A new hybrid collocation method for solving nonlinear two-point boundary value problems

Razieh Delpasand

Department of Applied Mathematics,
Faculty of Mathematics and Computer,
Shahid Bahonar University of Kerman,
Kerman, 76169-14111, Iran

and

Faculty of Mathematics,
Department of Applied Mathematics,
Yazd University,
Yazd, 89195-74, Iran
Email: r.delpasand@yahoo.com

Mohammad Mehdi Hosseini*

Mahani Mathematical Research Center,
Shahid Bahonar University of Kerman,
Kerman, 76169-14111, Iran

and

Department of Applied Mathematics,
Faculty of Mathematics and Computer,
Shahid Bahonar University of Kerman,
Kerman, 76169-14111, Iran
Email: mhosseini@uk.ac.ir
*Corresponding author

Farid Mohammad Maalek Ghaini

Faculty of Mathematics,
Department of Applied Mathematics,
Yazd University,
Yazd, 89195-74, Iran
Email: maalek@yazd.ac.ir

Abstract: In this paper, numerical solution of boundary value problems (BVPs) of nonlinear ordinary differential equations (ODEs) by the collocation method is considered. Of course, to avoid solving systems of nonlinear algebraic equations resulting from the method, residual function of the boundary value problem is considered and an unconstrained optimisation model is suggested. Particle swarm optimisation (PSO) algorithm is used for solving the unconstrained optimisation problem. In addition, convergence properties of the Chebyshev expansion are studied. The scheme is tested on some interesting examples and the obtained results demonstrate reliability and efficiency of the proposed hybrid method.

Keywords: nonlinear boundary value problems; ordinary differential equations; collocation method; Chebyshev polynomials; particle swarm optimisation; convergence analysis.

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Biographical notes: Razieh Delpasand is working on spectral methods. She has also introduced a modified version of pseudospectral method based on optimisation metaheuristic algorithms for solving nonlinear differential equations. She got her BSc and MSc from Shiraz University and her PhD from Yazd University.

Mohammad Mehdi Hosseini is a well-known mathematician working on various numerical methods and is the author of more than 90 research papers in international journals. He has also introduced a modified version of spectral method for solving differential-algebric equations.

Farid Mohammad Maalek Ghaini is a Professor of Applied Mathematics in Yazd University. He got his BSc and MSc from Sharif University of Technology, and his PhD from the University of Science and Technology. His studies are in the fields such as numerical solution of integral, ordinary and partial differential equations.

1 Introduction

Ordinary differential equations (ODEs) appear in modelling various physical phenomena (Cheng et al., 2011), and depending on their conditions, they are classified under the two categories of initial value problems (IVPs) and boundary value problems (BVPs). Generally, it is extremely difficult to find analytical solutions for BVPs, and therefore some numerical methods must be utilised.

During the last decades, many numerical methods have been introduced to solve BVPs. Cuomo and Marasco (2008) proposed a numerical method to solve nonlinear two-point BVPs by using the finite difference method. In Sgura (2013), finite difference method was introduced for the numerical solution of BVPs with non-smooth coefficients. Also, Erdogan and Ozis (2011) proposed a new kind of finite difference method for special second order nonlinear two-point BVPs. Adomian decomposition method has been used for obtaining approximate solutions of BVPs (Ebaid, 2011). A collocation method based on B-Splines was introduced to solve nonlinear two-point BVPs in (Rashidinia and Ghasemi, 2011). Temimi and Ansari (2011) proposed an iterative method for solving nonlinear second order BVPs. Also, variational iteration method was introduced to obtain solutions of BVPs (Dumitru et al., 2013; Lu, 2007; Mohyud-Din et al., 2009; Zhang, 2009). Some works about quasi-linearisation method have been presented in (Ahmad et al., 2001; Vatsala and Melton, 2006). A numerical method based on uniform Haar wavelets was introduced for numerical solution of second order BVPs (Ul-Islam et al., 2010). In Zhang and Lin (2015) a numerical method based on a combination of the reproducing kernels and least square method was introduced for solving nonlinear BVPs. Homotopy analysis method (HAM) (Yildrim and Mohyud-Din, 2010], and homotopy perturbation method (Mohyud-Din et al., 2012; Sikander et al., 2017) used to solve BVPs. During the last years, some spectral methods have been proposed in order to solve BVPs numerically (Doha et al., 2011; Ezz-Eldien, 2018, 2019; Ezz-Eldien and Doha, 2019). To see more results, see Ezz-Eldien et al. (2018), Ezz-Eldien and El-Kalaawy (2018), Merdan et al. (2012), Merdan et al. (2013), Mohyud-Din et al. (2015), Shakeel et al. (2014), Ul-Hassan and Mohyud-Din (2016).

In this paper we consider the following two-point BVPs:

$$u'' = f(x,u,u'), \quad x \in [a,b] \tag{1}$$

$$G(u(a),u(b),u'(a),u'(b)) = 0, \tag{2}$$

where f and G are given nonlinear functions.

Here, for solving the above BVPs, the collocation method is applied by using the Chebyshev polynomials. Also, by considering residual function of BVP, an unconstrained optimisation problem is introduced and we use particle swarm optimisation (PSO) algorithm to select appropriate coefficients for the Chebyshev series expansion of the solution. Furthermore, convergence properties of the Chebyshev expansion are studied. Moreover, the efficiency of the proposed hybrid method is shown by some examples.

This paper is organised as follows: In Section 2, we discuss about the Chebyshev polynomials and their spectral accuracy in approximation theory. In Section 3, we review a summary of PSO algorithm. Then, in Section 4, the hybrid collocation method for solving nonlinear BVPs is introduced. In Section 5, we present the results of numerical experiments.

2 Orthogonal Chebyshev polynomials and their properties

Consider the first kind orthogonal Chebyshev polynomials $\{T_k\}_{k=0}^{\infty}$, which are eigenfunctions of the singular Sturm-Lioville problem:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{1-x^2}\,\frac{\mathrm{d}}{\mathrm{d}x}T_n(x)\right)+\frac{n^2}{\sqrt{1-x^2}}T_n(x)=0, \quad -1<x<1,\, n=0,1,\ldots \tag{3}$$

The Chebyshev polynomials are orthogonal with respect to the L_w^2 inner product on the interval $[-1,1]$ by the weight function $w(x)=1\big/\sqrt{1-x^2}$, i.e.:

$$\left\langle T_n(x),T_m(x)\right\rangle_w=\int_{-1}^{1}T_m(x)T_n(x)w(x)dx=\frac{\pi\gamma_m}{2}\delta_{mn}, \tag{4}$$

where δ_{mn} denotes the Kronecker delta and

$$\gamma_m = \begin{cases} 2 & m = 0 \\ 1 & m \geq 1 \end{cases}. \tag{5}$$

Let u be a real arbitrary function on $[-1,1]$, the Chebyshev expansion of which is given by:

$$u(x) = \sum_{n=0}^{\infty} a_n T_n(x), \tag{6}$$

where $a_n = \frac{2}{\pi\gamma_n}\langle T_n(x), u(x)\rangle_w$. If the infinite series in equation (6) is truncated, then it can be written as,

$$u(x) \simeq P_N u(x) = \sum_{n=0}^{N} a_n T_n(x),$$

where $N \in \mathbb{N} \cup \{0\}$.

Now, we investigate the convergence analysis of the Chebyshev expansion.

Theorem 2.1 (Mason and Handscome, 2003): *If u is continuous and either is of bounded variation or satisfies a Dini-Lipschitz condition on $[-1,1]$, then its Chebyshev series expansion is uniformly convergent.*

Theorem 2.2: *If the Chebyshev series expansion of a continuous function u be a uniformly convergent series, then the series converges to u in $L_w^2[-1,1]$.*

Proof: Suppose

$$g(x) = \sum_{n=0}^{\infty} a_n T_n(x), \tag{7}$$

where

$$a_n = \frac{2}{\pi\gamma_n}\langle T_n(x), u(x)\rangle_w.$$

By multiplying both sides of expansion (7) in $T_m(x)\big/\sqrt{1-x^2}$, where $m \in \{0,1,...\}$ and integration on $(-1,1)$ we have:

$$\langle T_m(x), g(x)\rangle_w = \int_{-1}^{1} g(x)\frac{T_m(x)}{\sqrt{1-x^2}}\mathrm{d}x = \int_{-1}^{1}\sum\nolimits_{n=0}^{\infty} a_n T_n(x)\frac{T_m(x)}{\sqrt{1-x^2}}\mathrm{d}x$$
$$= \sum\nolimits_{n=0}^{\infty} a_n \int_{-1}^{1} T_n(x)T_m(x)\frac{1}{\sqrt{1-x^2}}dx = a_m\frac{\pi\gamma_m}{2}.$$

So $a_m = (2/\pi\gamma_m)\langle T_m(x), g(x)\rangle_w$. It implies that u and g have the same expansion with respect to the Chebyshev polynomials, and so $u = g$. □

Theorem 2.3: *Let u be an arbitrary function on $[-1,1]$ with bounded second derivative, i.e., $|u''(x)| \le M$, then the Chebyshev series expansion of u is uniformly convergent to u, i.e.,. $\sum_{n=0}^{\infty} a_n T_n(x) = u(x)$. Moreover, we have $\|u - P_N u\| \le \sum_{n=N+1}^{\infty} \frac{2M}{n^2-1}$.*

Proof: By considering the Chebyshev expansion for $n = 0$, we have

$$a_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{u(x)T_0(x)}{\sqrt{1-x^2}}\mathrm{d}x, \text{ so } |a_0| = \left|\frac{1}{\pi}\int_{-1}^{1} \frac{u(x)T_0(x)}{\sqrt{1-x^2}}\mathrm{d}x\right| \le \frac{1}{\pi}\int_{-1}^{1} \frac{|u(x)|}{\sqrt{1-x^2}}\mathrm{d}x < \infty.$$

Also, for $n = 1$,

$$a_1 = \frac{2}{\pi}\int_{-1}^{1} \frac{u(x)T_1(x)}{\sqrt{1-x^2}}\mathrm{d}x, \text{ so } |a_1| = \left|\frac{2}{\pi}\int_{-1}^{1} \frac{u(x)T_1(x)}{\sqrt{1-x^2}}\mathrm{d}x\right| \le \frac{2}{\pi}\int_{-1}^{1} \frac{|xu(x)|}{\sqrt{1-x^2}}\mathrm{d}x < \infty.$$

Now for $n \in N$ and $n \ge 2$, we have

$$a_n = \frac{2}{\pi}\int_{-1}^{1} \frac{u(x)T_n(x)}{\sqrt{1-x^2}}\mathrm{d}x$$

by considering $x = \cos(\theta)$ we have

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} u(\cos(\theta))\cos(n\theta)\,\mathrm{d}\theta.$$

So by using integration by parts, we get:

$$\begin{aligned}
|a_n| &= \left|\frac{2}{\pi}\int_{0}^{\pi} u(\cos(\theta))\cos(n\theta)\,\mathrm{d}\theta\right| \\
&= \left|\frac{1}{n\pi}\int_{0}^{\pi} \frac{1}{1-n} u''(\cos(\theta))\sin((1-n)\theta)\sin(\theta)\mathrm{d}\theta - \frac{1}{n\pi}\int_{0}^{\pi} \frac{1}{1+n} u''(\cos(\theta))\sin((1+n)\theta)\sin(\theta)\mathrm{d}\theta\right| \\
&\le \left|\frac{1}{n\pi}\int_{0}^{\pi} \frac{1}{1-n} u''(\cos(\theta))\sin((1-n)\theta)\sin(\theta)\mathrm{d}\theta\right| + \left|\frac{1}{n\pi}\int_{0}^{\pi} \frac{1}{1+n} u''(\cos(\theta))\sin((1+n)\theta)\sin(\theta)\mathrm{d}\theta\right|,
\end{aligned}$$

since $|u''(x)| \le M$, we have:

$$|a_n| \le \frac{M}{n(n-1)} + \frac{M}{n(n+1)} = \frac{2M}{n^2-1},$$

and

$$\sum_{n=2}^{\infty} |a_n| \le \sum_{n=2}^{\infty} \frac{2M}{n^2-1} < \infty,$$

which shows that $\sum_{n=2}^{\infty} a_n$ is absolutely convergent. Also, we have:

$$|\sum_{n=2}^{\infty} a_n T_n(x)| \le \sum_{n=2}^{\infty} |a_n||T_n(x)| \le \sum_{n=2}^{\infty} |a_n| < \infty.$$

By considering theorem 2.2, the series $\sum_{n=0}^{\infty} a_n T_n(x)$ is convergent to $u(x)$. Moreover, we have:

$$\|u - P_N u\| = \max_{x\in[-1,1]} \left| \sum_{n=N+1}^{\infty} a_n T_n(x) \right| \leq \max_{x\in[-1,1]} \sum_{n=N+1}^{\infty} |a_n| \, |T_n(x)| \leq \sum_{n=N+1}^{\infty} |a_n| \leq \sum_{n=N+1}^{\infty} \frac{2M}{n^2 - 1} < \infty,$$

which completes the proof. □

Corollary: *If* u *be an arbitrary function on* $[-1,1]$, *with bounded kth derivative, i.e.,* $\left|u^{(k)}(x)\right| \leq M$, *where* $k \in \{2,3,\ldots\}$, *then the Chebyshev series expansion of* u *is uniformly convergent to* u *and* $|u(x) - P_N u(x)| = O(N^{-k})$ *for all* $x \in [-1,1]$.

It must be noted that if $u \in C^{\infty}[-1,1]$, then the truncation error $\|u - P_N u\|$ approaches zero faster than any negative power of the N number, as that N tends to infinity (Canuto et al., 1998). This phenomenon is usually referred to as 'spectral accuracy' (Gottlieb and Orzag, 1979).

In continue, the Chebyshev series expansion of the first and the second derivatives of a real function u are presented.

Theorem 2.4 (Canuto et al., 1998): *Suppose that* $u(x) = \sum_{n=0}^{\infty} a_n T_n(x)$, *then we have* $u'(x) = \sum_{n=0}^{\infty} a_n^{(1)} T_n(x)$ *and* $u''(x) = \sum_{n=0}^{\infty} a_n^{(2)} T_n(x)$, *where* $a_n^{(1)} = \frac{2}{\gamma_n} \sum_{\substack{p=n+1 \\ p+n \text{ odd}}}^{\infty} p a_p$ *and* $a_n^{(2)} = \frac{1}{\gamma_n} \sum_{\substack{p=n+2 \\ p+n \text{ even}}}^{\infty} p(p^2 - n^2) a_p$.

3 Particle swarm optimisation

PSO is a population based stochastic optimisation technique developed by Kennedy and Eberhart (1995), inspired by social behaviour of bird flocking or fish schooling (Eberhart and Kennedy, 1995; Kennedy and Eberhart, 1995, 2001).

In PSO, each single solution is a bird in the search space, which is called a particle. Each particle has a fitness value, which is evaluated by the fitness function to be optimised and velocities which directed the flying of the particles.

In PSO, each particle will change its position according to its personal experience and the experiences of the whole society. Social sharing information between particles has a series of evolutionary advantages, a hypothesis which is the basis of PSO algorithm.

PSO is initialised with a group of random particles or solutions. In every iteration, each particle needs its best fitness, which it has achieved so far. This value is called pbest. Also, another best value is needed which is the best value, obtained so far by any particle in the population. This best value is the global best value and is called gbest.

Suppose we have m particles and each particle is treated as a point in the D-dimensional searching space. We will show the position, velocity and the best position of ith particle in searching space, respectively, by $X_i = (X_{i1}, X_{i2}, ..., X_{iD})$, $V_i = (V_{i1}, V_{i2}, ..., V_{iD})$ and $P_i = (P_{i1}, P_{i2}, ..., P_{iD})$ for $i = 1, ..., m$, and the global position in searching space by $P_g = (P_{g1}, P_{g2}, ..., P_{gD})$.

Velocity and position of each particle is updated at each time step by the recursive relations:

$$V_i(t+1) = V_i(t) + c_1 r_1 (P_i - X_i(t)) + c_2 r_2 (P_g - X_i(t)), \tag{8}$$

and

$$X_i(t+1) = X_i(t) + V_i(t+1), \tag{9}$$

where c_1 and c_2 are learning factors, the recommended choice for both of them is 2 (Kennedy and Eberhart, 1995), r_1 and r_2 are two random numbers in $(0,1)$.

The pseudo code of the general PSO algorithm is as follows (Das et al., 2008):

Algorithm 1 The pseudo code of the general PSO algorithm

Generate an initial population by random position and velocity;
While the termination condition is not seen do
  For $i = 1$ to number of particles;
  Evaluate the fitness;
  Update P_i and P_g;
Update the velocity and position of each particle by (8) and (9);
  Increase i;
  End for
End while

4 Proposed hybrid collocation method and PSO algorithm for nonlinear BVPs

In this section, the implementation of the collocation method for nonlinear BVP (1) and (2) is presented.

The spectral methods for solving this class of equations is based on the expansion of the solution function u for (1) and (2) as a finite sum in terms of smooth basis functions, as below:

$$\tilde{u}(x) \approx \sum_{n=0}^{N} a_n \phi_n(x),$$

in which $\{\phi_i\}_i$ represents a family of orthogonal polynomials on $[a,b]$. In this paper, we consider the Chebyshev polynomials of the first kind on $[-1,1]$.

Now, we should compute the coefficients of the Chebyshev series expansion,

$$\tilde{u}(x) \approx \sum_{i=0}^{N} a_i T_i(x), \tag{10}$$

where $\{T_i\}_{i=0}^{N}$ are the Chebyshev polynomials as mentioned in Section 2. By substituting equation (10), respectively in equation (1) and its boundary conditions we define the residual function:

$$F(a_0, a_1, \ldots, a_N, x) = \sum_{i=0}^{N} a_i^{(2)} T_i(x) - f\left(x, \sum_{i=0}^{N} a_i T_i(x), \sum_{i=0}^{N} a_i^{(1)} T_i(x) \right). \tag{11}$$

Moreover, we assume:

$$G\left(\sum_{i=0}^{N} a_i T_i(a), \sum_{i=0}^{N} a_i T_i(b), \sum_{i=0}^{N} a_i^{(1)} T_i(a), \sum_{i=0}^{N} a_i^{(1)} T_i(b) \right) = 0. \tag{12}$$

In the standard collocation method, by considering the residual function (11), the boundary condition (12) and choosing $\{x_k\}_{k=0}^{N}$ as a set of collocation points, we try to obtain a nonlinear algebraic system with $N+1$ equations and $N+1$ unknown parameters (Hosseini, 2006a, 2006b). Since solving a nonlinear algebraic system is facing many problems including the choice of a suitable starting point, in this paper, we introduce a nonlinear unconstrained optimisation problem for finding the coefficients of the Chebyshev series expansion, as bellow.

For given large natural number M, we choose $\{x_k\}_{k=1}^{M}$ as a set of collocation points and define a general residual function by:

$$V = \sqrt{\frac{1}{M} \sum_{k=1}^{M} F^2(a_0, a_1, \ldots, a_N, x_k) + \left(G\left(\sum_{i=0}^{N} a_i T_i(a), \sum_{i=0}^{N} a_i T_i(b), \sum_{i=0}^{N} a_i^{(1)} T_i(a), \sum_{i=0}^{N} a_i^{(1)} T_i(b) \right) - 0 \right)^2}. \tag{13}$$

Now, according to equation (13), we define the nonlinear unconstrained optimisation problem:

$$\begin{aligned} &\min \ V, \\ &\text{s.t.,} \ \ a_i \in \mathbb{R} \end{aligned} \tag{14}$$

This optimisation problem is solved by using PSO algorithm, and we find the appropriate coefficients for the Chebyshev series.

5 Numerical results

In this section, we present some examples and use the proposed hybrid method to solve them.

In our study, we choose the initial population for PSO with 20 particles, and each particle is random numbers for coefficients in the Chebyshev series expansion. It should be noted that N is the number of basis functions and we set $M = 50$ and

$$x_k = \cos\left(\frac{k\pi}{M} \right), \ k = 0, 1, \ldots, 50,$$

also, we test the proposed method 20 times for all examples.

Example 1: Consider the following nonlinear Dirichlet BVP (Cuomo and Marasco, 2008):

$$y'' = -\cos y \sin y' + 2y + \cos(1-t^2)\sin(2t) - 2(t^2-1) + 2, \quad t \in [-1,1] \tag{15}$$

$$y(-1) = 0, \quad y(1) = 0, \tag{16}$$

We solve this example by the proposed hybrid method with $N = 2$, and reach $y(t) = t^2 - 1$, which is also the exact solution for this problem.

Example 2: Consider the following BVP (Cuomo and Marasco, 2008):

$$y'' = -\sin y' - \cos y + \cos(4t^2 - 1) - \sin(1 - 12t^2) + 24t, \quad t \in [-\frac{1}{2}, \frac{1}{2}] \tag{17}$$

$$y\left(-\frac{1}{2}\right) - y'\left(-\frac{1}{2}\right) = -2, \quad y\left(\frac{1}{2}\right) + y'\left(\frac{1}{2}\right) = 2. \tag{18}$$

We solve this problem with $N = 3$, and we obtain $y(t) = t(4t^2 - 1)$, which is the exact solution for this problem.

Example 3: Consider the following BVP (Cuomo and Marasco, 2008):

$$y'' = -(1 + \alpha^2 (y')^2), \quad t \in [0,1] \tag{19}$$

$$y(0) = 0, \quad y(1) = 0. \tag{20}$$

The exact solution of this problem is:

$$y(t) = \frac{\ln\left(\frac{\cos\alpha(t-0.5)}{\cos(0.5\alpha)}\right)}{\alpha^2}. \tag{21}$$

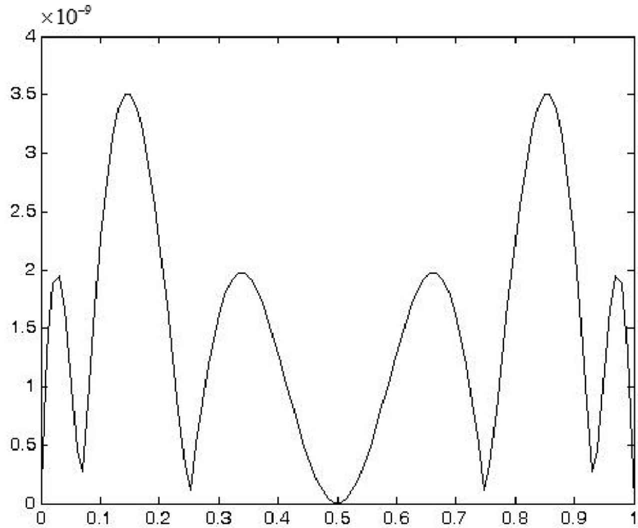
We consider $\alpha = 1/7$ to solve this problem. Absolute error of mean and the best solution for different number of basis function which have been obtained by the proposed method and the absolute error obtained by the standard colocation method are shown in Table 1.

Table 1 Results of absolute error for Example 3

N	*Absolute error of mean solution value*	*Absolute error of the best solution value*	*Absolute error of the standard collocation method*
4	1.17×10^{-5}	6.57×10^{-7}	5.42
8	1.06×10^{-6}	4.38×10^{-8}	1.13
12	6.46×10^{-8}	3.50×10^{-9}	5.11×10^{-3}

The graph of the absolute error of the best solution obtained by the proposed hybrid method for $N = 12$ is shown in Figure 1.

Figure 1 Absolute error of the best solution for $N = 12$, in example 3



Example 4: Consider the following BVP (Jain et al., 2016):

$$\left(\frac{1}{1+t}y'\right)' = 2e^{3y}, \quad t \in [0,1] \tag{22}$$

$$y(0) = 0, \quad y(1) = -\ln(2). \tag{23}$$

The exact solution is:

$$y(t) = \ln\left(\frac{1}{1+t}\right). \tag{24}$$

Absolute error of mean and the best solution for different number of basis function and the absolute error obtained by the standard collocation method are shown in Table 2. The graph of the absolute error of the best solution for $N = 7$ is shown in Figure 2.

Example 5: Consider the following BVP (Kumar, 2003; Zhang and Lin, 2015):

$$ty'' + y' + te^{y} = 0, \quad t \in (0,1) \tag{25}$$

$$y'(0) = 0, \quad y(1) = 0. \tag{26}$$

The exact solution is:

$$y(t) = 2\ln\left(\frac{4-2\sqrt{2}}{(3-2\sqrt{2})t^2+1}\right). \tag{27}$$

Absolute error of mean and the best solution for different number of basis function, the absolute error obtained by the standard collocation method and the absolute error of the method (Zhang and Lin, 2015) which was based on combination of the reproducing kernel method and least square method are shown in Table 3.

Table 2 Results of absolute error for Example 4

N	*Absolute error of mean solution value*	*Absolute error of the best solution value*	*Absolute error of the standard collocation method*
3	2.50×10^{-3}	1.41×10^{-3}	0.01
5	3.91×10^{-5}	1.43×10^{-5}	17.76
7	1.77×10^{-6}	3.79×10^{-7}	25.13

Figure 2 Absolute error of the best solution for $N = 7$, in example 4

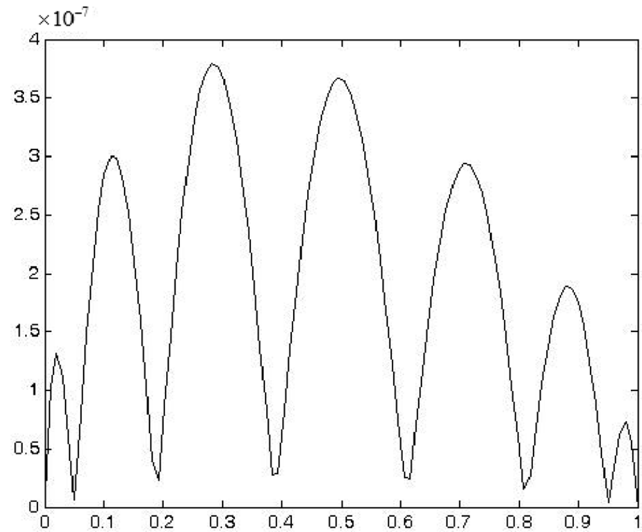


Table 3 Results of absolute error for Example 5

	Our method		*The standard collocation method*		*Method (Zhang and Lin, 2015)*	
N	*Absolute error of mean solution value*	*Absolute error of best solution value*	N	*Absolute error*	n	*Absolute error*
3	5.73×10^{-4}	3.95×10^{-4}	3	6.53	4	1.9×10^{-3}
5	7.58×10^{-6}	2.15×10^{-6}	5	3.55	32	1.4×10^{-5}
7	1.69×10^{-7}	2.09×10^{-9}	7	3.52	128	7.1×10^{-7}

The graph of the absolute error of the best solution which is obtained by the proposed hybrid method for $N = 7$ is shown in Figure 3.

Example 6: Consider the following BVP (Cuomo and Marasco, 2008; Handerson and Thompson, 2002; Thompson, 1996):

$$y'' = -\sin t - 2\cos(t - y^2)\sin y - (y')^5, \quad t \in [0,1] \tag{28}$$

$$y(0) - y'(0) + \frac{[y^2(1) - (y'(1))^2]}{10} = 0, \tag{29}$$

$$y(0)+y'(0)+6y(1)+y'(1)+\sin(y(0)-y'(1))=0. \tag{30}$$

The graphs of the approximate solutions obtained by the proposed algorithm for different number of basis functions for this example are shown in Figure 4.

Figure 3 Absolute error of the best solution for $N=7$, in example 5

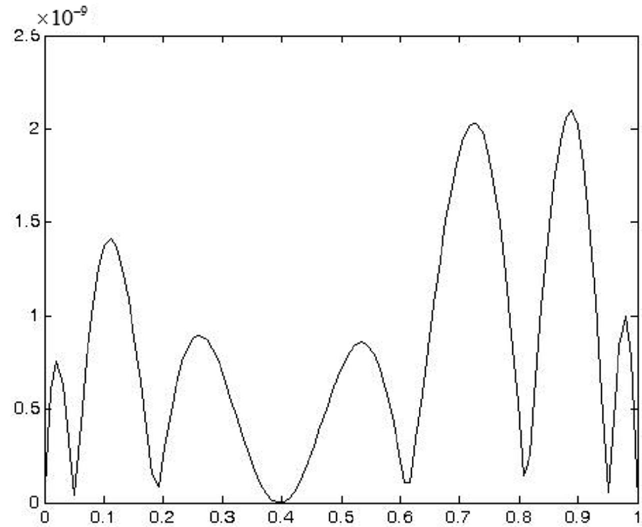
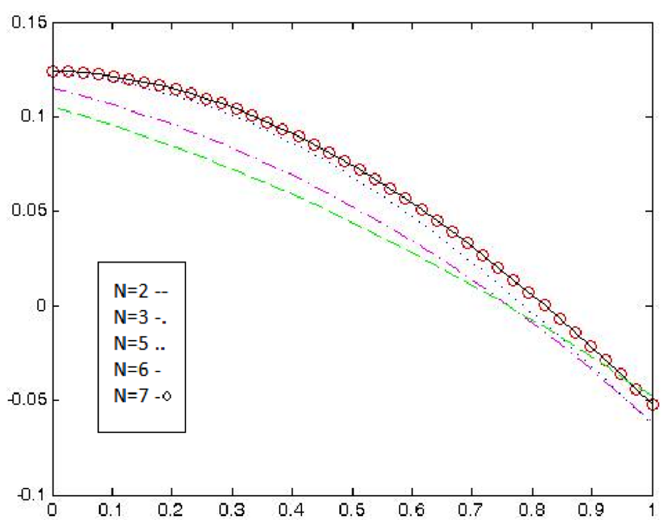


Figure 4 The approximate solution for different number of basis functions for example 6 (see online version for colours)



As can be seen, the approximate solutions which are obtained by the proposed method are convergent, specially the approximate solutions for $N=6$ and $N=7$ are coincided with each other.

6 Conclusion

It is well known that to solve nonlinear BVPs of ODEs by the collocation methods is generally difficult. To remove this difficulty, we defined a general residual function for the nonlinear BVP and then introduced an appropriate unconstrained optimisation model. Also, to solve this optimisation problem we used PSO algorithm. The merit of our hybrid method is its simplicity of implementation, such that there is no need to solve systems of nonlinear algebraic equations. Furthermore, the proposed method can be usually obtained spectral accuracy to solve nonlinear two-point boundary ODEs. To illustrate the efficiency of our method, some interesting examples have been solved. Numerical results show the high accuracy and efficiency of the proposed hybrid method.

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