A stochastic lot sizing model with partial backordering and imperfect production processes

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Abstract: This research presents an inventory system with partial backordering and imperfect process with a stochastic numbers of products that are defective per order. This problem is known and there are some customers that do not wait for their orders to be fulfilled so a particular proportion of backordered items become lost sales. This paper considers both mentioned situations simultaneously while the number of defective items follows a uniform distribution and the proportion of backordering is constant. This condition is modelled. The cost function of this inventory model includes order cost, holding cost and two types of shortage costs, one of them is related to backordered items and the other one is related to lost sales. This paper also provides a solution method to obtain optimum values for the decision variables, order quantity and total shortage, then we derive the value of the total cost function according to the optimum values obtained for the decision variables. Finally, some numerical results and diagrams are provided to show how some parameters affect the values of the decision variables and the cost function.

Keywords: inventory; EOQ model; defective items; partial backordering.


Biographical notes: Ata Allah Taleizadeh is an Assistant Professor in the School of Industrial Engineering at the University of Tehran in Iran. He received his PhD in Industrial Engineering from Iran University of Science and Technology. Moreover, he received his BSc and MSc both in Industrial Engineering from Azad University of Qazvin and Iran University of Science and Technology, respectively. His research interest areas include inventory control and SCM, pricing and revenue optimisation and game theory. He has published several papers and chapter books in reputable journals and he serves as the editor/editorial board member for a number of international journals.

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1 Introduction

Inventory models are necessary to control demand and supply because they are not ever completely balanced in real world. With these models we can put the sources aside for uses in the future. Inventories can be in different levels of the supply chain from raw material to final products and it is common to all sectors such as industry, agriculture and business, among others. These models help the managers to minimise total inventory cost and maximise sales and responsiveness for their customers. There are many situations that should be considered to model the inventory system to balance demand and supply perfectly. One of these situations is that customers are separated in two groups; one group is the ones that their requirements are not critical and when the vendor does not provide their order quantity in the right time they would wait to fulfil their orders. On the other hand, there are some customers that do not want to be patient to fulfil their orders, or their requirement is critical and it should be accessible in the right time. This last group of customers does not wait for the vendor to fulfil their order and choose other vendor. Besides, in reality the quantity of products that customers order contains some defective products. In the following part we give a review on some articles that consider these two assumptions: backordering and imperfect items.

There are many researchers that work on inventory system and provide various models. Chikan (1990) analysed many inventory models. Also the analysis of inventory system provided by Hadley and Whitin (1963) is one of the important models of inventory system, EOQ/EPQ, that is given under different conditions in various articles until now. The primary model of inventory system is provided by Harris (1913). Although this inventory model has many assumptions that are not realistic, it is the foundation of many other models that consider one or more of those assumptions. For instance, the EOQ considerations of perfect process are not realistic in most sections so if managers ignore it in their inventory models they would not determine the level of supply and demand perfectly. Karlin (1958) was one of the first authors that worked on these assumptions. He worked on the inventory system regarding to the cost structure and provided newsvendor models with three single stages to determine the optimal policy for ordering while supply is random. Silver (1976) provided the EOQ model considering different conditions to improve the yield; one of these considerations is presence of imperfect products. Porteous (1986) worked on the impacts of defective products on the Harris EOQ model in a condition that a system becomes out-of-control. Rosenblatt and Lee (1986) considered producing with lots that are smaller while results of system are imperfect. They considered that the duration from controlled state and not controlled state has an exponential distribution and imperfect products are reworked. Later researchers consider this proportion of defective products as uncertain parameters and defined various functions for it. For instance Gerchak et al. (1988) considered the stochastic demand and variable yield in a problem with single period; then this model was developed as an n-period problem. Yano and Lee (1995) provided a review for lot sizing models with stochastic production or yields of procurement. Inderfurth (2004) provided a certain policy for optimal producing for a demand with uniform distribution and yield rate. Wright and Mehrez (1998) provided a review of the articles that work on the association between inventory and quality. The mentioned articles considered defective units as scrapped or reworked items. Compared to these works, Salemeh and Jaber (2000) developed EOQ model while defective units are sold with decreased price. Maddah and Jaber (2008) changed the formulation of expectation for annual gain that is provided by
Salemeh and Jaber (2000) because they believe it is not accurate and for calculating annual profit they used the theorem that is named renewal-reward. Yoo et al. (2009) provided the model including a sales return for customers that get imperfect items because of errors in inspection that are committed by the seller. Paknejad et al. (1995) provided a paper in which a model was given with continuous time and demand is constant while the yield is uncertain with uniform distribution. They give these models to make the rate of yield better and decrease the variability of yield. Also provided the models in them mean and variance of a mentioned uniform distribution and the relationship between them play a main role to improve the yield. Sarkar and Moon (2014) provide an article in which he has worked on quality improvement, reduction of setup cost, and variable backorder costs.

Other assumption that is not considered in EOQ and EPQ model is that customers are separated in two groups, one group is the ones that their requirements are not critical and in a situation that the vendor does not provide their order quantity in the right time they would wait to fulfil their order. On the other hand there are some customers that do not be patient to fulfil their orders or their requirement is critical and it should be accessible in the right time. This group of customers does not wait for the vendor to fulfil their order and choose other vendor. In this situation a proportion of backordered demand that is associated to the second group becomes lost demand. Fabryck and Banks (1976) provided the primary models related to this assumption on the primary EOQ model – backlogging or partial backlogging or backordering. Montgomery et al. (1973) were the first ones that extended the model for primary EOQ model with the situation that backordering are partial also a solving method is provided for this model. Mak (1987) provided a paper in which particular optimal policies for inventory system are given while the quantity of backordered is uncertain. There are some papers that their authors assumed that the percentage of backorders is based on the replenish time. For instance Abad (1996, 2000, 2001, 2003, 2008) supposed that in real world, more customers wait to fulfill their order if the waiting time is not long and provided five papers regarding to this assumption. Papachristos and Skouri (2000) considered partial backordering based on the replenish time. In addition, some authors considered that the percentage of backlogging is based on the backlog size. For instance, Padmanabhan and Vrat (1990, 1995), Ouyang et al. (2003), Chu and Chung (2004), and Dye et al. (2006) considered that the backordering probability has a negative relation with existed backlog size. Pentico and Drake (2009) proposed a method for modelling the deterministic EOQ in the situation with partial backordered items which leads to the presence of some equations which are more like the ones associated with the primary EOQ and its backordering development. Sarkar and Sarkar (2013) provided an extended inventory model with partial backlogging, variation of deterioration time; also they considered the demand as a parameter depending on stock. Taleizadeh (2014) improved an efficient order quantity model regarding to evaporating item and its related payments and partial backordering.


In this paper we consider that there are always some defective items in each order that is considered as random parameter with mean $\mu$ and variance $\nu$, also a particular fraction of shortage is lost sale and the rest of it is backordered. This paper is the first one that considers both these assumptions simultaneously: backordering and imperfect products.
Table 1  Brief literature review

<table>
<thead>
<tr>
<th>Type</th>
<th>Article</th>
<th>Remarks</th>
</tr>
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<tr>
<td>Models for defective products</td>
<td>Salemeh and Jaber (2000)</td>
<td>Consider decrease price for imperfect products</td>
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<tr>
<td></td>
<td>Huang (2002)</td>
<td>Seller – purchaser supply chain</td>
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<td></td>
<td>Chan et al. (2003)</td>
<td>Different scenarios to get rid of imperfect products</td>
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<td>Chang (2004)</td>
<td>Using triangular fuzzy number</td>
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<td>Tsou (2007)</td>
<td>Probable quality with normal distribution</td>
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<td></td>
<td>Konstantaras et al. (2007)</td>
<td>Imperfect products are reworked</td>
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<tr>
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<td>Wee et al. (2007)</td>
<td>Shortage with full backordered items</td>
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<tr>
<td></td>
<td>Eroglu and Ozdumir (2007)</td>
<td>Shortage with full backordered items</td>
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<td></td>
<td>Maddah and Jaber (2008)</td>
<td>Defectives are carried for more than one cycles</td>
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<tr>
<td></td>
<td>Tsou et al. (2009)</td>
<td>Imperfect products are reworked</td>
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<tr>
<td></td>
<td>Khan and Jaber (2009)</td>
<td>Supply chain with three rank and a window for proportion of imperfect products</td>
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<td></td>
<td>El-Kassar (2009)</td>
<td>Defective and perfect items have continuous demand</td>
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<td></td>
<td>Wahab and Jaber (2010)</td>
<td>various costs to hold a unit of defective and perfect products</td>
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<td>Sana (2011)</td>
<td>Three-rank chain while there is difference between shipment size</td>
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<td></td>
<td>Khan et al. (2010)</td>
<td>A survey of the development of a altered EOQ model for defective units</td>
</tr>
<tr>
<td>Models with partial backordering</td>
<td>Mak (1987)</td>
<td>Backordering rate is an uncertain parameter</td>
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<td></td>
<td>Padmanabthan and Vrat (1990, 1995)</td>
<td>Backordering rate depends on shortage quantity</td>
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<td></td>
<td>Abad (1996)</td>
<td>Backordering rate depends on replenishment cycle</td>
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<td></td>
<td>Ouyang et al. (2003)</td>
<td>Backordering rate depends on shortage quantity</td>
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<td></td>
<td>Chu and Chung (2004)</td>
<td>Backordering rate depends on replenishment cycle</td>
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<tr>
<td></td>
<td>Dye et al. (2006)</td>
<td>Backordering rate depends on replenishment cycle</td>
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<td>Pentico and Drake (2009)</td>
<td>A survey of the EOQ and EPQ models with partial backordering</td>
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<td></td>
<td>Sarkar and Sarkar (2013)</td>
<td>EOQ for deteriorating item with stock dependent demand rate</td>
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<td></td>
<td>Taleizadeh (2014)</td>
<td>EOQ with partial backordering and advanced payment</td>
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2 Problem definition

This paper considers that the customers are separated in two groups; one group is the ones that their requirements are not critical when the vendor does not provide their order quantity in the right time they would wait to fulfil their order. On the other hand, there are some customers that do not be patient to fulfill their orders, or their requirement is critical and it should be accessible in the right time. This group of customers does not
wait for the vendor to fulfil their order and choose other vendor. In this situation a proportion of backordered demand that is associated to the second group becomes lost sales so shortage cost consists of two types of penalty. One of them is related to backordered demands and another one is related to lost sales. Besides, in reality the quantity of products that customers order contains some defective products. Also this proportion is random and we consider a particular probability density function for it in this paper. The purpose of this paper is minimising the annual cost for customer regarding to both mentioned situations. This cost involves order cost, shortage cost and holding cost. We will obtain a certain formulation for two decision variables, optimum order quantity and optimum shortage. In the following part of this paper we model this condition. As it is mentioned, in each order a proportion of it is defective and this proportion is a continuous random variable with mean $\mu$ and variance $v$. Purchasers inspect all products one by one to realise that it is acceptable for them or not. The cost of inspection for each ordered product is on the vendor because in this situation vendor can attract purchaser to himself and the purchaser becomes a long term customer for him. Also it is possible that purchaser cannot fulfil all demand in the end of the cycle so according to the previous part he had two types of shortage and regarding to it two types of shortage cost. According to these conditions the objective function will be defined according to the notations provided below.

We consider the notations below for our parameters and variables:

- $D$: demand for products
- $Q$: order quantity
- $S$: number of backordered units
- $K$: cost for one ordering
- $I(t)$: level of inventory at time $t$
- $c_h$: cost of holding a unit in a year
- $c_b$: cost per backordered unit in a year
- $\lambda$: the fraction of non-defective products in each order $d \leq \lambda \leq 1$
- $f(\lambda)$: function of probability density for $\lambda$ with mean, $\mu$, and variance, $v$
- $T = \frac{y}{D}$: cycle time, $y = \lambda Q$
- $E(.)$: expected value
- $c(Q, s)$: expected total cost in a year
- $p$: purchasing cost for a unit
- $r$: selling price for a unit
- $\omega_0$: cost of a backordered item with dependence of time
- $\omega$: cost of a backordered item in a unit of time
- $\pi_0$: cost of penalty for a unit that is lost, excluding the lost profit and independent of time
cost of penalty for a unit that is lost, dependent of time

proportion of backordered shortage

average cost for shortage with independency of time that is a convex function of \( \omega_0 \) and \( \pi_0 + r - p \) with including the lost profit

average cost for shortage that is a convex function of \( \omega \) and \( \pi \), dependent of time.

Considering this condition can be very useful in many industries. For example, consider a dairy product seller that gets particular amount of different types of dairy products every morning. These products are transported from their manufactory to this vendor and it is very possible that they are damaged during this time that are distributed between different sellers for instance their package can be hurt or lose their standard shape so these ones are not acceptable for vendors. Besides, it is possible that during packaging these products in their manufactory workers that have this job do not be careful to do their duty and packaging of some products are not done perfectly so this is the other reason that can make the products imperfect. Accordingly, random numbers of products are defective in every order that is given to the vendor. In addition to this situation that are related to products, there are some customers that do not patient to wait for their order for example, when they see there is a long queue for the products they prefer to leave the store or when their requirements are not critical they decide to buy their demand other days so proportion of shortage becomes lost sales because of these customers.

3 Mathematical modelling

Instead of full backordering we have partial backordering because of some inpatient customers that choose other purchaser so shortage cost converts to two shortages cost that are related to lost demands and backordered demands. According to this diagram, holding cost that is computed according to the blue shaded area of Figure 1 equals to:

\[
\text{Holding cost} = \frac{\left( y - s \right)^2}{2D} c_h \quad (1)
\]

Also shortage cost in one period is computed according the orange and purple shaded areas of the diagram that is shown in Figure 1 and the formulations that are proposed for \( \zeta_0 \) and \( \xi \), so we have:

\[
\text{Shortage cost} = \zeta_0 s + \xi \frac{s^2}{2D} \quad (2)
\]

So the total cost of a period equals to cost of holding a unit + shortage cost + ordered cost and the formulation below is proposed for it:

\[
c(y, s) = k + \frac{\left( y - s \right)^2}{2D} c_h + \frac{\zeta_0 s + \xi \frac{s^2}{2D}}{2D} \quad (3)
\]
Now we obtain the expected average cost per cycle regarding to $\lambda$ that is from 1 to $d$, so we integrate the total cost formulation as proposed below:

$$E(c) = \int_d^1 \left[ k + \frac{(\lambda Q - s)^2}{2D} c_h + \xi_0s + \xi s^2 \right] f(\lambda) \, d\lambda$$

$$= k + \int_d^1 \frac{(\lambda Q - s)^2}{2D} c_h f(\lambda) \, d\lambda + \xi_0s + \xi s^2$$

So we have:

$$\int_1^d \frac{(\lambda Q - s)^2}{2D} c_h f(\lambda) \, d\lambda = \int_1^d \frac{(v + \mu^2)Q^2 + s^2}{2D} c_h f(\lambda) \, d\lambda = \left[ \frac{(v + \mu^2)Q^2 + s^2}{2D} - \frac{\mu sQ}{D} \right] c_h$$

$$E(c) = \left[ \frac{(v + \mu^2)Q^2 + s^2}{2D} - \frac{\mu sQ}{D} \right] c_h + \xi_0s + \xi s^2$$

Also we have:

$$T = \frac{Q}{D} = \frac{\lambda Q}{D}$$

The cycle expected length is provided by

$$E(T) = \frac{\mu Q}{D}$$

We divide (8) by (10) to provide the expectation of total cost in a year:

$$c(Q, s) = \frac{kD}{\mu Q} + \frac{(v + \mu^2)Q}{2\mu} c_h + \frac{s^2 (c_h + \xi)}{2\mu Q} + \xi_0 \frac{sD}{\mu Q} - sc_h$$

In the following part we will compute the optimum order quantity $Q^*$, optimum shortage $s^*$ and optimum expected total annual cost $c(Q, s)$ according to above formulation.
4 Solution method

If in the cost formulation $c(Q, s)$ the decision variable $s$ is considered as constant, we obtain the univariate function in $Q$ which is similar to the cost function of EOQ model provided by Harris. So, it is convex, (see appendix A) and we can obtain the minimum quantity for $Q$ as below:

$$\frac{\partial c}{\partial Q} = -\left(\frac{kD}{\mu} + \frac{s^2(c_h + \xi)}{2\mu} + \frac{SD}{\mu} - sc_h + \frac{(v + \mu^2)}{2\mu} c_h = 0 \right)$$

(12)

So,

$$Q^* = \sqrt{\frac{2Dk + s^2(c_h + \xi) + 2\xi\xi_0 D}{(v + \mu^2)c_h}}$$

(13)

By substituting $Q^*$ in the $c(Q, s)$ so we have

$$c(Q^*, s) = \sqrt{\left(v + \mu^2\right)/\mu^2} \sqrt{2Dk_c + 2\xi\xi_0 Dc_h + (c_h + \xi)^2 c_h} - sc_h$$

(14)

It is obviously that $Q^* > s$ so

$$c(Q^*, s) = \sqrt{\left(v + \mu^2\right)/\mu^2} c_h \left(Q^* - \frac{s}{\sqrt{(v + \mu^2)/\mu^2}}\right) > 0$$

(15)
Lemma 1: Consider $s$ as decision variable so,

1.1 The function $c^2(Q, s)$ is continuous.

1.2 $c(Q^*, 0) = \sqrt{2kDc_h}\sqrt{\left(\frac{v + \mu^2}{\mu^2}\right)}$ (16)

1.3 $\lim_{{s \to \infty}} c(Q^*, s) = \infty$

Proof: See Appendix B.

1.4 $\frac{\partial c(Q^*, s)}{\partial s} = \frac{\varphi(s)}{\delta(s)} \left[ \sqrt{\left(\frac{v + \mu^2}{\mu^2}\right)} \left( c_h + \xi \right) + \sqrt{\left(\frac{v + \mu^2}{\mu^2}\right)} c_h + D + \delta(s) \right]$ (17)

That we have:

$$\varphi(s) = \left[ \frac{v + \mu^2}{\mu^2} \left( c_h + \xi \right)^2 - c_h \left( c_h + \xi \right) \right] s^2 + \left[ 2 \frac{v + \mu^2}{\mu^2} c_h \left( c_h + \xi \right) - 2c_hD + \xi \right] s + \frac{v + \mu^2}{\mu^2} c_h^2D^2 - 2Dck_h$$ (18)

and

$$\delta(s) = \sqrt{2Dck_h + 2c_hD + \left( c_h + \xi \right)s^2c_h}$$ (19)

Proof: See Appendix C.

Since $\delta(s) > 0$, we have $\text{sign} \left( \frac{dc(Q^*, s)}{ds} \right) = \text{sign}(\varphi(s))$. We should provide an answer for equation $\varphi(s) = 0$ for finding $s^*$, next we have to compute $Q^*$ from (13) and compute $c(Q^*, s^*)$ from (14). Function $\varphi(s)$ is a quadratic function. As we know we can obtain $s^*$ according to the coefficients of $s$ and $s^2$ in the function and below formulation:

$$\Delta = \left[ 2 \frac{v + \mu^2}{\mu^2} c_h \left( c_h + \xi \right) - 2c_hD + \xi \right]^2 - 4 \left[ \frac{v + \mu^2}{\mu^2} \left( c_h + \xi \right)^2 - c_h \left( c_h + \xi \right) \right] \left[ \frac{v + \mu^2}{\mu^2} c_h^2D^2 - 2Dck_h \right]$$ (20)

Observation 1: According to different quantity of $\Delta$ and $\zeta$ the optimum quantity of $s$ is obtained in following ways:

1. If $\Delta > 0$ then the function $c(Q^*, s^*)$ is an ascending function so it attains its minimum at minimum quantity of $s$ ($s^* = 0$) with value $\sqrt{2Dck_h}\sqrt{\left(\frac{v + \mu^2}{\mu^2}\right)}$. 

83
2 If $\Delta = 0$ and $\xi > 0$ then the function $c(Q^*, s^*)$ is a constant function and achieves its optimum quantity at $s^* = 0$ with value $\sqrt{2Dk_c} \sqrt{(v + \mu^2) / \mu^2}$.

3 If $\Delta = 0$ and $\xi = 0$ then the function $c(Q^*, s^*)$ is a constant function and achieves its optimum quantity at $s^* = 0$ with value $\sqrt{2Dk_c} \sqrt{(v + \mu^2) / \mu^2}$.

4 If $\Delta < 0$ and $\xi > 0$ so function $c(Q^*, s^*)$ obtains its optimum quantity at:

$$s^* = \frac{-2 \left[ (v + \mu^2) / \mu^2 \right] \xi_0 D (c_h + \xi) - 2 \xi_0 Dc_h}{2 \left[ (v + \mu^2) / \mu^2 \right] (c_h + \xi)^2 - c_h (c_h + \xi)}$$

or

$$s^* = \frac{-2 \left[ (v + \mu^2) / \mu^2 \right] \xi_0 D (c_h + \xi) - 2 \xi_0 Dc_h}{2 \left[ (v + \mu^2) / \mu^2 \right] (c_h + \xi)^2 - c_h (c_h + \xi)}$$

The second quantity is negative so the first formulation when it is positive or zero is the optimum quantity for $s$, also if its quantity is negative according to the first formulation we consider zero for its optimum quantity. Accordingly we have:

$$s^* = \frac{2 \xi_0 Dc_h - 2 \left[ (v + \mu^2) / \mu^2 \right] \xi_0 D (c_h + \xi) + 4v^2 \frac{4v^2}{\mu^2} c_h^2 + \frac{8v}{\mu^2} Dk_c - \frac{4v^2}{\mu^2} c_h^2}{2 \left[ (v + \mu^2) / \mu^2 \right] - c_h (c_h + \xi)}$$

That is obtained of simplifying the first formulation. Also we compute $Q^*$ according to the optimum quantity for $s$:

$$Q^* = \frac{A + B}{\sqrt{c_h (c_h + \xi) (v + \mu^2) (\xi_0^2 + c_h \mu + \xi_0 v)^2}}$$

where,

$$A = c_h^3 \left( 4kD \xi_0^2 v + (1 + 6kD \xi - \xi_0^2 D^2) v^2 \right) - v^3 \left( 2 \xi_0^2 D^2 \mu^2 \xi^2 + \xi_0^2 D^2 \mu^2 - 4kD \mu^3 \xi^3 \right)$$

$$B = c_h^3 \left( 98kD \xi_0^2 v + 2kD \mu^2 - 2 \xi_0^2 D^2 \mu^2 \xi^2 \right)$$

By substituting $Q^*$ and $s^*$ we have,
A stochastic lot sizing model with partial backordering

\[
c(Q^*, s^*) = \sqrt{\left(\frac{v+\mu^2}{\mu^2} \right)^2 \left(\frac{2Dk\xi + c_h (c_h + \xi)}{2c_h (c_h + \xi) - 4(c_h + \xi)^2} + \frac{v+\mu^2}{\mu^2} \right)} - 2Dc_\xi E + 2c_h (c_h + \xi) - \\
\sqrt{\left(\frac{2(c_h + \xi)^2}{\mu^2} + c_h E\right) - 2c_h (c_h + \xi) \left(\frac{v+\mu^2}{\mu^2}\right)}
\]

(27)

\[
E = \frac{\sqrt{c_h (8kD\mu^2 v + 4c_h v^2) - \xi_0^2 D (4v\mu^2 + 4v^2)} + 2Dc_\xi \xi_0 - 2D(c_h + \xi) \left(\frac{v+\mu^2}{\mu^2}\right)}}{\mu^4}
\]

(28)

If \(\Delta < 0\) and \(\xi = 0\) then we calculate \(c(Q^*, s^*)\) from the above formulation by substituting 0 instead of \(\xi\).

In the following part we provide numerical result to illustrate how the solution method works.

5 Numerical result and sensitivity analysis

In this section we consider a particular case to conduct several experiments and we obtain some numerical results according to its data. This case is a diary store that is explained in detail in Section 2. As mentioned before this store has both mentioned situations regarding to these factors that per day it gets its order and random number of this order is imperfect, also there are some customers that do not be patient to wait for receiving their demand so they leave the queue of dairy store. These numerical results are shown in three tables according to different quantities that are related to three parameters – \(\rho\), \(\mu\) and \(v\) – used to conduct the sensitivity analysis, and to illustrate how the solution method works. We consider \(\mu = 0.7\), \(V = 0.01\), \(D = 200\) items per year, \(k = \$500\) per order, \(c_h = $3\) per unit per year, \(\omega = $1\) per unit per year and \(m_0 + s - p = $2\) per unit, \(\omega_0 = $1.5\) per unit and \(\pi = $0\) per unit per year. In Table 2 according to quantity of \(\rho\) that is consider for it, the quantity for \(\xi_0\) and \(\xi\) are determined. Also in all cases the condition for convexity is fulfilled.

### Table 2
The results of sensitivity analysis for the case with different quantity for \(\rho\)

<table>
<thead>
<tr>
<th>(\rho)</th>
<th>(\xi_0)</th>
<th>(\xi)</th>
<th>(\Delta)</th>
<th>(s^*)</th>
<th>(Q^*)</th>
<th>(c(Q^<em>, s^</em>))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>1.00</td>
<td>0.00</td>
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<td>0</td>
<td>298.1424</td>
<td>798.5957</td>
</tr>
<tr>
<td>0.8</td>
<td>0.80</td>
<td>0.80</td>
<td>-5.7388e+05</td>
<td>0</td>
<td>298.1424</td>
<td>798.5957</td>
</tr>
<tr>
<td>1.00</td>
<td>0.75</td>
<td>1.00</td>
<td>-5.7704e+05</td>
<td>0</td>
<td>298.1424</td>
<td>798.5957</td>
</tr>
</tbody>
</table>
As we observe, in Table 1 by increasing $\rho$, $\xi_0$ decrease and $\xi$ increase, for each quantity we obtain $\Delta$ and according to $\Delta$ and $\xi$ we obtain optimum quantity for decision variables $s$ and $Q$. For instance in number 1, 2 and 3, $\Delta$ is negative and the optimum $s$ is positive, so according to the fourth case in the solution method we obtain optimum $Q$ and optimum $c$. On the other hand, in numbers 4, 5 and 6 though $\Delta$ is negative, $s$ is obtained with the negative quantity so we have the fifth case in the solution method and we consider $s$ as zero and the optimum decision variable $Q$ and optimum cost $c$ are obtained and all these quantities are the same in all numbers according to their formula. We show the effect of $\rho$ on $s^*$, $Q^*$ and $c(Q^*, s^*)$ in Figures 2 to 4.

**Figure 2** Impact of $\rho$ on $s^*$ (see online version for colours)

![Figure 2](image1.png)

**Figure 3** Impact of $\rho$ on $Q^*$ (see online version for colours)

![Figure 3](image2.png)

**Figure 4** Impact of $\rho$ on $c(Q^*, s^*)$ (see online version for colours)

![Figure 4](image3.png)
Table 3 is provided according to different quantity for $\mu$ and all the other parameters are the same as the previous table. In this table all numbers have the fulfilled conditions for convexity and $\Delta$ is negative and $s$ is positive so the optimum quantity for decision variables and cost function is obtained according to the fourth case mentioned in the solution method. As $\mu$ is increased both $s^*$ and $Q^*$ are decreased. On the other hand, as we observe in figure 8, cost is decreased at first, then it is increased. We show the effects of $\mu$ on $s^*$, $Q^*$ and $c(Q^*, s^*)$ in Figures 5 to 7.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\Delta$</th>
<th>$s^*$</th>
<th>$Q^*$</th>
<th>$c(Q^<em>, s^</em>)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>-5.6092e+05</td>
<td>93.7456</td>
<td>1230.0928</td>
<td>724.5342</td>
</tr>
<tr>
<td>0.4</td>
<td>-5.6164e+05</td>
<td>86.9979</td>
<td>994.5620</td>
<td>696.3965</td>
</tr>
<tr>
<td>0.6</td>
<td>-5.6290e+05</td>
<td>75.8891</td>
<td>447.9817</td>
<td>646.6809</td>
</tr>
<tr>
<td>0.8</td>
<td>-5.6333e+05</td>
<td>57.8996</td>
<td>247.3950</td>
<td>715.7638</td>
</tr>
<tr>
<td>1</td>
<td>-5.6354e+05</td>
<td>43.6542</td>
<td>183.4221</td>
<td>748.6829</td>
</tr>
</tbody>
</table>

Figure 5  Impact of $\mu$ on $s^*$ (see online version for colours)

Figure 6  Impact of $\mu$ on $Q^*$ (see online version for colours)
Table 4 is according to different quantity for \( v \) and all the other parameters are the same as the first table. In this table for all numbers the condition for convexity is fulfilled and similar to the previous table in all numbers and \( \Delta \) is negative and \( s \) is positive so the optimum quantity for decision variables and cost function is obtained according to the fourth case mentioned in the solution method. As \( v \) is increased both \( s^* \) and \( Q^* \) are decreased. On the other hand cost is increased. We show the effects of \( v \) on \( s^* \), \( Q^* \) and \( c(Q^*, s^*) \) in Figures 8 to 10.

**Table 4**  
The results of sensitivity analysis for the case with different quantity for \( \mu \)

<table>
<thead>
<tr>
<th>( V )</th>
<th>( \Delta )</th>
<th>( s^* )</th>
<th>( Q^* )</th>
<th>( c(Q^<em>, s^</em>) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>-5.6383e+05</td>
<td>92.6324</td>
<td>463.5965</td>
<td>604.1783</td>
</tr>
<tr>
<td>0.025</td>
<td>-5.6139e+05</td>
<td>86.7624</td>
<td>438.9691</td>
<td>650.2169</td>
</tr>
<tr>
<td>0.050</td>
<td>-5.5889e+05</td>
<td>75.7266</td>
<td>404.1612</td>
<td>696.6317</td>
</tr>
<tr>
<td>0.075</td>
<td>-5.5638e+05</td>
<td>63.3869</td>
<td>373.0032</td>
<td>744.8419</td>
</tr>
<tr>
<td>0.090</td>
<td>-5.5426e+05</td>
<td>54.2365</td>
<td>339.9504</td>
<td>779.2689</td>
</tr>
</tbody>
</table>
In this part we provide some numerical results according to the data related to a diary store; analyse these results regarding to the sensitive parameters to find out how this model can impact on the cost function and decision variables, also find out how it can improve the process. Accordingly this model could be so applicable for practitioners in different aspects as we mention below:

1. Quality improvement of managerial activities
2. Increasing of process flexibility to respond to changes of the system
3. Providing service with higher level
4. Being more responsiveness to customers’ requirements
5. Perfect determination of demand and supply to minimise cost and maximise sales.

6 Conclusions

In this research two situations simultaneously are considered to analyse an inventory system. In one of such situations, we consider some customers that do not be patient to
wait to fulfil their orders or their requirement is critical and it should be accessible in the right time. This group of customers does not wait for the vendor to fulfil their order and they choose another vendor. In this situation, a proportion of backordered demand that is associated to this group of customers becomes lost demand and the shortage costs consist of two type of penalty. One of them is related to backordered demand and another one is related to lost demand. In another situation, the quantity of products that customers order contains some defective products. Also this proportion is random and we consider a particular probability density function for it. The aim of this paper is minimising the annual cost for customer regarding to both mentioned situations. This cost involves order cost, shortage cost and holding cost. We have obtained a certain formulation for two decision variables; optimum order quantity and optimum shortage. After it, we provided numerical results to show how the given solution works. In addition we provide a sensitivity analysis to show how the sensitive parameters affect the values of the decision variables and cost function.

References


A stochastic lot sizing model with partial backordering


Appendix A

Proof of convexity of cost function

For convexity we should prove that

$$\frac{\partial^2 c}{\partial Q^2} > 0$$  \hspace{1cm} (A1)

Now we obtain $\frac{\partial^2 c}{\partial Q^2}$ to prove that it is greater than zero:

$$\frac{\partial c}{\partial Q} = -\frac{1}{Q^2} \left( \frac{kD}{\mu} + \frac{s^2}{2\mu} \left( c_h + \xi \right) + \frac{sD}{\mu} \right) - sc_h + \frac{(v + \mu^2)}{2\mu} c_h$$  \hspace{1cm} (A2)

$$\frac{\partial^2 c}{\partial Q^2} = \frac{1}{Q^3} \left( \frac{2kD}{\mu} + \frac{s^2}{\mu} \left( c_h + \xi \right) + \frac{sD}{\mu} \right) = \frac{1}{\mu Q^3} \left( 2kD + s^2 (c_h + \xi) + 2\xi_0 sD \right)$$  \hspace{1cm} (A3)

As we see $\frac{\partial^2 c}{\partial Q^2} = \frac{1}{\mu Q^3} \left( 2kD + s^2 (c_h + \xi) + 2\xi_0 sD \right)$ that is greater than zero so the first condition is fulfilled.

$$\left( \frac{\partial^2 c}{\partial Q^2} \right)^2 - \left( \frac{\partial^2 c}{\partial \xi \partial Q} \right)^2 > 0$$  \hspace{1cm} (A4)

To fulfill this condition we obtain $\left( \frac{\partial^2 c}{\partial Q^2} \right)^2 - \left( \frac{\partial^2 c}{\partial \xi \partial Q} \right)^2$ so we have:

$$\frac{\partial^2 c}{\partial Q^2} = \frac{1}{\mu^2 Q^4} \left[ 2kD \left( c_h + \xi \right) + s^2 \left( c_h + \xi \right)^2 + 2\xi_0 sD \left( c_h + \xi \right) \right]$$  \hspace{1cm} (A5)

$$\left( \frac{\partial^2 c}{\partial \xi \partial Q} \right)^2 = \frac{1}{\mu^2 Q^4} \left[ s^2 \left( c_h + \xi \right)^2 + 2\xi_0 sD \left( c_h + \xi \right) + \xi_0^2 D^2 \right]$$  \hspace{1cm} (A6)

$$\left( \frac{\partial^2 c}{\partial \xi \partial Q} \right)^2 = \frac{1}{\mu^2 Q^4} \left( 2kD \left( c_h + \xi \right) - \xi_0^2 D^2 \right)$$  \hspace{1cm} (A7)

So $\frac{1}{\mu^2 Q^4} \left( 2kD \left( c_h + \xi \right) - \xi_0^2 D^2 \right)$ should be greater than 0 and to fulfill it the below inequality should be satisfied:

$$2k \left( c_h + \xi \right) > \xi_0^2 D$$  \hspace{1cm} (A8)
Appendix B

Proof of Lemma 1.3

\[
\lim_{s \to \infty} e(Q^*, s) = \lim_{s \to \infty} \sqrt{\frac{\nu + \mu^2}{\mu^2}} 2c_h D(k + \tilde{z}_0 s) + (c_h + \tilde{z}) s^2 c_h - \\
\sqrt{\frac{\nu + \mu^2}{\mu^2}} 2c_h D(k + \tilde{z}_0 s) + (c_h + \tilde{z}) s^2 c_h + sc_h
\]

\[
\sqrt{\frac{\nu + \mu^2}{\mu^2}} 2c_h D(k + \tilde{z}_0 s) + (c_h + \tilde{z}) s^2 c_h + sc_h
\]

\[
\frac{\nu + \mu^2}{\mu^2} - \frac{2Dk_c h + \frac{\nu + \mu^2}{\mu^2} (c_h + \tilde{z}) c_h - s^2 c_h}{\sqrt{\frac{\nu + \mu^2}{\mu^2} 2Dk_c h + \frac{\nu + \mu^2}{\mu^2} D_c s c_h + (c_h + \tilde{z}) s^2 c_h + sc_h}}
\]

\[
\lim_{s \to \infty} = \infty
\]

Appendix C

Proof of Lemma 1.4

\[
c(Q^*, s) = \sqrt{\frac{\nu + \mu^2}{\mu^2}} 2Dk_c h + \frac{\nu + \mu^2}{\mu^2} D_c s + (c_h + \tilde{z}) s^2 c_h - sc_h
\]

\[
\frac{\partial c(Q^*, s)}{\partial s} = \sqrt{\frac{\nu + \mu^2}{\mu^2}} \frac{2\tilde{z}_0 D_c h + (c_h + \tilde{z}) s c_h}{\sqrt{2Dk_c h + 2\tilde{z}_0 D_c s + (c_h + \tilde{z}) s^2 c_h}} - c_h
\]

\[
\frac{\partial c(Q^*, s)}{\partial s} = \sqrt{\frac{\nu + \mu^2}{\mu^2}} \frac{\tilde{z}_0 D_c h + (c_h + \tilde{z}) s c_h}{\sqrt{2Dk_c h + 2\tilde{z}_0 D_c s + (c_h + \tilde{z}) s^2 c_h}} - c_h
\]

\[
\frac{\partial c(Q^*, s)}{\partial s} = c_h \frac{\frac{\nu + \mu^2}{\mu^2} \tilde{z}_0 D + (c_h + \tilde{z}) s}{\sqrt{2Dk_c h + 2\tilde{z}_0 D_c s + (c_h + \tilde{z}) s^2 c_h}} - 1
\]

\[
\frac{\partial c(Q^*, s)}{\partial s} = c_h \frac{\frac{\nu + \mu^2}{\mu^2} \tilde{z}_0 D + (c_h + \tilde{z}) s - \sqrt{2Dk_c h + 2\tilde{z}_0 D_c s + (c_h + \tilde{z}) s^2 c_h}}{\sqrt{2Dk_c h + 2\tilde{z}_0 D_c s + (c_h + \tilde{z}) s^2 c_h}}
\]
\[
\frac{\partial c(Q', s)}{\partial s} = \frac{\partial c(Q', s)}{\partial s} \\
\frac{\sqrt{\frac{\nu + \mu^2}{\mu^2}} (z_0 D + (c_h + \xi)s) + \sqrt{2Dk_c h + 2z_0^2 Dsc_h + (c_h + \xi)s^2 c_h}}{\sqrt{\frac{\nu + \mu^2}{\mu^2}} (z_0 D + (c_h + \xi)s) + \sqrt{2Dk_c h + 2z_0^2 Dsc_h + (c_h + \xi)s^2 c_h}}
\]

(C6)

\[
\frac{\partial c(Q', s)}{\partial s} = c_h \left[ \frac{\left( \frac{\nu + \mu^2}{\mu^2} z_0 D + (c_h + \xi)s \right)^2 + \left( \frac{\nu + \mu^2}{\mu^2} z_0 D + (c_h + \xi)s \right)^2}{\sqrt{2Dk_c h + 2z_0^2 Dsc_h + (c_h + \xi)s^2 c_h}} \right] \\
\left[ \frac{\sqrt{\frac{\nu + \mu^2}{\mu^2} (c_h + \xi)^2 - c_h (c_h + \xi)} s^2 + \left( \frac{\nu + \mu^2}{\mu^2} \right) z_0 D (c_h + \xi) - 2z_0 Dc_h \right] s + \sqrt{2Dk_c h + 2z_0^2 Dsc_h + (c_h + \xi)s^2 c_h}
\]

(C7)

\[
\frac{\partial c(Q', s)}{\partial s} = \frac{\nu + \mu^2}{\mu^2} z_0^2 D^2 - 2Dk_c h
\]

(C8)

\[
\frac{\partial c(Q', s)}{\partial s} = \frac{\phi(s)}{\delta(s)} \left[ \frac{\nu + \mu^2}{\mu^2} (c_h + \xi)s + \sqrt{\frac{\nu + \mu^2}{\mu^2} z_0 D + \delta(s)} \right]
\]

(C9)

\[
\phi(s) = \left[ \frac{\nu + \mu^2}{\mu^2} (c_h + \xi)^2 - c_h (c_h + \xi) \right] s^2 + \left[ \frac{\nu + \mu^2}{\mu^2} \right] z_0 D (c_h + \xi) - 2z_0 Dc_h \]

\[
\delta(s) = \sqrt{2Dk_c h + 2z_0^2 Dsc_h + (c_h + \xi)s^2 c_h}
\]

(C10)

(C11)