Numerical simulation for fractional phi-4 equation using homotopy Sumudu approach

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Abstract: In this study, we utilise the homotopy analysis method and the Sumudu transform to find the solution of fractional order partial differential equations. We focus primarily on the employment of the method for solving the fractional phi-4 equation in one-dimensional spatial domain. The method can be extended easily for more general nonlinear equations. Convergence and error analysis are given. Numerical experiments show that the method is both effective and accurate, and suits such type of applications.

Keywords: Sumudu transform; homotopy analysis method; HAM; phi-4 equation; convergence; unique solution.

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1 Introduction

Fractional partial differential equations (FPDEs) are generalisations of the classical models for which the integer-order derivatives are replaced by fractional-order ones. In the past few decades, FPDEs have gained growing interest due to a promising wide range of applications in physics, geophysics, computer sciences, and finance (Podlubny, 1998; Hilfer et al., 2000; Diethelm, 2010). They eliminate many deficiencies in the classical models and are found to be more adequate models to describe real-life phenomena such as the anomalous diffusion, protein diffusion within cells, traffic flow, movement of a material along fractals (Kilbas et al., 2006; Uchaikin, 2013; Metzler and Klafter, 2000). And other new applications can be found in Morales-Delgado et al. (2019a), Morales-Delgado et al. (2019b), Saad (2018), Saad and Al-Sharif (2019), Gómez-Aguilar et al. (2019), Shukla and Sharma (2018) and Vaidyanathan et al. (2018).

Nonlinear dispersive equations constitute a large class of equations which can be adequately model various phenomena in physics and engineering. Well known models of this kind are the nonlinear Schrödinger, Korteweg-de Vries (KdV) equation, and phi-4 equations, which are core tools in the field of nuclear and elementary particle physics.

The classical phi-4 equation is a wave equation with cubic nonlinearity which is given as

$$u_{tt}(x,t) = u_{xx}(x,t) - m^2 u(x,t) - \lambda u^3(x,t). \tag{1.1}$$

The solution of the phi-4 equation have been proposed by several analytical and numerical methods. For example, the Jacobi elliptic expansion method (Alquran et al., 2018), the trigonometric B-spline collocation method (Zahra, 2017), the generalised Kudryashov method (Mahmud et al., 2017), the Jacobi elliptic function method (Djob et al., 2016), the finite difference method (Encinas et al., 2015), the collocation method (Bhrawy et al., 2013). See also Akbulut et al. (2016), Triki and Wazwaz (2013), Khajeh et al. (2010) and Wazwaz (2007) for other methods.

In this study, we consider the fractional counterpart of the phi-4 equation:

$$D_t^\alpha u(x,t) = u_{xx}(x,t) - m^2 u(x,t) - \lambda u^3(x,t) \tag{1.2}$$

where $D_t^\alpha u$ is taken in the Caputo sense with order $1 < \alpha \leq 2$. We further impose the side conditions

$$u(x,0) = f_0(x), \quad u_t(x,0) = f_1(x). \tag{1.3}$$

The fractional version of the phi-4 equation was considered in several research articles. For example, Rezazadeh et al. (2018) used the conformable fractional derivative and applied the extended direct algebraic method, Alquran et al. (2017) used the Caputo fractional derivatives and applied the residual power series method, Tariq and Akram (2017) applied the fractional complex transform and tanh method.

Recently, new analytical techniques for fractional differential equations via the Sumudu transform have been emerged. In Rathore et al. (2012), introduced a hybrid method of Sumudu transform and homotopy analysis method (HAM) for solving the linear Fokker-Planck equation. The method provides a series solution that approaches to the solution. Sushila and Shishodia (2014) used the HASTM to solve the thin film flow problem. Their method is also adopted to solve other types of fractional equations. For instance, Kumar et al. (2016) solved the fractional model of differential-difference equation, Pandey and Mishra (2017b) solved time-space fractional heat and wave equations, and others (Choi et al., 2017; Pandey and Mishra, 2015; Singh et al., 2017; Pandey and Mishra, 2017a; Singh et al., 2018).

The proposed method by this paper utilises the Sumudu transform and the HAM to find convergent series solution for the nonlinear fractional phi-4 equation. The results show that the method can calculate higher terms of the series solution because it reduces the volume of the computational work, which leads to high accuracy of the numerical results.

2 Preliminaries

In this section, definitions and results concerning fractional calculus and the Sumudu transform are provided.

Non-integer orders of differentiation and integration are called fractional calculus. It can be traced back to the year 1695 in a letter from L'Hôpital to Leibniz asking for the meaning of $d^n y/dx^n$ with $n = 1/2$. Since then several definitions have been given trying to accommodate the meaning of non-integer order derivatives and integrals. To date, however, there is no unified definition of fractional integrals and derivatives, but there are some accepted and commonly used definitions. These definitions include but are not limited to the Riemann-Liouville, the Caputo, and the Gränwald Letnikov definitions (Podlubny, 1998), see also Alomari et al. (2010), Alomari (2011), Alomari et al. (2013), Al-Sawalha et al. (2011), Al-Jamel (2019), Al-Jamel et al. (2018) and Saad et al. (2018) for more recent definitions and applications. The following definitions are needed for $\alpha > 0$.

1 Riemann-Liouville fractional integral

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

where $\Gamma(\cdot)$ stands for the usual Gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

By convention $J^0 f(t) := f(t)$.

2 Riemann-Liouville fractional derivative

$$D_R^\alpha f(t) = \frac{d^n}{dt^n} J^{n-\alpha} f(t), \quad n = \lceil \alpha \rceil.$$

3 Caputo fractional derivative

$$D^\alpha f(t) = J^{n-\alpha} f^{(n)}(t), \quad n = \lceil \alpha \rceil.$$

Contrary to the Riemann-Liouville definition, Caputo's definition of fractional derivative appears more frequently in the contexts of mathematical physics since it allows traditional (non-fractional) initial conditions to be incorporated in the mathematical model, which we adopt in the sequel.

Watugala (1993) introduced Sumudu, which is an integral transform, and applied it to solve differential equations and control engineering problems.

Definition 2.1: For a function $f(t)$ the Sumudu transform $F(u)$ is

$$F(u) = \mathbb{S}[f(t)] = \int_0^\infty e^{-t} f(ut)dt \tag{2.1}$$

provided the integral converges for some u.

Besides the scale and unit preserving properties, the Sumudu transform obeys several interesting properties. We compile the relevant properties in Table 1. Readers may refer to Rathore et al. (2012) and Asiru (2002) for a more comprehensive list.

Table 1 Basic properties of the Sumudu transform

General properties	1	$\mathbb{S}[\beta_1 f_1(t) + \beta_2 f_2(t)] = \beta_1 F_1(u) + \beta_2 F_2(u)$
	2	$\mathbb{S}[f(\beta t)] = F(\beta u)$
	3	$\lim_{t\to 0} f(t) = f(0) = \lim_{u\to 0} F(u)$
Differentiation formulas	4	$\mathbb{S}[f^{(n)}(t)] = u^{-n}[F(u) - \Sigma_{k=0}^{n-1} u^k f^{(k)}(0)]$
	5	$\mathbb{S}[tf(t)] = u^2 \frac{d}{du}F(u) + uF(u)$
Convolution formulas	6	$\mathbb{S}[\int_0^t f(\tau)g(t-\tau)d\tau] = uF(u)G(u)$
Polynomial formulas	7	$\mathbb{S}[\frac{t^\alpha}{\Gamma(\alpha+1)}] = u^\alpha,\ \alpha > 0$
Fractional formulas	8	$\mathbb{S}[J^\alpha f(t)] = u^{-\alpha}F(u).$
	9	$\mathbb{S}[D^\alpha f(t)] = u^{-\alpha}F(u) - \sum_{k=0}^{n-1} u^{-\alpha+k} f^{(k)}(0^+), \quad n = \lceil\alpha\rceil.$

3 Basic idea of the proposed method

To show the fundamental origination of the homotopy analysis Sumudu transform method (HASTM) for solving fractional partial differential equations, we consider a nonlinear equation in the form:

$$D_t^\alpha u(x,t) = N\left[u(x,t)\right], \tag{3.1}$$

subject to the initial conditions

$$\frac{\partial^k u}{\partial t^k}(x,0) = f_k(x),\ \ k = 0, \ldots, n-1, \quad n = \lceil\alpha\rceil. \tag{3.2}$$

Apply the Sumudu transform for equation (3.1), then invoking property 9 in Table 1 yield

$$v^{-\alpha}\mathbb{S}[u(x,t)] - \sum_{k=0}^{n-1} v^{-\alpha+k} \frac{\partial^k u}{\partial t^k}(x,0) = \mathbb{S}\left[N[u(x,t)]\right],$$

where v denotes the dummy variable in the definition of the Sumudu transform. Using equation (3.2), we get

$$\mathbb{S}[u(x,t)] = g(v,x) + v^\alpha \mathbb{S}[N[u(x,t)]], \tag{3.3}$$

where

$$g(v,x) = \sum_{k=0}^{n-1} v^k f_k(x).$$

The existence of the nonlinear term $N[u(x,t)]$ makes it difficult to solve equation (3.3) for u by simply invoking the inverse Sumudu transform. Therefore, we can utilise the HAM to resolve this issue. To this end, we define the zeroth-order deformation equation:

$$(1-q)\mathbb{S}[\Phi(x,t;q) - u_0(x,t)] = \hbar q \mathcal{N}[\Phi(x,t;q)], \tag{3.4}$$

where q is the embedding parameter between 0 and 1, $\hbar \neq 0$ is the convergent control parameter, u_0 is some initial guess, and $\mathcal{N}$ is the nonlinear operator given by

$$\mathcal{N}[\Phi(x,t;q)] = \mathbb{S}[\Phi(x,t;q)] - v^\alpha \mathbb{S}[N[\Phi(x,t;q)]] - g(v,x).$$

Evidently, when $q = 0$ it holds that $\Phi(x,t;0) = u_0(x,t)$ whereas $\Phi(x,t;1) = u(x,t)$ when $q = 1$. So that, when q varies from 0 to 1, the solution Φ of equation (3.4) varies (deforms) from the initial guess u_0 to the closed form of the solution u of the nonlinear equation (3.1).

Now, display $\Phi(x,t;q)$ as a Taylor series about $q = 0$, results in the series expansion

$$\Phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \tag{3.5}$$

where the coefficients u_m are given by

$$u_m = \left.\frac{1}{m!}\frac{\partial^m \Phi}{\partial q^m}\right|_{q=0}. \tag{3.6}$$

If $\hbar$ is chosen in such a way the series (3.5) is convergent at $q = 1$ then

$$u(x,t) = \Phi(x,t;1) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \tag{3.7}$$

which represents the solution of equation (3.1), see Theorem 4.1.

To determine the terms $u_m(x,t)$, we differentiate the zeroth-order deformation equation (3.4) m-times with respect to q, then we set $q = 0$, resulting in the equation

$$\mathbb{S}\left[u_m - \chi_m u_{m-1}\right] = \hbar \mathcal{R}_m, \tag{3.8}$$

where

$$\mathcal{R}_m = \left.\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[\mathcal{N}[\Phi(x,t;q)]\right]\right|_{q=0}.$$

Recalling the definition of $\mathcal{N}$, we get the so-called m^{th}-order deformation equation

$$\mathbb{S}\left[u_m - \chi_m u_{m-1}\right] = \hbar\left(\mathbb{S}\left[u_{m-1}\right] - v^\alpha \mathbb{S}\left[R_m\right] - (1-\chi_m)g(v,x)\right), \tag{3.9}$$

where

$$R_m = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[N[\Phi(x,t;q)]\right]\Big|_{q=0},$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

Finally, apply the inverse $\mathbb{S}^{-1}$ for both sides of equation (3.9), to achieve

$$\begin{aligned} u_m(x,t) &= (\hbar + \chi_m)u_{m-1}(x,t) \\ &\quad - \hbar\mathbb{S}^{-1}\left[v^\alpha \mathbb{S}[R_m] - (1-\chi_m)g(v,x)\right], \end{aligned} \tag{3.10}$$

which defines a recursive formula to determine the coefficients u_m. It is deserve to mention that, for initial-value problems, we can predict the initial guess as

$$u_0(x,t) = \sum_{k=0}^{n-1}\frac{t^k}{k!}f_k(x),$$

which has been proposed by Liao (2003) and Odibat (2010), and we adopt this convention throughout.

4 Convergence analysis

In this section, we presents some theorems concerning the convergence of the proposed HASTM.

First, we show that if the HASTM converges, then it converges to a solution of the original nonlinear problem (3.1)–(3.2).

Theorem 4.1: If the homotopy series solution $\sum_{m=0}^{\infty} u_m(x,t)$, where u_m are resulted from equation (3.6), is convergent, then the limit of the series is an exact solution of the nonlinear problems (3.1)–(3.2).

Proof: Assume that the series $\sum_{m=0}^{\infty} u_m$ is convergent. From equations (3.8)–(3.9), we get

$$\begin{aligned} \hbar\sum_{m=1}^{\infty}\mathcal{R}_m &= \lim_{k\to\infty}\sum_{m=1}^{k}\mathbb{S}\left[u_m - \chi_m u_{m-1}\right] \\ &= \mathbb{S}\left[\lim_{k\to\infty}\sum_{m=1}^{k}(u_m - \chi_m u_{m-1})\right] \\ &= \mathbb{S}\left[\lim_{k\to\infty} u_k\right]. \end{aligned}$$

Since $\hbar \neq 0$ and $u_m \to 0$, this implies $\sum_{m=1}^{\infty}\mathcal{R}_m = 0$. Now, expand $\mathcal{N}[\Phi(x,t;q)]$ about $q=0$, then set $q=1$, to see that

$$\mathcal{N}[\Phi(x,t;1)] = \sum_{m=1}^{\infty}\mathcal{R}_m = 0.$$

Consequently, if the series in equation (3.7) is convergent, it follows that

$$\mathcal{N}[\Phi(x,t;1)] = 0,$$

that is, $u(x,t) = \Phi(x,t;1) = \sum_{m=0}^{\infty} u_m(x,t)$ solves equation (3.3). Therefore, $\mathbb{S}[D_t^\alpha u(x,t)] = \mathbb{S}[N[u(x,t)]]$, showing that $u(x,t) = \sum_{m=0}^{\infty} u_m(x,t)$ solves the original nonlinear problem (3.1). Moreover, from properties (3) and (4) (refer to Table 1) we infer that

$$\frac{\partial^k u}{\partial t^k}(x,0) = \lim_{v\to 0}\frac{1}{k!}\frac{d^k}{dv^k}\mathbb{S}[u(x,t)],\ k = 0,\dots,n-1.$$

From this and equation (3.3) it follows that

$$\frac{\partial^k u}{\partial t^k}(x,0) = f_k(x),\ k = 0,\dots,n-1, \quad n = \lceil\alpha\rceil,$$

showing that u also satisfies the initial conditions (3.2). Hence u is a solution to the IVP (3.1)–(3.2), which ends the proof. □

For sufficient condition on the convergence of the series solution and error bounds on the truncated solution, we cite the following general results (Odibat, 2010).

Theorem 4.2: Let the solution component $u_0(x,t)$, $u_1(x,t)$, $u_2(x,t)$, ... be defined as equation (3.10). The series solution $\sum_{m=0}^{\infty} u_m(x,t)$ defined in equation (3.7) converges if $\exists\ 0 < \gamma < 1$ such that $\|u_{m+1}(x,t)\| \le \gamma\|u_m(x,t)\|$, $\forall m > m_0$, for some $m_0 \in \mathbb{N}$.

Theorem 4.3: Suppose that the series solution $\sum_{m=0}^{\infty} u_m(x,t)$ is convergent to the solution $u(x,t)$. If the truncated series $\sum_{m=0}^{k} u_m(x,t)$ is used as an approximation to the solution $u(x,t)$, then the truncated error satisfies

$$\left\| u(x,t) - \sum_{m=0}^{k} u_m(x,t)\right\| \le \frac{1}{1-\gamma}\gamma^{k+1}\|u_0(x,t)\|.$$

5 Fractional phi-4 equation

In this section, we apply the proposed approach to find approximate analytic solution for the fractional order phi-4 equation (Alquran et al., 2017):

$$\begin{aligned} D_t^\alpha u(x,t) &= u_{xx}(x,t) - m^2 u(x,t) \\ &\quad - \lambda u^3(x,t),\ 1 < \alpha \le 2, \end{aligned} \tag{5.1}$$

subject to the initial conditions:

$$u(x,0) = f_0(x),\ u_t(x,0) = f_1(x). \tag{5.2}$$

Apply the Sumudu transformation for both sides, to get

$$\begin{aligned} &\mathbb{S}[u(x,t)] - f_0(x) - vf_1(x) \\ &= v^\alpha \mathbb{S}\left[u_{xx}(x,t) - m^2 u(x,t) - \lambda u^3(x,t)\right]. \end{aligned} \tag{5.3}$$

Then define the map

$$(1-q)\mathbb{S}[\Phi(x,t;q) - u_0(x,t)] = \hbar q\mathcal{N}[\Phi(x,t;q)],$$

where u_0 is some initial guess, and

$$\begin{aligned}\mathcal{N}[\Phi] &= \mathbb{S}[\Phi] - f_0 - v f_1 \\ &\quad - v^\alpha \mathbb{S}\left[\frac{\partial^2 \Phi}{\partial x^2} - m^2\Phi - \lambda\Phi^3\right]. \end{aligned} \quad (5.4)$$

Hence, it is clear that

$$\mathcal{R}_m = \mathbb{S}[u_{m-1}] - v^\alpha \mathbb{S}[R_m],$$

with

$$\begin{aligned}R_m &= \frac{\partial^2 u_{m-1}}{\partial x^2} - m^2 u_{m-1} \\ &\quad - \lambda \sum_{i=0}^{m-1}\sum_{j=0}^{i} u_j u_{i-j} u_{m-1-j}.\end{aligned}$$

Therefore, the m^{th}-order deformation equation can be written as

$$\begin{aligned}\mathbb{S}[u_m - \chi_m u_{m-1}] &= \hbar\left(\mathbb{S}[u_{m-1}] - v^\alpha \mathbb{S}[R_m]\right. \\ &\quad \left. - (1-\chi_m)(f_0 + v f_1)\right).\end{aligned}$$

Apply the inverse Sumudu transform, to obtain

$$\begin{aligned}u_m &= (\chi_m + \hbar)u_{m-1} + \hbar\mathbb{S}^{-1}[(1-\chi_m)(f_0 + v f_1)] \\ &\quad - \hbar\mathbb{S}^{-1}\left[v^\alpha \mathbb{S}\left[\frac{\partial^2 u_{m-1}}{\partial x^2} - m^2 u_{m-1}\right.\right. \\ &\quad \left.\left. - \lambda\sum_{i=0}^{m-1}\sum_{j=0}^{i} u_j u_{i-j} u_{m-1-j}\right]\right]. \end{aligned} \quad (5.5)$$

In practice, we only consider a finite number of terms u_m. Thus, the solution of equation (5.1) is approximated by the so-called M^{th}-order solution:

$$\tilde{u}_M(x,t) = u_0(x,t) + \sum_{m=1}^{M} u_m(x,t).$$

As a particular demonstration of the HASTM, we consider the phi-4 equation (5.1) with parameters $m = 1$ and $\lambda = -1$, subject to the initial conditions:

$$u(x,0) = \tanh\left(\frac{x}{4}\right), \quad u_t(x,0) = \frac{-3}{4}\text{sech}^2\left(\frac{x}{4}\right). \quad (5.6)$$

According to the HASTM, we choose initial guess as

$$\begin{aligned}u_0(x,t) &= u(x,0) + u_t(x,0)t \\ &= \tanh\left(\frac{x}{4}\right) - \frac{3t}{4}\text{sech}^2\left(\frac{x}{4}\right).\end{aligned}$$

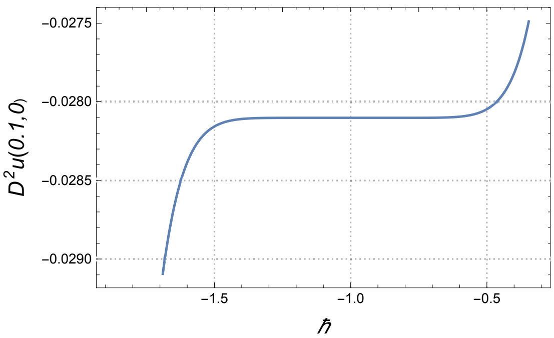
The subsequent terms $u_1, u_2, \ldots,$ are determined using equation (5.5). We utilised Mathematica software to simplify the computations.

In the first experiment, we set $\alpha = 2$, whence, the exact solution is given by

$$u(x,t) = \tanh\left(\frac{x - 3t}{4}\right). \quad (5.7)$$

We use the 5th-order approximation $\tilde{u}_5$. As suggested by most literature in HAM, a proper value for the control parameter $\hbar$ can be obtained using the $\hbar$-curve; for more detail see Liao (2003). The $\hbar$-curve of $\tilde{u}_{tt}(x,0)$ when $x = 0.1$ is shown in Figure 1. We observe that the valid region of $\hbar$, that is, the region where the the curve is a horizontal line, is $-0.7 \le \hbar \le -1.3$. The absolute errors at various nodes (x,t) corresponding to $\hbar = -1$ are reported in Table 2.

Figure 1 The $\hbar$-curve when $\alpha = 2$ (see online version for colours)



Next, we consider the fractional phi-4 equation when $\alpha = 1.9, \pi/2, 1.2$. We set $\hbar = -0.75$ and $M = 19$. Plots for the approximate solution $\tilde{u} = \tilde{u}_{19}$ as well as the residual error

$$\begin{aligned}\mathrm{R}(x,t) &= D_t^\alpha \tilde{u}(x,t) - \tilde{u}_{xx}(x,t) + m^2\tilde{u}(x,t) \\ &\quad + \lambda\tilde{u}^3(x,t),\end{aligned}$$

are shown in Figure 2.

6 Discussion and further results

Comparing our results for the case $\alpha = 2$ with results obtained via the modified residual power series method (Alquran et al., 2017), we see that the HASTM is very accurate. In fact, the results reported in Alquran et al. (2017), show that the error is roughly of magnitude 10^{-6} (using the same number of series terms), while ours is of magnitude 10^{-12}. Therefore, we conclude the method is very powerful.

In the fractional case, the residual errors are of magnitudes 10^{-10}, 10^{-8}, 10^{-6}, corresponding to the the fractional orders $\alpha = 1.9, \pi/2, 1.2$, respectively, which indicate the accuracy of the HASTM applied to the fractional phi-4 equation.

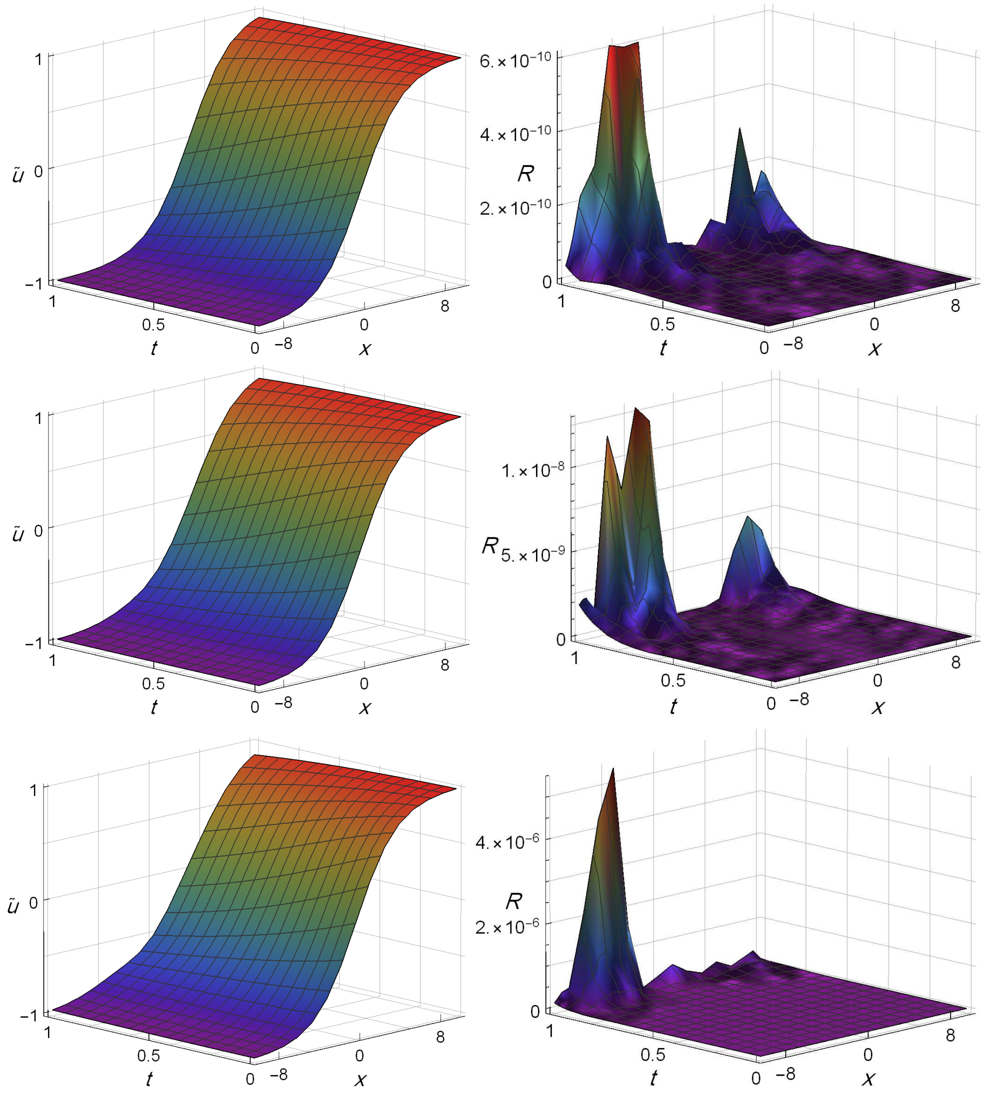
Finally, we conclude with the following result which gives partial answer regarding the uniqueness of the solution of the fractional phi-4 IVP given by equations (5.1)–(5.2). For this purpose, we set $J(u) = u_{xx} - m^2u - \lambda u^3$.

Theorem 6.1: Suppose that J is Lipschitz with Lipschitz constant γ, and $\hbar$ is optimally chosen. The solution of the IVP (5.1)–(5.2) by mean of HASTM, $0 \le t \le T$, is unique whenever $K = (1 + \hbar) - \hbar\gamma T^\alpha/\Gamma(\alpha + 1) < 1$.

Table 2 Absolute error $|u(x,t) - \tilde{u}_5(x,t)|$ when $\alpha = 2$

$x\|t$	*0.1*	*0.15*	*0.2*	*0.25*	*0.3*
–4	1.11×10^{-16}	1.11×10^{-16}	3.66×10^{-15}	6.75×10^{-14}	7.32×10^{-13}
–2	1.11×10^{-16}	4.44×10^{-16}	1.33×10^{-14}	2.15×10^{-13}	2.08×10^{-12}
0	0	1.39×10^{-17}	8.33×10^{-16}	1.47×10^{-14}	1.49×10^{-13}
2	5.55×10^{-17}	5.55×10^{-17}	6.11×10^{-16}	1.35×10^{-14}	2.95×10^{-13}
4	0	1.11×10^{-16}	4.88×10^{-15}	8.48×10^{-14}	8.81×10^{-13}

Figure 2 (a) The 19th-order solutions (b) The corresponding residual errors, when α = 1.9, α = π/2, and α = 1.2, respectively (see online version for colours)



Proof: Suppose that u and u^* both solve the equations (5.1)–(5.2) for $0 \le t \le T$. By adding u_0 for both side of equation (5.5) and take the sum, the convolution property in Table 1, it follows that

$$\begin{aligned} u - u^* &= (1+\hbar)(u-u^*) \\ &\quad - \hbar \mathbb{S}^{-1}\left[v^{\alpha}\mathbb{S}\left[J(u) - J(u^*)\right]\right] \qquad (6.1) \\ &= (1+\hbar)(u-u^*) \\ &\quad - \hbar\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left(J(u) - J(u^*)\right)d\tau. \qquad (6.2) \end{aligned}$$

Then from the Lipschitz condition we see that

$$\begin{aligned} &\max|u-u^*| \le (1+\hbar)\max|u-u^*| \\ &-\hbar\gamma\max|u-u^*|\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}d\tau \le K\max|u-u^*|. \end{aligned}$$

Since $K < 1$, it must be that $u = u^*$ showing the uniqueness of the solution. □

7 Conclusions

Our aim was to construct an accurate solution to the fractional phi-4 equation via a relatively new analytical technique, the homotopy-Sumudu transformation method. The method does not require the calculations of the fractional derivatives or the integrals in each terms. The results compared with residual power series method yield the efficiency of the presented algorithm. Moreover, the computations via this technique are very simple, straightforward and reduce the volume of calculations.

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