Transient analysis of a discrete-time infinite server queue with system disaster

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Abstract: This paper studies a discrete-time infinite server queue subject to system disaster. The exact time-dependent probabilities of the number of customers present in the system are obtained using generating functions, continued fractions and confluent hypergeometric functions. Further, the results are extended to obtain closed form expression for busy period distribution and steady state system size probabilities. Numerical illustrations are provided to visualise the effect of system size probabilities, both in steady state and transient state, and busy period distribution for different parameter values.

Keywords: discrete queue; time-dependent probability; generating functions; busy period; confluent hypergeometric functions; continued fractions.


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This paper is a revised and expanded version of a paper entitled ‘Transient analysis of a discrete-time infinite server queue with system disasters’ presented at International Conference on Mathematics and its Applications, University College of Engineering Villupuram, Villupuram, India, 15–17 December 2014.

1 Introduction

Queues subject to disaster induces negative customers that remove all the customers in the system (including the customer being processed) upon its arrival. A disaster is also called a catastrophe, mass exodus, or queue flushing (see Atencia and Moreno, 2004;
Chen and Renshaw, 1997; Wang et al., 2011). In many applications, it might be natural to assume the dependence between positive arrival and negative arrival. Queues with disasters are very close to the stochastic clearing systems and have found applications in computer communications and manufacturing systems (see Baumann and Sandmann, 2012; Jain and Sigman, 1996; Stidham, 1974).

The well known infinite server queues, denoted by \( M/M/\infty \) queues have been studied by Takacs (1980) (classic papers), Smith (1973) as well as Liu et al. (1990) (extensions of the shot noise approach), Glynn and Whitt (1991) (heavy-traffic results). They are often used to analyse manufacturing processes and to model phenomena in telecommunication networks. In the context of broadband integrated services digital networks based upon the asynchronous transfer mode (ATM), this system has been pointed out to be of interest while studying loop statistical multiplexing of data connections on an ATM network (Guillemin and Phincon, 1998).

In recent years, there is a growing interest in the analysis of discrete-time queues due to their applications in communication systems and other related areas. This paper studies a discrete-time infinite server queue subject to system disaster. The exact time-dependent probabilities of the number of customers present in the system are obtained using generating functions, continued fractions and confluent hypergeometric functions. Further, the results are extended to obtain closed form expression for busy period distribution and steady state system size probabilities. Numerical illustrations are provided to visualise the effect of system size probabilities, both in steady state and transient state, and busy period distribution for different parameter values.

2 Model description

We consider a discrete-time infinite server queueing model with system disaster where the time axis is divided into equal intervals called slots. All queueing activities such as arrivals, departures and disasters occur only at the slot boundaries. An early arrival system (EAS) (Hunter, 1983; Takagi, 1993) policy is assumed wherein arrivals occur just after the beginning of the slots and departures take place just before the end of the slots.

Let \( X_m \) be a discrete random variable denoting the number of customers in the system at the time epoch \( m \). Then

\[
\{ X_m : m = 0, 1, 2, \ldots \}
\]

is a discrete time Markov chain with state space \( \{0, 1, 2, \ldots\} \).

In the system, the arrivals occur according to a Bernoulli process with parameter \( \alpha_n \), \( n \geq 0 \) and service completions occur according to a geometric distribution with parameter \( \beta_n \), \( n \geq 0 \) where \( \beta c = \beta_0 \) during any time slot. We assume that the probability of more than one arrival and/or departure during a given slot is zero and the events in different slots are independent. When the system is not empty, disaster occurs according to a geometric distribution with rate \( \xi \). A disaster event would make the system instantly empty.
3 Transient and steady state system probabilities

In this section, we derive the time-dependent and steady state system probabilities in closed form for the infinite server queue described in Section 2.

Let \( P_m(n) = P(X_m = n | X_0 = 0) \), \( m, n = 0, 1, 2, \ldots \) be the probability that there are \( n \) customers in the system at time epoch \( m \) given that the system is empty at the initial epoch.

**Theorem 1:** The system size probabilities \( P_m(n) \) are given by

\[
P_m(n) = A(n, m) + \varepsilon \sum_{r=0}^{m-1} A(n, m-(r+1)), \quad \text{if } m-(r+1) \geq 0 \quad n = 0, 1, 2, \ldots
\]

where

\[
A(n, m) = \sum_{k=0}^{m} \left( -1 \right)^k \frac{\lambda^n}{k!} \sum_{l=0}^{n+k} \left( -1 \right)^l \frac{\mu^l}{l} (1-\xi-\beta)^m.
\]

**Proof:** The difference equations governing the model described in Section 2 are given as follows:

\[
P_{m+1}(0) = (1-\alpha_0) P_m(0) + \left[ \alpha_0 \beta_0 (1-\xi) \right] P_m(0) + \alpha_0 \mu P_m(0)
\]

\[
+ \beta_1 (1-\alpha_1)(1-\xi) P_m(1) + \xi \alpha_1 \beta_1 + \alpha_1 (1-\beta_1)
\]

\[
+ \beta_2 (1-\alpha_2) + (1-\alpha_2)(1-\beta_2) \sum_{n=1}^{\infty} P_{m}(n)
\]

\[
= \left[ 1-\alpha_0 (1-\beta_0)(1-\xi-\zeta) \right] P_m(0) \left[ \beta_1 (1-\alpha_1)(1-\xi) \right] P_m(1) + \zeta,
\]

\[
P_{m+1}(n) = \alpha_{n+1}(1-\beta_{n+1})(1-\zeta) P_m(n-1)
\]

\[
+ \left[ \alpha_n \beta_n (1-\xi) + (1-\alpha_n)(1-\beta_n)(1-\xi) \right] P_m(n)
\]

\[
+ \beta_{n+1}(1-\alpha_{n+1})(1-\xi) P_{m+1}(n), \quad n = 1, 2, 3, \ldots
\]

where we assume that the system is empty at \( m = 0 \).

If we assume that \( \lambda_n = \alpha_0 (1-\beta_0)(1-\xi) \) and \( \mu_n = \beta_0 (1-\alpha_0)(1-\xi) \), then

\[
P_{m+1}(0) = (1-\lambda_0 - \xi) P_m(0) + \mu_0 P_m(1) + \zeta,
\]

\[
P_{m+1}(n) = \lambda_{n+1} P_m(n-1) + (1-\lambda_n - \mu_n - \xi) P_m(n) + \mu_{n+1} P_{m+1}(n+1).
\]

Define the generating function as follows:

\[
G_z(n) = \sum_{m=0}^{\infty} P_m(n) z^m, \quad n = 0, 1, 2, 3, \ldots \mid z \mid < 1.
\]

On applying (3.7) in (3.5) and (3.6), we get

\[
G_z(0) \left( \frac{1}{z} - 1 + \lambda_0 + \xi \right) = \frac{P_0(0)}{z} + \mu_0 G_z(1) + \frac{\zeta}{1-z}.
\]
\[ G_z(n) \left( \frac{1}{z} - 1 + \lambda z + \mu z + \zeta \right) - \mu z \cdot G_z(n+1) - \lambda z \cdot G_z(n-1) = \frac{P_n(n)}{z}. \] (3.9)

For the sake of simplicity we take the transformation as \( s = \frac{1}{z} - 1 \), then the above system becomes

\[ G_z(0) = \frac{(s + 1) \left( 1 + \frac{\zeta}{s} \right)}{s + \lambda z + \mu z - \mu_G(1)} \left( \frac{G_z(1)}{G_z(0)} \right) \] (3.10)

and

\[ \frac{G_z(n)}{G_z(n-1)} = \frac{\lambda_{n+1}}{s + \lambda z + \mu z + \zeta} - \mu_{n+1} \frac{G_z(n+1)}{G_z(n)} \], \( n \geq 1 \).

The iteration of above equation yields the following continued fraction, for \( n = 1, 2, 3, \ldots \),

\[ \frac{G_z(n)}{G_z(n-1)} = \frac{\lambda_{n+1}}{s + \lambda z + \mu z + \zeta} - \mu_{n+1} \frac{\lambda_{n+1}}{s + \lambda z + \mu z + \zeta} - \mu_{n+1} \frac{\lambda_{n+1}}{s + \lambda z + \mu z + \zeta} - \ldots \] (3.11)

Using (3.11) in (3.10), we get

\[ G_z(0) = \frac{(s + 1) \left( 1 + \frac{\zeta}{s} \right)}{s + \lambda z + \mu z - \mu_G(1)} \left( \frac{\lambda_{n+1}}{s + \lambda z + \mu z + \zeta} - \mu_{n+1} \frac{\lambda_{n+1}}{s + \lambda z + \mu z + \zeta} - \mu_{n+1} \frac{\lambda_{n+1}}{s + \lambda z + \mu z + \zeta} - \ldots \right) \] (3.12)

where the notation for continued fraction is used as

\[ \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \ldots = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \ldots \]

Now, the above state-dependent discrete time queue can be reduced to infinite server queue by using the arrival and service rates

\[ \alpha_n = \frac{1}{2(1 - \zeta)} \left[ \lambda - \eta \mu + 1 + \zeta + \sqrt{\left( \lambda - [n \mu - (1 - \zeta)] \right)^2 - 4 \lambda (1 - \zeta)} \right] \],

\[ \beta_n = \frac{1}{2(1 - \zeta)} \left[ \lambda - \eta \mu + 1 + \zeta + \sqrt{\left( \lambda - [n \mu - (1 - \zeta)] \right)^2 - 4 \lambda (1 - \zeta)} \right] \]

in \( \lambda_n \) and \( \mu_n \) and we obtain the new arrival and departure rates as

\[ \lambda_n = \lambda, \quad n = 0, 1, 2, \ldots; \quad \mu_n = n \mu, \quad n = 1, 2, \ldots \]

and these rates coincide with infinite server queueing model.

Hence (3.12) becomes,
Transient analysis of a discrete-time infinite server queue

\[ G_s(0) = \frac{(s+1) \left(1 + \frac{\xi}{s}\right)}{s + \lambda + \xi} \frac{\mu_s}{s + \lambda + \mu + \xi - s + \lambda + 2\mu + \xi} - \cdots \]  

(3.13)

If we take the transformation \( s + \xi = y \) in the above equation and denote \( G_s(n) \) by \( G_s(n) \) then the above equation reduces to

\[ G_s(y) = \frac{(1 + y - \xi) \left(1 + \frac{\xi}{y - \xi}\right)}{\lambda + y - \xi} \frac{\lambda \mu}{\lambda + \mu + y - \lambda + 2\mu + y - \lambda + 3\mu + y} - \cdots \]

Using the identity from (Lorentzen and Waadeland, 1992),

\[ _{1}F_{1}(a; c; z) = \frac{(a)_{n}}{(c)_{n}} \frac{z^{n}}{n!} \]

then we have

\[ G_s(0) = \left(\frac{y - \xi + 1}{y - \xi}\right) \frac{\Lambda}{\mu} \left[ _{1}F_{1} \left(1, \frac{y}{\mu} + 1; \frac{\lambda}{\mu}\right) \right] \]  

(3.14)

Similarly making use of the identities from (Andrews, 1992),

\[ (c - a) _{1}F_{1}(a; c + 1; z) + a _{1}F_{1}(a + 1; c + 1; z) = c _{1}F_{1}(a; c; z) \]

\[ c _{1}F_{1}(a + 1; c; z) - c _{1}F_{1}(a; c; z) = z _{1}F_{1}(a + 1; c + 1; z) \]

\[ (c - a) _{1}F_{1}(a - 1; c; z) + (2a - c + z) _{1}F_{1}(a; c; z) = a _{1}F_{1}(a + 1; c; z) \]

we write (3.11) as,

\[ \frac{G_s(n)}{G_s(n-1)} = \left(\frac{\lambda}{y + n\mu}\right) \left[ _{1}F_{1} \left(n + 1, \frac{y}{\mu} + n + 1; \frac{\lambda}{\mu}\right) \right] \]

\[ = \left(\frac{\lambda}{y + n\mu}\right) \left[ _{1}F_{1} \left(n + 1, \frac{y}{\mu} + n + 1; \frac{\lambda}{\mu}\right) \right], \quad n \geq 1. \]

Iterating the above equation and applying (3.14), we obtain


Applying partial fraction in the above equation, we get

\[ G_y(n) = y \left( \frac{y - \xi + 1}{y - \xi} \right) \sum_{k=0}^{\infty} \frac{(-1)^k (n+k)!}{k!n!} \left( \frac{\lambda}{\mu} \right)^{n+k} \frac{Z^{n+k}}{k!} \prod_{l=0}^{n+k-1} (y + l\mu) \]

After some calculation, replacing \( y \) by \( \frac{1}{z} - 1 + \xi \) leads to

\[ G_z(n) = \frac{1 - z(1 - \xi)}{1 - z} \sum_{k=0}^{\infty} \frac{(-1)^k (n+k)!}{k!n!} \left( \frac{\lambda}{\mu} \right)^{n+k} \sum_{l=0}^{n+k} \frac{(-1)^l}{l} \sum_{m=0}^{\infty} (1 - \xi - l\mu)^m z^m. \]

Further simplification of \( G_z(n) \) yields

\[ G_z(n) = (1 + z\xi + z^2\xi + z^3\xi + \ldots) \sum_{m=0}^{\infty} A(n, m) z^m, \]

where \( A(n, m) \) are given by (3.2).

Comparing the coefficient of \( z^m \) on both sides of the above equation, we obtain the system size probabilities as in (3.1). Hence the theorem.

**Remark 1:** Since \( s = \frac{1}{z} - 1 \), using the final value theorem of \( z \)-transforms, we obtain the steady state probabilities \( \pi_n \) for \( n = 0, 1, 2, \ldots \), as

\[ \pi_n = \lim_{m \to \infty} P_m(n) = \lim_{s \to 0} sG_z(n) = \left( \frac{\lambda}{\mu} \right)^n \sum_{k=0}^{\infty} \frac{(n+k)!}{n!} \frac{(-\xi)^k}{k!} \frac{1}{\mu^{n+k}} \prod_{l=0}^{n+k-1} (\xi + l\mu) \]

**Remark 2:** The steady state probabilities of infinite server queue in the absence of disaster (i.e.) when \( \xi \to 0 \) is given by

\[ \pi_n = \frac{\left( \frac{\lambda}{\mu} \right)^n e^{-\lambda}}{n!}, \quad n = 0, 1, 2, \ldots \]
4 Busy period analysis

The busy period is the time period measured between the instant a customer arrives to an empty system until the instant a customer departs leaving behind an empty system. In the busy period analysis, we modify the actual model in Section 2 by making the state 0 as an absorbing state. Let $T$ be a random variable denoting the time until the process reaches the absorbing state 0, starting initially from state 1. That is, the variable $T$ denotes the duration of a busy period.

Let $B(m) = P(T = m), m = 1, 2, 3, \ldots$ denote the probability mass function of $T$.

**Theorem 2:** The busy period probability $B(m)$ is given by

$$B(m) = \mu q_m(1) + \zeta \left[1 - q_m(0)\right], \quad m = 1, 2, 3, \ldots,$$

where

$$q_m(n) = P\left[X_m = n, \left| X_0 = 1\right|\right]$$

with $X_m$ denoting the number of customers in the system at time $m$. In particular,

$$q_m(0) = \zeta \sum_{j=0}^{m-1} (1 - \zeta)^j + \mu \sum_{j=0}^{\infty} \sum_{i=0}^{m-1} (1 - \zeta - \mu)^{n-j}(1 - \zeta)^j$$

and

$$q_m(1) = \sum_{i=1}^{\infty} \lambda(1 - \zeta - \mu)^i$$

**Proof:** Clearly, $q_m(0) = P(T \leq m)$.

We derive the busy period probability $B(m)$ through intuitive arguments as follows: a busy period will come to end after $m$ time units [with probability $B(m)$] due to either one of the following situations:

1. at time $m$, a service completion takes place in the system with one customer [with probability $\mu q_m(1)$]
2. at time $m$, a disaster strikes when the system is non-empty [with probability $\zeta \sum_{m=1}^{\infty} q_m(n)$].

The function $q_m(n)$ satisfies the following difference equations:

$$q_{m+1}(0) = (1 - \zeta)q_m(0) + \mu q_m(1) + \zeta$$

$$q_{m+1}(1) = (1 - \lambda - \mu - \zeta)q_m(1) + 2\mu q_m(2)$$

$$q_{m+1}(n) = \lambda q_m(n-1) + (1 - \lambda - \mu - \zeta)q_m(n) + (n+1)\mu q_m(n+1),$$

$$n = 2, 3, 4, \ldots$$

If $G_z(n) = \sum_{m=0}^{\infty} q_m(n)z^n, \ |z| < 1$, then the above equations lead to,
\[ G_z(1) = \frac{1}{1 - z(1 - \mu - \xi)} \binom{2, 1}{\frac{1 - z(1 - \xi)}{\mu \xi} + 2, \frac{-\lambda}{\mu}} \binom{1}{\frac{1 - z(1 - \xi)}{\mu \xi} + 1, \frac{-\lambda}{\mu}} \quad (4.8) \]

and

\[ G_z(0) = \xi \sum_{m=0}^{\infty} (1 - \xi)^j \sum_{j=0}^{m} \mu \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} z^n (1 - \xi - l \mu)^{m - j} (1 - \xi)^j. \quad (4.9) \]

On comparing the coefficients of \( z^m \) in (4.8) and (4.9), we get the probabilities \( q_m(0) \) and \( q_m(1) \) as in (4.3) and (4.4), respectively. For \( m = 1, 2, 3, \ldots \),

\[ B(m) = P(T = m) = P(T \leq m) - P(T \leq m - 1) \]

\[ = q_m(0) - q_{m-1}(0), \quad m = 1, 2, 3, \ldots, \]

\[ = \mu q_m(1) + \xi [1 - q_m(0)]. \]

The last equation follows from (4.5). Hence the theorem.

**Remark 3:** The average busy period duration \( E[T] \) is given by

\[ E[T] = \sum_{m=1}^{\infty} mB(m). \quad (4.10) \]

### 5 Numerical illustrations

To get the insight of the foregoing discussion, we plot the graphs for busy period, system size probabilities for number of customers in the system (both in transient- and steady-state) for the varying values of \( \lambda, \mu \) and \( \xi \) from the analytical expressions derived in the previous sections.

**Figure 1** Busy period vs. time (m) for various values of \( \mu \) (see online version for colours)
In Figure 1, the busy period values decrease as the service rate $\mu$ increases. This is because larger values of $\mu$ represent more departures which in turn result in shorter busy periods. The busy period is stable when $m \geq 100$.

**Figure 2** Busy period vs. time ($m$) for various values of $\xi$ (see online version for colours)

In Figure 2, the busy period decreases when $\xi$ increases where $\xi = 0.001, 0.002, 0.003, 0.004, 0.005$ because large $\xi$ means the chance for occurrence of disaster is more which in turn results in shorter busy periods.

**Figure 3** Transient system size probabilities for $\lambda = 0.2, \mu = 0.3$ and $\xi = 0.1$ (see online version for colours)
Consider the model with arrival, service completion, and disaster probabilities $\lambda = 0.2$, $\mu = 0.3$ and $\xi = 0.1$, respectively. Using (3.1), the system size probabilities $P_m(n)$ for $n = 0, 1, 2, \ldots, 4$ are calculated and are plotted in Figure 3 for different values of $m$. For clarity sake, the $x$- and $y$-axes are adjusted to show times from 0 to 4 and probabilities from 0 to 1, respectively. Because the system starts with 0 customers at time 0, the graph of $P_m(0)$ decreases from 1 while the graphs of $P_m(n)$ for $n \geq 1$ increase from 0, and reach the steady-state around $m = 4$ time units. Note that for $n \geq 1$, the graphs of $P_m(n)$ remain zero until $m < n$ as we expected.

**Figure 4**  Steady state system size probabilities vs. $\lambda$ (see online version for colours)

**Figure 5**  Steady state system size probabilities vs. $\mu$ (see online version for colours)
Figures 4 and 5 represent the steady-state system size probabilities versus the arrival rate $\lambda$ and service rate $\mu$ respectively. As expected, when the arrival rates ($\lambda$) increases, $\pi_0$ values decreases whereas the increase in service rate ($\mu$) results the increment values of $\pi_0$. The other probabilities increase in the beginning values of $\lambda$ and $\mu$ and then decreases.

6 Conclusions

In this paper, explicit expressions for the time dependent system size probabilities of a discrete time infinite server queue subject to system disaster are derived in the form of finite summations using generating functions, continued fractions and confluent hypergeometric functions. Numerical computations are performed to visualise the effect of the closed form expressions obtained against varying values of the parameters without truncation errors.

References


