, \leq)$  admits quantifier elimination, we have that every definable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is piecewise given by terms, that is, for each such  $f$  there exist  $k \in \mathbb{N}$  and terms  $t_1, \dots, t_k$  such that

$$(\mathbb{R}, +, \times, 0, 1, \leq) \models \forall x (f(x) = t_1(x) \vee \dots \vee f(x) = t_k(x))$$

The terms in the language of real closed fields are polynomial functions. It means that the graph of a first-order definable function can be constructed by gluing together a finite number of sections of polynomial functions. The graph of an exponential growing function cannot be constructed that way. Thus, we have that exponential growing functions are not first-order definable.  $\square$

**Proposition 14:** *If  $A \subset \mathbb{R}^n$  is semialgebraic, the boundary of  $A$ , which we denote with the symbol  $\delta(A)$ , is also semialgebraic.*

*Proof:* Let  $\psi(x_1, \dots, x_n)$  be a first-order formula defining  $A$  and let  $\varphi(y_1, \dots, y_n)$  be the formula

$$\forall (\epsilon > 0) \exists (x_1 \dots x_n) \exists (z_1 \dots z_n) (\psi(x_1, \dots, x_n) \wedge \neg \psi(z_1, \dots, z_n) \wedge \alpha(\vec{x}, \vec{y}, \vec{z}))$$

where  $\alpha(\vec{x}, \vec{y}, \vec{z})$  is equal to

$$\left( \sum_{i=1}^n (x_i - y_i)^2 < \epsilon \right) \wedge \left( \sum_{i=1}^n (z_i - y_i)^2 < \epsilon \right)$$

It is easy to see that  $\varphi(y_1, \dots, y_n)$  defines the boundary of  $A$ .  $\square$

**Proposition 15:** *Given a semialgebraic set  $A \subset \mathbb{R}^n$ , the function  $d_A : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by*

$$d_A(\vec{x}) = \inf \{d(\vec{a}, \vec{x}) : \vec{a} \in A\}$$

*is a semialgebraic function, where the symbol  $d(\vec{a}, \vec{x})$  denotes the euclidean distance between  $\vec{a}$  and  $\vec{x}$*

*Proof:* We have to show that the graph of  $d_A$  is first-order definable. Suppose that  $A$  is defined by the formula  $\psi(z_1, \dots, z_n)$ . Let  $\alpha(x, x_1, \dots, x_n)$  be the formula given by:

$$\alpha(x, x_1, \dots, x_n) = \forall y (\beta(y, x, x_1, \dots, x_n)) \wedge \neg \exists u (\gamma(u, x, x_1, \dots, x_n))$$

where  $(\beta(y, x, x_1, \dots, x_n))$  is equal to

$$(y > 0) \Rightarrow \exists (z_1 \cdots z_n) \left( \psi(z_1, \dots, z_n) \wedge \left( \sum_{i=1}^n (x_i - z_i)^2 < x + y \right) \right)$$

and  $\gamma(y, x, x_1, \dots, x_n)$  is equal to

$$(u < x) \wedge \exists (z_1 \cdots z_n) \left( \psi(z_1, \dots, z_n) \wedge \left( \sum_{i=1}^n (x_i - z_i)^2 = u \right) \right)$$

It is easy to check that  $\alpha(x, x_1, \dots, x_n)$  defines the graph of  $d_A$ . □

**Theorem 16:** *Let  $A \subset \mathbb{R}^n$  be a closed, non-empty set, let  $P$  be a plane with rational parameters, and let  $R$  be the ray  $P_0 + tv$ , where  $P_0, v \in \mathbb{Q}^n$ . Suppose that  $R$  is contained in  $P \setminus (P \cap A)$  and define a function  $\beta(t)$  by*

$$t \mapsto d_{\delta(A \cap L)}(P_0 + tv)$$

*If  $\beta(t)$  is an exponential decaying function, the set  $A$  is not semialgebraic.*

*Proof:* Suppose that  $A$  is semialgebraic. We have that  $A \cap P$  and  $\delta(A \cap P)$  are first-order definable and hence semialgebraic. Therefore, we have that the function  $d_{\delta(A \cap P)}$  is semialgebraic. Notice that  $\beta(t)$  is the restriction of  $d_{\delta(A \cap P)}$  to the set  $R$ , which is semialgebraic. Restrictions of semialgebraic functions to semialgebraic sets are also semialgebraic. Thus,  $\beta(t)$  is a semialgebraic non-vanishing function. We can use the first-order definition of  $\beta(t)$  to define the function  $\frac{1}{\beta(t)}$ . Notice that the later function is a semialgebraic function of exponential growth. Thus, we have arrived to a contradiction and the theorem is proved. □

Let  $(\overline{\Gamma}_4^*)^\circ \cap P$  be a two-dimensional section of  $(\overline{\Gamma}_4^*)^\circ$ . Given a ray  $R$  contained in  $P$ , we say it is a *forbidden ray*, if and only if,  $R \cap (\overline{\Gamma}_4^*)^\circ = \emptyset$ . Notice that, for achieving our goal, it is sufficient to find a sequence of information inequalities, which is contained in a two-dimensional section of  $(\overline{\Gamma}_4^*)^\circ$ , and which approaches a forbidden ray at exponential speed.

#### 4 Looking for a non-semialgebraic, two-dimensional section of $(\overline{\Gamma_4^*})^\circ$

We will try to use, in this section, the analytical machinery developed so far, it means that we will try to determine a two-dimensional section of  $(\overline{\Gamma_4^*})^\circ$  containing an infinite object that does not behave semialgebraically.

We proved, following Matúš, that there exists an infinite sequence of information inequalities called  $\mathcal{M}$ , which is contained in a two-dimensional plane (the plane  $P$ ), which behaves *quadratically*, and which is close to the boundary of  $(\overline{\Gamma_4^*})^\circ$ . Being close to the boundary of  $(\overline{\Gamma_4^*})^\circ$  can be interpreted in the following way: The curve  $\delta\left((\overline{\Gamma_4^*})^\circ \cap P\right)$  approaches the sequence  $\mathcal{M}$  well enough. Then, we concluded that  $(\overline{\Gamma_4^*})^\circ$  cannot be polyhedral, given that the boundary of a two-dimensional section of a polyhedral cone is a polygonal curve, and polygonal curves cannot approach a quadratic sequence well enough.

Notice that the above paragraph encodes a general strategy that could be applied in some other settings. Thus, for instance, if one wants to prove that  $(\overline{\Gamma_4^*})^\circ$  is non-semialgebraic, he could try to prove that there exists a two-dimensional sequence of information inequalities, which is close to the boundary of  $(\overline{\Gamma_4^*})^\circ$ , and which behaves non-semialgebraically (and which behaves in such a way that it cannot be approximated by a semialgebraic curve well enough). The canonical example of non-semialgebraic behaviour is exponential behaviour. Then, to begin with, we could look for a sequence of information inequalities that behaves exponentially and which is close to the boundary of  $(\overline{\Gamma_4^*})^\circ$ . Being close to the boundary is somewhat equivalent to being sharp. Therefore, we have decided to look for a sharp, exponential, two-dimensional sequence of information inequalities.

There are some infinite sequences of information inequalities registered in the literature (Dougherty, Freiling and Zeger, 2011), all of them, but one, behaves polynomially. The remaining sequence is an exponential decaying, two-dimensional sequence discovered by Dougherty, Freiling and Zeger (2011). We will study, in the remaining of the paper, the sequence of Dougherty, Freiling and Zeger. The question that we could not answer is the question about the sharpness of the sequence.

Now, we introduce the sequence of Dougherty et al. Set:

- $\mathcal{J}_{\frac{1}{2}}^* = v_5 + v_7$
- $\mathcal{J} = \mathcal{J}_{\frac{1}{2}} + \mathcal{J}_{\frac{1}{2}}^*$ .

The sequence

$$\left\{ \mathcal{I} - \mathcal{J}_{\frac{1}{2}}^* + \left(\frac{1}{2^s - 1}\right) v_3 + \left(\frac{s2^{s-1}}{2^s - 1}\right) \mathcal{J} \right\}_{s \geq 1}$$

is an infinite sequence of information inequalities discovered by Dougherty, Freiling and Zeger (2011). We use the symbol  $dfz_s$  to denote the inequality

$$\mathcal{I} - \mathcal{J}_{\frac{1}{2}}^* + \left(\frac{1}{2^s - 1}\right) v_3 + \left(\frac{s2^{s-1}}{2^s - 1}\right) \mathcal{J}$$

and we use the term DFZ-sequence to denote the sequence  $\{dfz_s\}_{s \geq 1}$ .

Let  $Q$  be the plane spanned by the vectors  $v_3$  and  $\mathcal{J}$ , and let  $L$  be equal to  $Tr_{\mathcal{I} - \mathcal{J}_{\frac{1}{2}}^*}(Q)$ .

Notice that the DFZ sequence is included in  $(\overline{\Gamma_4^*})^\circ \cap L$ . If we use  $\mathcal{I} - \mathcal{J}_{\frac{1}{2}}^*$  as the

origin of  $L$ , and the lines  $\left\{ \mathcal{I} - \mathcal{J}_{\frac{1}{2}}^* + sv_3 : s \geq 0 \right\}$  and  $\left\{ \mathcal{I} - \mathcal{J}_{\frac{1}{2}}^* + s \cdot \mathcal{J} : s \geq 0 \right\}$  as its coordinate axis, we get a parametrisation of  $\{dfz_s\}_{s \geq 1}$ , over the plane  $L$ , that looks like  $\left( \frac{1}{2^s-1}, \frac{s2^{s-1}}{2^s-1} \right)_{s \geq 1}$  and that asymptotically behaves like  $\left( \frac{1}{2^s}, \frac{s}{2} \right)$ . Thus, the DFZ sequence is a plane sequence that approaches the ray

$$R = \left\{ \mathcal{I} - \mathcal{J}_{\frac{1}{2}}^* + \lambda \cdot \mathcal{J} : \lambda \geq 0 \right\}$$

at exponential speed. It only remains to be proved that  $R$  is contained in the complement of  $(\Gamma_4^*)^\circ \cap L$ .

Let  $R^*$  be the ray  $\{\mathcal{I} + \lambda \cdot \mathcal{J} : \lambda \geq 0\}$ , we have

**Lemma 17:** *If  $R$  is a forbidden ray, then  $R^*$  is a forbidden ray.*

*Proof:* First, we note that  $\mathcal{J}_{\frac{1}{2}}^*$  is a Shannon inequality. Then, if  $\mathcal{I} + \lambda \cdot \mathcal{J}$  is not an information inequality, the vector  $\mathcal{I} - \mathcal{J}_{\frac{1}{2}}^* + \lambda \cdot \mathcal{J}$  cannot be an information inequality.  $\square$

Given the above lemma, we focus our attention on the following problem

**Problem 18:** *Prove that for all  $\lambda > 0$ , the vector  $\mathcal{I} + \lambda \cdot \mathcal{J}$  is not an information inequality.*

In despite of all our efforts, we have been unable of solving the above problem. Let us discuss some facts related to it.

Kaced and Romashchenko (2013) introduced the notion of conditional information inequality, which is defined below

**Definition 19:** Given  $v, w \in \mathbb{R}^{15}$ , the pair  $(v, w)$  is a conditional information inequality, if and only if the implication

$$\langle v, h \rangle = 0 \text{ implies that } \langle w, h \rangle \geq 0$$

holds true for any entropic vector  $h$ .

We have

**Lemma 20:** *The pair  $(\mathcal{J}, \mathcal{I})$  is a conditional information inequality.*

*Proof:* Suppose that  $h$  is an Ingleton violating polymatroid, and suppose that  $h$  is orthogonal to  $\mathcal{J}$ . Then,  $h$  must be orthogonal to  $I(1 : 3 | 2)$ ,  $I(2 : 3 | 1)$ ,  $I(1 : 4 | 2)$  and  $I(2 : 4 | 1)$ . It implies that  $h$  is orthogonal to  $\mathcal{J}_{\frac{1}{2}}^*$ , given that  $\mathcal{J}_{\frac{1}{2}}^* = I(1 : 4 | 2) + I(2 : 4 | 1)$ . Then, we have that

$$\begin{aligned} \langle dfz_s, h \rangle &= \left\langle \mathcal{I} - \mathcal{J}_{\frac{1}{2}}^* + \left( \frac{1}{2^s-1} \right) v_3 + \left( \frac{s2^{s-1}}{2^s-1} \right) \mathcal{J}, h \right\rangle \\ &= \langle \mathcal{I}, h \rangle + \frac{\langle v_3, h \rangle}{2^s-1} \end{aligned}$$

and then, it is clear that there must exist  $s$  such that  $\langle dfz_s, h \rangle < 0$ . If  $h$  is an entropic polymatroid, this last inequality cannot hold for  $h$ , given that  $dfz_s$  is an information inequality. Thus, if  $h$  is an Ingleton violating entropic polymatroid, it cannot be orthogonal to  $\mathcal{J}$ . It means that  $(\mathcal{J}, \mathcal{I})$  is a conditional information inequality.  $\square$

A more interesting notion is the notion of essentially conditional information inequality introduced by the same authors (Kaced and Romashchenko, 2013).

**Definition 21:** Given a conditional information inequality, say  $(v, w)$ , it is *essentially conditional*, if and only if for all  $N \geq 0$ , the expression  $N \cdot v + w$  is not an information inequality.

Kaced and Romashchenko (2013) proved that  $(\mathcal{J}_{\frac{1}{2}}, \mathcal{I})$  is essentially conditional. Notice that we used this last fact in our proof of Matúš' theorem. It is easy to prove that  $\mathcal{J}_{\frac{1}{2}}^*$  is also an essentially conditional information inequality, this fact follows from Kaced and Romaschenko result by switching the variables  $X_3$  and  $X_4$ . Observe that problem 18 is equivalent to prove that  $(\mathcal{J}, \mathcal{I})$  is an essentially conditional information inequality. Thus, we have

**Theorem 22:** *If  $(\mathcal{J}, \mathcal{I})$  is an essentially conditional information inequality, the almost-entropic regions of order larger than 3 are not semialgebraic.*

This last theorem, together with our proof of Matúš' theorem, indicates that the existence of two-dimensional sections of  $(\overline{\Gamma}_4^*)^\circ$  exhibiting a complex geometrical structure (non-polyhedral or non-semialgebraic structure) is closely related to the existence of essentially conditional information inequalities.

There is a recent result of Liu and Walsh (2015) claiming that the almost-entropic region of order 4 can be defined by a single non-linear inequality. This result seems to refute our conjecture that this region is not semialgebraic. It is important to remark that the authors of the aforementioned work are using a very general notion of non-linear function: A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is non-linear, if and only if it is not linear. Let  $\chi$  be the characteristic function of  $co\text{-}(\overline{\Gamma}_4^*)$  (the set theoretical complement of  $\overline{\Gamma}_4^*$ ). Notice that  $h \in \overline{\Gamma}_4^*$ , if and only if  $-\chi(h) \geq 0$ . Thus, the almost-entropic region of order 4, as well as any other subset of  $\mathbb{R}^{15}$ , can be easily defined by a single non-linear (and useless) inequality. There is one reason that makes the result of Liu and Walsh a non-trivial result: The function that occurs in their inequality, which we denote with the symbol  $LW$ , is a concrete function, and it is not an abstract function like our function  $\chi$ . How concrete is function  $LW$ ? Is it a polynomial function? Function  $LW$  has a complex definition as the minimum of a certain optimisation problem, it is not a polynomial function, and the inequality it determines is not a polynomial inequality.

We can prove that the set  $\overline{\Gamma}_4^*$  cannot be defined by a single polynomial inequality. Then, the inequality of Liu and Walsh is not a polynomial inequality and their result does not refute our conjecture. Next theorem shows that  $(\overline{\Gamma}_4^*)^\circ$  cannot be defined by a single polynomial inequality. One can use the same argument to prove that  $\overline{\Gamma}_4^*$  cannot be defined by a single polynomial inequality. The reader will notice that the proof is based on the fact that  $(\overline{\Gamma}_4^*)^\circ$  is either non-semialgebraic or that its boundary contains a line segment. The same is true of  $\overline{\Gamma}_4^*$ . It is the case given that the boundary of  $\overline{\Gamma}_4^*$  contains at least one extreme

ray, but it does not contain the whole line determined by this ray. We decided to make the proof for  $(\overline{\Gamma}_4^*)^\circ$  because this proof uses the two-dimensional section that we have studied in this section, and hence it shows that our two-dimensional section can be useful in the study of the almost-entropic region.

**Theorem 23:** *The set  $(\overline{\Gamma}_4^*)^\circ$  cannot be defined by a single polynomial inequality.*

*Proof:* Recall that  $R$  is the ray  $\{\mathcal{I} - \mathcal{J}_{\frac{1}{2}}^* + \lambda \cdot \mathcal{J} : \lambda \geq 1\}$ . The set  $(\overline{\Gamma}_4^*)^\circ$  either intersects  $R$  or it does not intersect  $R$ . If  $(\overline{\Gamma}_4^*)^\circ$  does not intersect  $R$ , we know that  $(\overline{\Gamma}_4^*)^\circ$  is not semialgebraic and that it cannot be defined by a finite list of polynomial inequalities. Suppose that  $(\overline{\Gamma}_4^*)^\circ$  intersects  $R$ . There exists  $\lambda_0 > 0$  such that the equality

$$\lambda_0 = \inf \left\{ \lambda : \left( \mathcal{I} - \mathcal{J}_{\frac{1}{2}}^* + \lambda \cdot \mathcal{J} \right) \in (\overline{\Gamma}_4^*)^\circ \right\}$$

holds. We notice that for all  $\beta > \lambda_0$  the vector  $(\mathcal{I} - \mathcal{J}_{\frac{1}{2}}^* + \beta \cdot \mathcal{J}) \in (\overline{\Gamma}_4^*)^\circ$ . Moreover, the set

$$R_{\lambda_0} = \left\{ \mathcal{I} - \mathcal{J}_{\frac{1}{2}}^* + \lambda \cdot \mathcal{J} : \lambda \geq \lambda_0 \right\}$$

is contained in the boundary of  $(\overline{\Gamma}_4^*)^\circ$ . Now, we suppose that  $(\overline{\Gamma}_4^*)^\circ$  is defined by a single polynomial inequality, say  $p(X_1, \dots, X_{15}) \geq 0$ . Let  $q(t)$  be the univariate polynomial  $p(\mathcal{I} - \mathcal{J}_{\frac{1}{2}}^* + t \cdot \mathcal{J})$ , we have that  $q(t)$  vanishes over the set  $\{t : t \geq \lambda_0\}$  and that  $q(0) \neq 0$ . It means that the polynomial  $q(t)$  is a non-null polynomial with infinitely many zeros, and it is clearly a contradiction. Thus, we can conclude that  $(\overline{\Gamma}_4^*)^\circ$  cannot be defined by a single polynomial inequality, and the theorem is proved.  $\square$

We feel that we have provided strong evidence in favour of our conjecture asserting that the almost-entropic region of order four is not semialgebraic. The big conjecture is that this region is not decidable. It means that it cannot exist an algorithm for the recognition of the almost-entropic vectors of order four. It is important to remark that the polar set of this region is semidecidable, which means that there exists an algorithm allowing one to recognise the elements of  $co\text{-}(\overline{\Gamma}_4^*)^\circ$ . The algorithm that we have in mind is a naive brute force algorithm, which searches the infinite set of all the rational valued four-tuples of jointly distributed random variables, looking for an entropic vector that violates the inequality encoded by the input vector. Take into account that the searched set is infinite. It means that the elements of  $(\overline{\Gamma}_4^*)^\circ$  give place to infinite computations, while the elements of  $co\text{-}(\overline{\Gamma}_4^*)^\circ$  give place to (possibly arbitrarily long, but) halting computations. Thus, if a vector like  $\mathcal{I} - \mathcal{J}_{\frac{1}{2}}^* + \mathcal{J}$  does not encode a sound information inequality, a counterexample can be found in finite time. This last remark suggests an experimental approach to deal with the problem of proving that for all  $\lambda > 0$  the vector  $\mathcal{I} - \mathcal{J}_{\frac{1}{2}}^* + \lambda \cdot \mathcal{J}$  is not an information inequality. We have tried such an experimental approach, and it must be said that the results obtained so far are somewhat frustrating: We could not find a single counterexample to the inequality encoded by  $\mathcal{I} - \mathcal{J}_{\frac{1}{2}}^* + \mathcal{J}$ , which is, in some sense, the easiest to refute element of  $R$ . Thus, the computational experiments carried out up to the date seems to indicate that  $R$  is not a forbidden ray, and that our hypothesis is false. It must

be clear that those experimental results are not conclusive. Those results cannot be used to conclude that  $R$  is not a forbidden ray, and they cannot be used to conclude that  $\mathcal{I} - \mathcal{J}_{\frac{1}{2}}^* + \mathcal{J}$  is not an information inequality: If our algorithm has not found a counterexample to  $\mathcal{I} - \mathcal{J}_{\frac{1}{2}}^* + \mathcal{J}$ , the reason could be that it has not been running for long enough time.

## 5 Concluding remarks

We know that the almost-entropic regions cannot be defined by finite lists of linear inequalities. We conjecture that those regions cannot be defined by finite lists of polynomial inequalities, and we believe that a proof of this fact will be found in short time. If we are right, we have that the almost-entropic regions cannot be defined in first order logic over the real closed field. Which is the next level of complexity to try? We think that the next question to be investigated is the decidability of the almost-entropic regions, we conjecture that the set of four-dimensional, integer valued almost-entropic vectors is not decidable.

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## Note

<sup>1</sup>We cannot use the same idea with the set  $\overline{\Gamma_3^*}$ , given that it is polyhedral, and hence semialgebraic