Nonlinear Steklov eigenvalue problem with variable exponents and without Ambrosetti–Rabinowitz condition

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Abstract: In this paper, we study a nonlinear Steklov eigenvalue problem involving the $p(x)$-Laplacian on a bounded domain. We introduce a new variational technic that allows us to investigate this problem without need of the Ambrosetti–Rabinowitz condition on the nonlinearity.

Keywords: critical point; $p(x)$-Laplacian; Steklov problem; variable exponent Lebesgue–Sobolev spaces.


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1 Introduction and main result

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary $\partial \Omega$, $\nu$ is the outward unit normal vector on $\partial \Omega$, $p \in C_+ (\Omega) := \{ h \in C (\overline{\Omega}) / \min_{x \in \Omega} h(x) > 1 \}$. Motivated by the work that has studied the Dirichlet case in Benouhiba and Saker (2013), we consider the following Steklov problem

$$\begin{cases}
\Delta_{p(x)} u = (1 - \mu) |u|^{p(x)-2} u & \text{in } \Omega, \\
|\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} + \mu |u|^{p(x)-2} u = \lambda f(x, u) & \text{on } \partial \Omega
\end{cases} \quad (1.1)$$

where $\mu \in \{0, 1\}$, $\Delta_{p(x)} u = \text{div}(\nabla |u|^{p(x)-2} u)$ is the $p(x)$-Laplacian and $f : \partial \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a Carathéodory function fulfilling appropriate conditions. The considered eigenvalue problem involves variable exponent growth conditions. The study of such kind of equations is a very active field of investigation in the last decade since they can serve as models for different physical phenomena. We refer to Růžička (2002) for a model of partial differential equation with non standard growth in electrorheological fluids.

The main idea of this work is to study equation (1.1) in both cases $\mu = 0$ and $\mu = 1$ without assuming the Ambrosetti–Rabinowitz condition (Ambrosetti and Rabinowitz, 1973) on $f$, namely,

$$(\text{AR}) \; \exists M > 0, \tau > p^+ \text{ such that } 0 < \tau F(x, s) < f(x, s)s, |s| \geq M, x \in \partial \Omega,$$

where $f$ is a nonlinear term such that $F(x, t) := \int_0^t f(x, s)ds$ and $p^+ = \max_{x \in \Omega} p(x)$, since it is a restrictive condition eliminating many nonlinearities.

In Deng (2008), the problem (1.1) has been considered in the case where $\mu = 0$ and $f(x, u) = \lambda |u|^{q(x)}-2 u$. The author showed the existence of infinitely many positive eigenvalue sequences. While in Anane et al. (2014), we have studied the problem (1.1) in the case where $\mu = 0$ and $f(x, u) = \lambda |u|^{q(x)}-2 u$ with $p(x) \neq q(x)$.

The familiar approach to solve problem (1.1) is to search critical points of the functional

$$\Phi_{\lambda,0}(u) = \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} \, dx + \int_{\Omega} \frac{|u|^{p(x)}}{p(x)} \, dx - \lambda \int_{\partial \Omega} F(x, u) \, d\sigma,$$

in the case $\mu = 0$ and the functional

$$\Phi_{\lambda,1}(u) = \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} \, dx + \int_{\partial \Omega} \frac{|u|^{p(x)}}{p(x)} \, dx - \lambda \int_{\partial \Omega} F(x, u) \, d\sigma,$$

in the case $\mu = 1$, where $d\sigma$ is the $N - 1$ dimensional Hausdorff measure. In this method, some authors assume that the nonlinearity $f$ satisfies (AR), we cite for example
Karim et al. (2015). Others study the problem (1.1) without this condition in the superlinear case and with \( \mu = 0 \), we refer to Ayoujil (2014) and Zang (2008).

The author in Rother (1993) discovered a new functional whose critical point is a solution of the following equation

\[ \Delta_p = \lambda f(x,u), \quad \text{in } \Omega \]  

(1.2)

in the special case \( p = \text{constant} \) with Dirichlet boundary conditions. This case has been generalised in Benouhiba and Saker (2013), the authors use a more general functional and they get the existence of infinitely many solutions of problem (1.2) without assuming the (AR) condition.

Throughout this paper, we denote by \( h^+ := \max_{x \in \Omega} h(x), \ h^- := \min_{x \in \Omega} h(x) \) for any \( h \in C_+ (\Omega) \) and

\[ p^*_0(x) = \begin{cases} \frac{(N - 1)\mu(x)}{N - p(x)}, & \text{if } p(x) < N \\ \infty, & \text{if } p(x) \geq N \end{cases} \]

We enumerate now the hypotheses concerning the functions \( f \) and the variable exponents:

A1: \( p, q \in C_+ (\bar{\Omega}) \) and \( p(x) < q(x) < p^*_0(x) < +\infty \), for all \( x \in \bar{\Omega} \);

A2: there exists an open non empty \( \omega \subset \partial \Omega \), and \( 0 \leq \tau_1 \leq \tau_2 + \omega < +\infty \) such that

\[ f(x,t) > 0 \text{ on } \omega \times ]\tau_1, \tau_2[; \]

A3: there exists \( g(x) \in L^{s(x)} (\partial \Omega) \) such that

\[ f(x,t) \leq g(x)t^{q(x) - 1} \quad \text{for a.e. } x \in \partial \Omega \text{ and for all } t \in \mathbb{R}^+; \]

where the variable exponent \( s \in C_+ (\bar{\Omega}) \) and satisfies \( \frac{p^*_0(x)}{p^*_0(x) - q(x)} < s(x) \) for all \( x \in \bar{\Omega} \).

Example 1: we give an example of the functions \( p, q, f \) and \( g \), which satisfy the conditions (A1)–(A3), in the case where \( N = 2 \) and \( \Omega = [0,1] \times [0,1] \). For \( x = (x_1, x_2) \in \Omega \), we define \( p(x) = \frac{1}{10}(x_1 + x_2 + 11) \), thus \( p_0^*(x) = \frac{x_1 + x_2 + 11}{9 - (x_1 + x_2)} \).

We take \( q(x) = \frac{1}{4}(p(x) + p_0^*(x)), \ s(x) = \frac{p^*_0(x)}{p_0^*(x) - q(x)} + 1, \ g(x) = |x|^{1/s(x)} \) and \( f(x,t) = |x|^{1/s(x)} t^{q(x) - 1} \), where \(|.|\) is the usual norm of \( \mathbb{R}^2 \).

The following theorem is the main result in this work.

**Theorem 1.1:** Suppose that the assumptions (A1), (A2) and (A3) are fulfilled. Then the problem (1.1) has infinitely many positive eigenvalues.

This article is organised as follows. Section 1 contains an introduction and the main result. In Section 2, which has a preliminary character, we recall some important definitions and results of variable exponent Lebesgue and Sobolev spaces. The proof of our main theorem is given in Section 3.
2 Preliminaries

We recall in what follows some basic facts about the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$.

For $p \in C_{+}(\bar{\Omega})$, we introduce the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) := \left\{ u; u : \Omega \to \mathbb{R} \text{ measurable and } \int_{\Omega} |u|^{p(x)} \, dx < +\infty \right\},$$

endowed with the Luxemburg norm

$$|u|_{p(x)} := \inf \left\{ \alpha > 0; \int_{\Omega} \left| \frac{u(x)}{\alpha} \right|^{p(x)} \, dx \leq 1 \right\},$$

which is separable and reflexive Banach space (Kovácik and Rákosnik, 1991). Denote by

$$|u|_{p(x), \partial\Omega} := \inf \left\{ \alpha > 0; \int_{\partial\Omega} \left| \frac{u(x)}{\alpha} \right|^{p(x)} \, dx \leq 1 \right\},$$

the norm of $L^{p(x)}(\partial\Omega)$. Let us define the space

$$W^{1,p(x)}(\Omega) := \{ u \in L^{p(x)}(\Omega); |\nabla u| \in L^{p(x)}(\Omega) \},$$

equipped with the norm

$$\|u\| = \inf \left\{ \alpha > 0; \int_{\Omega} \left[ \left| \frac{\nabla u(x)}{\alpha} \right|^{p(x)} + \left| \frac{u(x)}{\alpha} \right|^{p(x)} \right] \, dx \leq 1 \right\}: \forall u \in W^{1,p(x)}(\Omega),$$

which is equivalent to the norm (Deng, 2009).

$$\|u\|_{1} = \inf \left\{ \alpha > 0; \int_{\Omega} \left| \frac{\nabla u(x)}{\alpha} \right|^{p(x)} \, dx + \int_{\partial\Omega} \left| \frac{u(x)}{\alpha} \right|^{p(x)} \, d\sigma \leq 1 \right\}: \forall u \in W^{1,p(x)}(\Omega).$$

**Proposition 2.1.** (Deng, 2008; Fan, 1996): If $q \in C_{+}(\bar{\Omega})$ and $q(x) < p^*_0(x)$ for any $x \in \bar{\Omega}$, then the embedding $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\partial\Omega)$ is compact and continuous.

**Proposition 2.2.** (Fan and Zhang, 2003; Fan and Zhao, 1998; Kovácik and Rákosnik, 1991): Hölder inequality holds, namely,

$$\int_{\Omega} |uv| \, dx \leq 2|u|_{p(x)}|v|_{p'(x)}, \forall u \in L^{p(x)}(\Omega), \forall v \in L^{p'(x)}(\Omega),$$

where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. 


An important role in manipulating the generalised Lebesgue–Sobolev spaces is played by the mapping $\rho$ and $\varrho$ defined by

$$
\rho(u) := \int_{\Omega} \left[ |\nabla u|^{p(x)} + |u|^{p(x)} \right] \, dx, \quad \forall u \in W^{1,p(x)}(\Omega).
$$

and

$$
\varrho(u) := \int_{\Omega} |\nabla u|^{p(x)} \, dx + \int_{\partial \Omega} |u|^{p(x)} \, d\sigma, \quad \forall u \in W^{1,p(x)}(\Omega).
$$

**Proposition 2.3.** (Fan and Han, 2003): For $u, u_k \in W^{1,p(x)}(\Omega); k = 1, 2, \ldots$, we have

1. $\|u\| \geq 1$ implies $\|u\| \leq \rho(u) \leq \|u\|$
2. $\|u\| \leq 1$ implies $\|u\| \geq \rho(u) \geq \|u\|$
3. $\|u_k\| \to 0$ if and only if $\rho(u_k) \to 0$
4. $\|u_k\| \to \infty$ if and only if $\rho(u_k) \to \infty$

**Proposition 2.4.** (Deng, 2009): For $u, u_k \in W^{1,p(x)}(\Omega); k = 1, 2, \ldots$, we have

1. $\|u\|_1 \geq 1$ implies $\|u\|_1 \leq \varrho(u) \leq \|u\|_1$
2. $\|u\|_1 \leq 1$ implies $\|u\|_1 \geq \varrho(u) \geq \|u\|_1$
3. $\|u_k\|_1 \to 0$ if and only if $\varrho(u_k) \to 0$
4. $\|u_k\|_1 \to \infty$ if and only if $\varrho(u_k) \to \infty$

**Remark 2.5:** Proposition 2.3 will be used in the proof of the main result in the case $\mu = 0$ while Proposition 2.4 will be devoted to the case $\mu = 1$.

### 3 Proof of main results

Consider at first the case $\mu = 0$. In this case the problem (1.1) becomes

$$
\begin{cases}
\Delta_{p(x)} u = |u|^{p(x)-2} u & \text{in } \Omega, \\
|\nabla u|^{p(x)-2} \frac{\partial u}{\partial n} = \lambda f(x,u) & \text{on } \partial \Omega
\end{cases}
$$

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (3.1) if there exists $u \in (W^{1,p(x)}(\Omega)) \setminus \{0\}$ such that

$$
\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx + \int_{\Omega} |u|^{p(x)-2} u v \, dx = \lambda \int_{\partial \Omega} f(x,u) v \, d\sigma,
$$

for any $v \in W^{1,p(x)}(\Omega)$. The eigenfunction $u$ is called weak solution of problem (3.1).
We denote by $E$ the set of nondecreasing $C^1(\mathbb{R}^+, \mathbb{R}^+)$ functions $\Phi$ that satisfy: there exists $c > 0$ such that

$$
\Phi(r) > cr^{s-} \quad \text{if } 0 < r < 1,
$$

$$
\Phi(r) > cr^{s+} \quad \text{if } r \geq 1.
$$

(3.3)

It is clear that $E \neq \emptyset$ since we can take $\Phi(r) = c[1 + (r + 1)^{s-} + (r + 1)^{s+}]$ for all $r \in \mathbb{R}^+$.

**Example 2:** In the case of Example 1, a simple framework shown that $p^- = \frac{13}{10}$, $p^+ = \frac{13}{7}$, $s^- = \frac{21}{3}$ and $s^+ = 21$. In this case, we can take $\Phi(r) = c(r + 1)^{\frac{21}{3}}[1 + (r + 1)^{21}]$.

We define on $E$ an equivalence relation $\sim$ by

$$
\forall \Phi_1, \Phi_2 \in E : \Phi_1 \sim \Phi_2 \iff \exists c > 0 : \Phi_1(r) = c\Phi_2(r), \forall r > 0.
$$

We denote by $\tilde{E}$ the elements of $E/\sim$. For any $\Phi \in \tilde{E}$, we define the functional $J_\Phi$ by

$$
J_\Phi : (W^{1,p(x)}(\Omega)) \setminus \{0\} \rightarrow \mathbb{R}^+
$$

$$
J_\Phi(u) = \frac{\int_{\partial \Omega} F(x,u)d\sigma}{\Phi(I(u))},
$$

where

$$
I(u) = \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)}dx + \int_{\Omega} \frac{|u|^{p(x)}}{p(x)}dx \quad \text{and} \quad F(x,s) = \int_0^s f(x,t)dt.
$$

**Lemma 3.1:** Assume that (A1) and (A3) hold. For all $\Phi \in \tilde{E}$, $J_\Phi$ is well defined and bounded from above in $(W^{1,p(x)}(\Omega)) \setminus \{0\}$.

**Proof:** For all $u \in (W^{1,p(x)}(\Omega)) \setminus \{0\}$, $I(u) \neq 0$. Thus by definition of $\Phi$, we have $\Phi(I(u)) \neq 0$. On the other hand, under (A3), the integral $\int_{\partial \Omega} F(x,u)d\sigma$ is well defined. Indeed, applying the Hölder inequality we get for any $u \in W^{1,p(x)}(\Omega)$

$$
\int_{\partial \Omega} F(x,u)d\sigma \leq \frac{2}{q^+} |g|_{s(x),\partial \Omega} |u|^{q(x)}|\nu(x),\partial \Omega|
$$

$$
= \frac{2}{q^-} |g|_{s(x),\partial \Omega} |u|^{q(x)}|\nu(x'),\partial \Omega|
$$

Thus

$$
\int_{\partial \Omega} F(x,u)d\sigma \leq \begin{cases}
\frac{2}{q^+} |g|_{s(x),\partial \Omega} |u|^{q(x)}|\nu(x),\partial \Omega| & \text{if } |u|^{q(x)}|\nu(x),\partial \Omega| > 1 \\
\frac{2}{q^-} |g|_{s(x),\partial \Omega} |u|^{q(x)}|\nu(x'),\partial \Omega| & \text{if } |u|^{q(x)}|\nu(x'),\partial \Omega| \leq 1
\end{cases}
$$
As \( \frac{p(x)}{p'(x) - q(x)} < s(x) \) for all \( x \in \Omega \), we have \( q(x) s'(x) < p(x) \). It follows, from Proposition 2.1, that there exists \( c > 0 \) such that for all \( u \in W^{1, p(x)}(\Omega) \)

\[
\int_{\partial \Omega} F(x, u) d\sigma \leq \begin{cases} \frac{2c}{q} |g|_{s(x), \partial \Omega} \| u \|^{s^+} & \text{if } |u|_{q(x) s'(x), \partial \Omega} > 1 \\ \frac{2c}{q} |g|_{s(x), \partial \Omega} \| u \|^{s^-} & \text{if } |u|_{q(x) s'(x), \partial \Omega} \leq 1 \end{cases}
\]

Hence there exists \( C > 0 \) such that for all \( u \in W^{1, p(x)}(\Omega) \)

\[
\int_{\partial \Omega} F(x, u) d\sigma \leq C (\| u \|^{s^+} + \| u \|^{s^-}). \tag{3.4}
\]

Finally, the functional \( J_\Phi \) is well defined. It remains to show the boundedness of \( J_\Phi \) to complete the proof. We distinguish two cases: if \( \| u \| < 1 \), then \( I(u) \leq \frac{1}{p} \rho(u) \leq \rho(u) < 1 \). It follows, from (3.3) and (3.4), that

\[
0 \leq J_\Phi \leq C \frac{\| u \|^{s^+} + \| u \|^{s^-}}{\Phi(I(u))}
\]

\[
\leq C \frac{\| u \|^{s^+} + \| u \|^{s^-}}{\frac{2}{5} (I(u))^{\frac{5}{2}}} + \frac{2}{5} (I(u))^{\frac{5}{2}}
\]

\[
\leq C (I(u))^{\frac{5}{2}} + \left( I(u) \right)^{\frac{5}{2}}
\]

\[
\leq C''
\]

since, by Proposition 2.3, we have \( \| u \|^{s^+} \leq \left( \rho(u) \right)^{\frac{s^+}{s^t}} \leq \left( p^+ \right)^{\frac{s^+}{s^t}} (I(u))^{\frac{s^+}{s^t}} = c(I(u))^{\frac{s^+}{s^t}} \).

We use the same argument for the case \( \| u \| \geq 1 \) to get the boundedness of \( J_\Phi \). \( \square \)

**Proposition 3.2:** Assume the hypotheses of Theorem 1.1 hold, then for all \( \Phi \in \hat{E} \), there exists \( u_\Phi \in (W^{1, p(x)}(\Omega)) \) \( \setminus \{0\} \) such that

\[
J_\Phi(u_\Phi) = \max_{u \in (W^{1, p(x)}(\Omega)) \setminus \{0\}} J_\Phi(u).
\]

**Proof:** By Lemma 3.1, the maximum in Proposition 3.2 is a well defined positive real number. Let \( \varphi \in C_0^\infty(\Omega) \) such that \( \text{supp} \varphi \subset \omega \) and \( \sup_{x \in \partial \Omega} \varphi(x) > \tau_1 \), where \( \tau_1 \) is the constant from assumption (A2). It is clear that under assumption (A2), we have

\[
F(x, t) > 0 \quad \text{holds on } \omega \times [\tau_1, +\infty[. \tag{3.5}
\]

According to (3.5), we see that \( F(x, \varphi(x)) > 0 \) for all \( x \in \omega \). Thus \( J_\Phi(\varphi) > 0 \).

Let \( (u_n) \subset (W^{1, p(x)}(\Omega)) \setminus \{0\} \) be an maximising sequence. We may assume that

\[
J_\Phi(\varphi) \leq J_\Phi(u_n) \leq \max_{u \in (W^{1, p(x)}(\Omega)) \setminus \{0\}} J_\Phi(u).
\]
Suppose, by contradiction, that \((u_n)\) is not bounded in \(W^{1,p(x)}(\Omega)\). So we can assume that 
\[ \|u_n\| \geq 1. \]

\[ J_\Phi(\varphi) \leq J_\Phi(u_n) \Leftrightarrow J_\Phi(\varphi)\Phi(I(u_n)) \leq \int_{\partial \Omega} F(x, u_n)d\sigma. \]

As \(\Phi(I(u_n)) \geq c(I(u_n)) \frac{s^+}{p^+} \geq c\|u_n\|^{s^+} \) and \(\int_{\partial \Omega} F(x, u_n)d\sigma \leq c'\|u_n\|^{q^+}\), we obtain

\[ J_\Phi(\varphi)\|u_n\|^{s^+} \leq c''\|u_n\|^{q^+}. \]

Since the constant \(s^+\) is greater than \(q^+\), it follows that \((u_n)\) is bounded. So we can find a subsequence still denoted by \((u_n)\) that converges weakly to some \(u_\Phi \in W^{1,p(x)}(\Omega)\).

By the Sobolev trace embedding (see Proposition 2.1), \((u_n)\) converges strongly in \(L^{s'(x)q'(x)}(\partial \Omega)\). It is known that under assumption \((A3)\), the Nemytskii operator \(F(x, \cdot)\) is continuous from \(L^{s'(x)q'(x)}(\partial \Omega)\) into \(L^{s'(x)}(\partial \Omega)\). Thus \(F(x, u_n)\) is strongly convergent in \(L^{s'(x)}(\partial \Omega)\). It follows that

\[ \int_{\partial \Omega} F(x, u_n)d\sigma \rightarrow \int_{\partial \Omega} F(x, u_\Phi)d\sigma. \]

Since \(\Phi\) is a nondecreasing continuous function and \(I\) is weakly lower semi-continuous, we get

\[ \Phi(I(u_\Phi)) \leq \lim \inf \Phi(I(u_n)). \]

Consequently, we have

\[
\max_{u \in (W^{1,p(x)}(\Omega))\setminus\{0\}} J_\Phi(u) \geq J_\Phi(u_\Phi) \\
\geq \lim \sup_{u \in (W^{1,p(x)}(\Omega))\setminus\{0\}} J_\Phi(u) \\
= \max_{u \in (W^{1,p(x)}(\Omega))\setminus\{0\}} J_\Phi(u).
\]

Finally, we conclude that \(J_\Phi\) achieves its maximum at \(u_\Phi\). We affirm that \(u_\Phi \neq 0\), indeed:
as \(J_\Phi(u_\Phi) > 0\), we deduce that \(\int_{\partial \Omega} F(x, u_\Phi)d\sigma > 0\) and then \(u_\Phi \neq 0\) since \(F(x, 0) = 0\).

\(\square\)

Proof of Theorem 1.1: Standard arguments show that for all \(\Phi \in \hat{E}\), the functional \(J_\Phi \in C^1(W^{1,p(x)}(\Omega), \mathbb{R})\). Furthermore, it yields that

\[ \langle J'_\Phi(u_\Phi), v \rangle = 0 \quad \text{for all } v \in W^{1,p(x)}(\Omega), \]

since by Proposition 2.1 the functional \(J_\Phi\) achieves its maximum at \(u_\Phi\). By simple calculation, we get, for all \(v \in W^{1,p(x)}(\Omega)\)

\[ \Phi'(I(u_\Phi))(I'(u_\Phi), v) \int_{\partial \Omega} F(x, u_\Phi)d\sigma = \Phi(I(u_\Phi)) \int_{\partial \Omega} f(x, u_\Phi)v d\sigma, \]
where

\[
\langle I'(u_\Phi), v \rangle = \int_\Omega |\nabla u_\Phi|^{p(x)-2} \nabla u_\Phi \nabla vdx + \int_{\partial \Omega} |u_\Phi|^{p(x)-2} u_\Phi v d\sigma.
\]

Thus

\[
\langle I'(u_\Phi), v \rangle = \Phi(I(u_\Phi)) \frac{\Phi'(I(u_\Phi))}{\Phi'(I(u_\Phi))} \int_{\partial \Omega} F(x, u_\Phi) d\sigma \int_{\partial \Omega} f(x, u_\Phi) v d\sigma.
\]

This is equivalent to

\[
\int_\Omega |\nabla u_\Phi|^{p(x)-2} \nabla u_\Phi \nabla vdx + \int_{\partial \Omega} |u_\Phi|^{p(x)-2} u_\Phi v d\sigma = \lambda \int_{\partial \Omega} F(x, u_\Phi) d\sigma.
\]

where

\[
\lambda_\Phi = \frac{\Phi(I(u_\Phi))}{\Phi'(I(u_\Phi))} \int_{\partial \Omega} F(x, u_\Phi) d\sigma.
\]

Since \( \Phi(I(u_\Phi)) > 0, \int_{\partial \Omega} F(x, u_\Phi) d\sigma > 0 \) and \( \Phi'(I(u_\Phi)) > 0 \), then \( \lambda_\Phi \) is well defined and \( \lambda_\Phi > 0 \). Thus \( u_\Phi \) is a weak solution of problem (3.1) and since \( u_\Phi \neq 0 \), \( \lambda_\Phi \) is a positive eigenvalue of problem (3.1).

We affirm that the application \( \Phi \rightarrow \lambda_\Phi \) is well defined. Indeed, let \( \Phi_1 = \Phi_2 \) then there exists \( c > 0 \) such that \( \phi_1(r) = c\phi_2(r) \), for all \( r \geq 0 \) where \( \phi_1 \in \Phi_1 \) and \( \phi_2 \in \Phi_2 \). So we have \( J_{\phi_1} = cJ_{\phi_2} \) and consequently \( J_{\phi_1} = cJ_{\phi_2} \). This signifies that \( J_{\phi_1} \) and \( J_{\phi_2} \) have the same critical points. So we conclude that \( \lambda_{\phi_1} = \lambda_{\phi_2} \). As \( \Phi \) is taken arbitrary in \( \hat{E} \), it yields that problem (3.1) has infinitely many positive eigenvalues. \( \square \)

Remark 3.3: As in the Dirichlet case, the invertibility of the application \( \Phi \rightarrow \lambda_\Phi \) is still an open problem. We do not know if all the solutions of Problem (1.1) can be considered as critical points of the functional \( J_\Phi \).

Remark 3.4: For the case \( \mu = 1 \), we follow the same approach to solve the following problem

\[
\begin{cases}
\Delta_{p(x)} u = 0 & \text{in } \Omega, \\
|\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} + |u|^{p(x)-2} u = \lambda f(x, u) & \text{on } \partial \Omega 
\end{cases}
\]

(3.6)

In this case, we take

\[
J_\Phi(u) = \frac{\int_{\partial \Omega} F(x, u) d\sigma}{\Phi(I_1(u))},
\]

where

\[
I_1(u) = \int_\Omega |\nabla u|^{p(x)} dx + \int_{\partial \Omega} |u|^{p(x)} dx.
\]

The importance of studying the problem (3.6) comes from the importance of the \( p(x) \)-harmonic functions in the applications.
References


