Reversion strategy for online portfolio selection with transaction costs

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Abstract: This paper is about the online portfolio selection problem with transaction costs, which is an unavoidable factor in real financial trading. By exploiting the mean reversion property of stock prices, we propose a portfolio selection strategy named ‘mean reversion strategy with transaction costs (MRTC)’. To avoid overmuch transaction costs, the strategy adaptively transfers a proper amount of capital between stocks to adjust the turnover. Furthermore, we conduct numerical experiments on several real market datasets, and show that our proposed algorithm outperforms the existing state-of-the-art ones when taking transaction costs into account.

Keywords: online portfolio selection; investment strategy; mean reversion; transaction costs.

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1 Introduction

Portfolio selection is a financial problem attracting studies from various fields, ranging from traditional finance theory (Alexander et al., 2017; Kallsen and Muhle-Karbe, 2015) and mathematical finance (Kenneth, 2012; Messaoudi and Rebaï, 2014; Bai et al., 2016) to artificial intelligence (Li and Xu, 2013; Omidi et al., 2016; Berutich et al., 2016). It requires investors to determine a portfolio which allocates capital to a set of stocks for certain objectives, such as fat cumulative return. There are two major investigation streams of portfolio selection, i.e., the mean-variance theory (Markowitz, 1952) and the capital growth theory (CGT) (Kelly, 1956). The former one aims to trade-off between the expected return and risk of a portfolio, which is suitable for single-period portfolio selection; while, the latter one, in scope of multiple-period or sequential portfolio selection, focuses on maximising expected log return in the long run. Recently, Kelly’s CGT is widely adopted to deal with the online portfolio selection problem for its sequential nature.

Online portfolio selection has been actively explored and many algorithms have been proposed in literatures, which can be classified into two categories. One category is universal portfolio selection algorithms (Cover, 1991; Helmobold et al., 1998; Gaivoronski, 2000), which theoretically approach to benchmark strategies, such as the BCRP strategy to be mentioned in Section 3. Algorithms in another category (Borodin et al., 2004; Li et al., 2012, 2013, 2015) try to exploit statisticlal properties in real market, especially mean reversion phenomenon. The idea behind the mean reversion property is that worse performing stocks will perform better on the subsequent periods and vice versa. This property is not only applied for online portfolio selection, but also for other financial problems (Kitapbayev and Leung, 2017; Sun and Su, 2017; Zhao and Palomar, 2016).

While promising theoretically, universal strategies appear to be conservative, which hinders them to make wealth from the fluctuation of stock markets. In contrast, mean reversion strategies tend to aggressively exploit the mean reversion property and often achieve more encouraging wealth than the universal ones. Thus, the mean reversion principle was widely applied, and many related algorithms were proposed. However, decision-making processes of these mean reversion algorithms do not take into account transaction costs, an unavoidable factor in financial trading, so their performances fade
rapidly as the rate of transaction costs increases. This problem inspires the investigation of this paper. We introduce a new online portfolio selection strategy named ‘mean reversion strategy with transaction costs (MRTC)’, which exploits the multi-period mean reversion property. Based on two ways of measuring recent performances of stocks, the algorithm can derive two realisations. To meet the challenge that existing mean reversion strategies tend to trade frequently leading to a large amount of transaction costs, our algorithm introduces a self-adaptive parameter to adjust the turnover. Moreover, to illustrate the effectiveness of MRTC, we conduct numerical experiments on several real market datasets, and the results indicate that the proposed algorithm can efficiently adjust the turnover and gain more cumulative return than the existing state-of-the-art algorithms in most cases.

As a summary, our contributions in this paper include:

1. we propose a new online portfolio selection strategy, which exploits multi-period mean reversion property
2. our strategy can efficiently trade-off between high cumulative return and low transaction cost by adaptively adjusting the turnover
3. numerical experiments are conducted to illustrate the superiority of our strategy.

The rest of this paper is organised as follows. Section 2 formally formulates the online portfolio selection problem with transaction costs. Section 3 discusses related existing works. Section 4 presents our proposed algorithm MRTC. Section 5 presents and analyses experiment results of our algorithm. Section 6 concludes this paper.

2 Problem setting

In this section, we define some related notations and formulate the online portfolio selection problem with transaction costs. Consider a financial market with $m$ stocks to be invested for $n$ periods. We denote the price relatives of $m$ stocks on the $t$th trading period by $x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,m})^T$, where $x_{t,i}$ represents the ratio of the closing price $p_{t,i}$ to last closing price $p_{t-1,i}$ of the $i$th stock. At the end of the $(t-1)$th period, the investor determines a portfolio $b_t = (b_{t,1}, b_{t,2}, \ldots, b_{t,m})^T$ for the next trading period according to the price relative sequence $\{x_{t-1}\}_{t=1}^{t-1}$, where $b_{t,i}$ represents the proportion of the capital invested in the $i$th stock on the $t$th trading period. We assume all the portfolios are self-financed and no margin/short sale is allowed, that is, $b_t \in \Delta_m$, where

$$\Delta_m = \{b = (b_1, b_2, \ldots, b_m)^T : b_i \geq 0, \sum_{i=1}^{m} b_i = 1\}.$$

On the $t$th trading period, in the case of zero transaction cost, the investment based on portfolio $b_t$ gains wealth by a factor of $b_t \cdot x_t = \sum_{i=1}^{m} b_{t,i} x_{t,i}$ when the price relative $x_t$ occurs. Thus, after $n$ trading periods, the cumulative wealth is

$$S_n = S_0 \prod_{t=1}^{n} b_t \cdot x_t,$$

where $S_0$ denotes the initial wealth, and is set to one for convenience in this paper.

Next we consider the case of non-zero transaction costs. Most of the existing works (Blum and Kalai, 1999; Das et al., 2013; Vanli et al., 2016) adopt the proportional transaction costs model, in which the incurred transaction costs are proportional to
the wealth transferred during trading. At the beginning of the \( t \)th period, the portfolio manager rebalances the portfolio from the closing price adjusted portfolio \( \hat{b}_{t-1} \) to a new portfolio \( b_t \) and transaction costs are incurred, where each element of \( \hat{b}_{t-1} \) is calculated as \( \hat{b}_{t-1,i} = \frac{b_{t-1,i}x_{t-1,i}}{b_{t-1}·x_{t-1}} \). Denote the remaining wealth after adjusting the portfolio by \( \hat{S}_{t-1} \). For the \( i \)th stock, \( \hat{b}_{t-1,i}S_{t-1} > b_{t,i}S_{t-1} \) indicates selling, while \( \hat{b}_{t-1,i}S_{t-1} < b_{t,i}S_{t-1} \) buying occurred. Clearly, \( S_{t-1} \) is divided into two parts, i.e., \( \hat{S}_{t-1} \) and transaction costs. So we have

\[
S_{t-1} = \hat{S}_{t-1} + \eta_s \sum_{i=1}^{m} (\hat{b}_{t-1,i}S_{t-1} - b_{t,i}S_{t-1})^+ + \eta_b \sum_{i=1}^{m} (b_{t,i}S_{t-1} - \hat{b}_{t-1,i}S_{t-1})^+,
\]

(2)

where \( \eta_s \) and \( \eta_b \) are the rates of transaction costs of selling and buying, respectively. For simplicity, we set \( \eta_s = \eta_b = \eta \), and then equation (2) can be simplified as

\[
S_{t-1} = \hat{S}_{t-1} + \eta \left\| S_{t-1} \hat{b}_{t-1} - \hat{S}_{t-1} b_t \right\|_1,
\]

(3)

where \( \left\| \cdot \right\|_1 \) denotes the L1 norm on \( \mathbb{R}^m \).

We denote by \( c_{t-1} = S_{t-1}/S_{t-1} \) the proportion of the wealth after adjusting to the wealth at the end of last period. Dividing equation (3) by \( S_{t-1} \), we get

\[ 1 = c_{t-1} + \eta \left\| \hat{b}_{t-1} - c_{t-1} b_t \right\|_1. \]

At the end of the period \( t \), the cumulative return is

\[ S_t = S_{t-1} \times c_{t-1} \times (b_t \cdot x_t). \]

While constructing portfolio selection algorithms, the turnover is also an important factor to be concerned, which is directly related to the transaction costs. The average turnover of \( n \) trading periods can be defined as

\[ \bar{T} = \frac{1}{2n} \sum_{t=1}^{n} \left\| b_{t-1} - c_{t-1} b_t \right\|_1. \]

The objective of online portfolio selection is to determine a sequence of portfolios \( \{b_t\}_{t=1}^{n} \) to make the final cumulative return \( S_n \) as much as possible.

To concentrate on the key issue discussed in this paper, we simplify the financial market with the following assumptions. First, we assume transaction costs are proportional to the turnover, and the transaction cost rates of buying and selling are the same. Second, we can buy and sell any desired amount and our trading behaviours will not impact the market.

The procedure of the online portfolio selection problem with transaction costs is presented in Algorithm 1.
Reversion strategy for online portfolio selection with transaction costs

Algorithm 1  Online portfolio selection with transaction costs

Input: \( x_1, x_2, \ldots, x_n \) : Price relative sequence; \( \eta \) : Rate of transaction costs.
Output: \( S_n \) : Final cumulative wealth.
1: Initialise \( S_0 = 1, b_1 = (\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m})^\top; b_0 = (0, 0, \ldots, 0)^\top; \)
2: for \( t = 1, 2, \ldots, n \) do
3: Adjust according to \( b_t \); Calculate \( c_{t-1} \) by solving the following equation
\[
1 = c_{t-1} + \eta \|b_{t-1} - c_{t-1}b_t\|_1;
\]
4: Receive price relatives: \( x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,m})^\top; \)
5: Update the cumulative return:
\[
S_t = S_{t-1} \times c_{t-1} \times (b_t \cdot x_t);
\]
6: Update closing price adjusted portfolio \( b_t \), where \( b_{t,i} = \frac{b_{t-1,i} + x_{t,i}}{b_{t-1,i} \cdot x_{t,i}}; \)
7: Determine portfolio \( b_{t+1} \).
8: end for

3 Related works

3.1 Benchmarks

Several benchmarks are often used to estimate whether a portfolio selection algorithm is remarkable. One simple benchmark is buy-and-hold (BAH) strategy, which means one invests his/her wealth among a pool of stocks with an initial portfolio and does not rebalance it later. A BAH strategy is called uniform BAH (U-BAH) strategy if its initial portfolio is \( b_1 = (1/m, \ldots, 1/m)^\top \). U-BAH is adopted as market strategy in this paper as in many literatures (Borodin et al., 2004; Li et al., 2012, 2015). We say that a strategy ‘beats the market’ if it outperforms the U-BAH strategy. The optimal BAH strategy, which only invests in the best stock, is denoted by best in this paper.

An alternative strategy to the static BAH strategy is constant rebalanced portfolios (CRP), which actively rebalances wealth among the stocks at the beginning of each trading period to keep a fixed portfolio. The best CRP (BCRP) strategy keeps the most profitable portfolio among all CRPs; unfortunately, it is only a strategy in hindsight.

3.2 Existing works

The portfolio selection problem has been extensively studied in various fields, which generated many strategies. One important type of strategies is universal strategies, which asymptotically approaches the same exponential growth rate as the BCRP strategy for arbitrary sequences of price relatives. The regret of an online algorithm \( Alg \) is defined as the gap between its logarithmic cumulative wealth and that of BCRP, that is,
\[
\text{Regret}_n(Alg) = \sum_{t=1}^{n} \log(b^* \cdot x_t) - \sum_{t=1}^{n} \log(b_t \cdot x_t).
\]
An algorithm $Alg$ is universal if for an arbitrary market sequence $\{x_t\}_{t=1}^{\infty}$, it holds that
\[
\lim_{n \to \infty} \frac{1}{n} \text{Regret}_n(Alg) \rightarrow 0.
\]

Cover (1991) proposed the first universal strategy UP (Universal Portfolios), where the portfolio on each trading period is the historical performance weighted average of all CRP experts in the simplex domain. For UP is time-consuming with time complexity of $O(n^m)$, Kalai and Vempala (2003) proposed a relatively time-efficient implementation for UP, which requires running time $O(m^7n^8)$. To improve the regret bound of UP, Hazan and Kale (2015) presented an algorithm with the regret bound of $O(\log Q)$, where $Q$ is the quadratic variation of the stock prices. For UP takes a long time to produce significant growth, O’Sullivan and Edelman (2015) proposed an adaptive universal portfolio algorithm, which retains much of the qualitative nature of UP while enhancing its early performance. Helmobold et al. (1998) proposed another classic universal strategy Exponential Gradient (EG), which tries to maximise the expected logarithmic portfolio daily return and minimise the deviation between portfolios on the neighbouring periods. Convex optimisation has also been applied to resolve the portfolio selection problem. Agarwal (2006) proposed online Newton step (ONS) strategy, which exploits the second order information of the log wealth function, and aims to maximise the expected log cumulative wealth and minimise the variation of the expected portfolio. Dochow et al. (2014) designed a risk-adjusted strategy incorporating the ‘trading risk’ in terms of the maximum observed fluctuation of the period wealth up to current trading period. Mohr and Dochow (2016) considered the portfolio selection problem from the perspective of online algorithms that process input piece-by-piece in a serial fashion, and proposed two risk-adjusted algorithms which incorporate risk management. Based on a machine learning algorithm, Li et al. (2016) designed a method that adaptively switches between two distinct asset allocation strategies. Zhang and Yang (2017) considered CRPs as expert advice and constructed a universal portfolio strategy WAAC by using weak aggregating algorithm (WAA).

One underlying assumption for the optimality of BCRP is that stock returns are i.i.d., which may not be satisfied. Although BCRP can achieve maximum performance theoretically (Cover, 1991), it cannot achieve outstanding performance in backtests, as empirical stock returns violate the i.i.d. assumption (Li et al., 2015). Although universal strategies asymptotically approach the growth rate of BCRP, they often perform worse than BCRP.

Another kind of strategies exploits natural volatilities in the stock markets, especially the mean reversion property. One simple strategy of this category is CRP, which tends to transfer wealth from the high performing stocks to the relatively low performing ones. Inspired by this property, Borodin et al. (2004) proposed anti-correlation (Anticor) strategy, which benefits from statistical properties of positive lagged cross-correlation and negative auto-correlation. Li et al. (2012) presented passive aggressive mean reversion (PAMR) strategy, which employs the idea of online passive aggressive learning. The basic principle of PAMR is to hold poorly performing portfolios passively and adjust well-performing portfolios aggressively, because an unprofitable portfolio on current period generally indicates that most of wealth is distributed in low-performing stocks, which will profit on the next period and vice versa. In detail, if $b_{t-1} \cdot x_{t-1} \leq \epsilon$, keep the current portfolio; otherwise, select a portfolio based on the following optimisation problem,
Reversion strategy for online portfolio selection with transaction costs

\[
b_t = \arg \min_{b \in \Delta_m} \frac{1}{2} \| b - b_{t-1} \|^2 \quad \text{s.t.} \quad b \cdot x_{t-1} \leq \epsilon,
\]

where \( \epsilon \geq 0 \) is a parameter.

Though outstanding in experiments on most datasets, PAMR would lose its superiority when the single-period mean reversion assumption is not satisfied. For example, PAMR performs relatively poorly on the dataset DJIA among state-of-the-art strategies, while Anticor, which exploits the multi-period statistical correlation, performs relatively well. To better take advantage of the mean reversion property, Li et al. (2015) proposed online moving average reversion (OLMAR), which assumes that the price on the next period will revert to moving average (MA) of historical prices, rather than the price on last period simply. Li et al. (2015) introduced two types of MAs for OLMAR, which can more effectively take advantage of the mean reversion phenomenon than PAMR empirically. Huang et al. (2016) exploited multiple period reversion via robust L1-median estimator, which can better handle noises in the price relative data. Lin et al. (2017) considered portfolio selection problem from a meta learning perspective and proposed a boosting method for price relative prediction, which integrates mean reversion strategies with various window sizes.

Though the above strategies are profitable theoretically or empirically, their profitabilities are based on the precondition of zero transaction cost. Ignoring the transaction cost issue, the strategies mentioned above, especially the mean reversion ones, tend to massively transfer wealth between stocks. However, a mass of turnover would lead to vast transaction costs, which reduces the cumulative wealth. Only a few existing algorithms take into account transaction costs. Blum and Kalai (1999) proved that UP is still universal when proportional transaction costs are considered; however, it does not take the transaction costs into account when determining the portfolio of each period. Albeverio et al. (2001) presented a strategy which trades off between predictive wealth increment and transaction costs and ensures investors to achieve at least the same exponential growth rate of wealth as that of the best stock for a long term theoretically. Iyengar (2005) proposed a universal strategy with transaction costs. However, the above two strategies are limited to investments among two stocks. Das et al. (2013) introduced online lazy updates (OLU) algorithm, which takes into account the transaction costs while computing an optimal portfolio and results in sparse updates to the portfolio vector. Bean and Singer (2012) developed a strategy that takes into account both side information and transaction costs. To solve the transaction cost issue caused by frequently adjusting portfolio in every period, Huang et al. (2015) attempted to avoid rebalancing when the transaction cost outweighs the benefit of trading. Tunc et al. (2013) reallocated the distribution of the funds only if the portfolio breaches certain thresholds. However, to the best of our knowledge, none of mean reversion-based algorithms consider the transaction cost issue, which leads to rapid weakening of profitabilities as the rate of transaction costs increases.

To effectively exploit the mean reversion property and avoid a large amount of transaction costs, for one thing, our proposed strategy employs the property of multi-period mean reversion; for another, it preferentially transfers capital between stocks which are more likely to revert and adaptively adjusts the turnover. The detail of the algorithm is presented in Section 4.
4 Mean reversion strategy with transaction costs

4.1 Motivation

The first motivation of this investigation is the mean reversion principle, which assumes that stocks with poor performances will perform well in the subsequent trading periods. The state-of-the-art algorithms (Borodin et al., 2004; Li et al., 2012, 2013, 2015) show that the mean reversion principle is valid empirically as they bring about fat profit. Table 1 presents a simple example which exhibits the profitability of CRP, a strategy adopting the mean reversion trading idea. Consider a market of two stocks A and B, with a sequence of price relatives (1/2, 2), (2, 1/2), (1/2, 2), .... At the beginning of each trading period, the wealth is rebalanced according to portfolio $\left(\frac{1}{2}, \frac{1}{2}\right)$, which results in daily return of $\frac{5}{4}$ (ignoring transaction costs). After $t$ periods, the cumulative return reaches $(\frac{5}{4})^t$, though the price of each stock is unchanged on the whole when $t$ is an even number.

<table>
<thead>
<tr>
<th>Period</th>
<th>Price relative</th>
<th>Portfolio</th>
<th>Cumulative return</th>
<th>Wealth proportion</th>
<th>Transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1/2, 2)</td>
<td>(1/2, 1/2)</td>
<td>5/4</td>
<td>(1/5, 4/5)</td>
<td>B2A</td>
</tr>
<tr>
<td>2</td>
<td>(2, 1/2)</td>
<td>(1/2, 1/2)</td>
<td>$(5/4)^2$</td>
<td>(4/5, 1/5)</td>
<td>A2B</td>
</tr>
<tr>
<td>3</td>
<td>(1/2, 2)</td>
<td>(1/2, 1/2)</td>
<td>$(5/4)^3$</td>
<td>(1/5, 4/5)</td>
<td>B2A</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Table 1: An example of mean reversion phenomenon

Though existing mean reversion-based strategies perform better than many benchmarks, they are far from being applied in real market, because when considering transaction costs, their profitabilities decline quickly, which inspires our investigation.

4.2 Formulation

To construct our algorithm, we adopt the mean reversion principle because of its profitability revealed by the existing related algorithms. Instead of exploiting the single-period mean reversion property as PAMR (Li et al., 2012), we propose a strategy for the multi-period reversion. What is more, our algorithm adaptively adjusts the turnover so as to withstand high transaction cost rates. This handles the issue that existing mean reversion algorithms often actively transfer capital between stocks leading to massive transaction costs. The pseudocode of our strategy MRTC is presented in Algorithm 2.

The decision-making process of the proposed strategy mainly consists of two steps. First, measure the recent performance of each stock by a proper metric with historical price relatives. Second, calculate the portfolio according to the recent performances of stocks.
Algorithm 2  Mean reversion strategy with transaction costs: MRTC

Input: $x_1, x_2, \ldots, x_n$ : Price relative sequence; $\eta$ : Rate of transaction costs; $\alpha$ : Weight.
Output: $S_n$ : Final cumulative wealth.

1: Initialise $S_0 = 1$, $b_1 = (\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m})^\top$, $b_0 = (0, 0, \ldots, 0)^\top$;
2: for $t = 1, 2, \ldots, n$ do
3:    Adjust according to $b_t$: Calculate $c_{t-1}$ by solving the following equation
4:        $1 = c_{t-1} + \eta \|b_{t-1} - c_{t-1} b_t\|_1$;
5:    Receive price relatives: $x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,m})^\top$;
6:    Update the cumulative return:
7:        $S_t = S_{t-1} \times c_{t-1} \times (b_t \cdot x_t)$;
8:    Update closing price adjusted portfolio $b_t$, where $\hat{b}_{t,i} = \frac{b_{t,i} x_{t,i}}{b_t^\top x_t}$;
9:    Select value of $r_{t+1}$:

$$r_{t+1} = \arg \max_{r \in M} \prod_{r=1}^t b_r^\top \cdot x_r,$$

where $b_r^\top = \text{PS}(x_1, x_2, \ldots, x_{r-1}, b_{r-1}, r, \alpha)$ and $M = \{0, 1, \ldots, m - 1\}$;

9: end for

We improve the stock performance metric of the strategy PAMR. PAMR implicitly assumes that price relatives will revert on the next period, that is, better-performing stocks will perform worse on the next period and vice versa. However, this assumption would lead to two limitations. First, PAMR would lose its effectiveness if consecutive increasing or decreasing of stock prices occurs, which is common in financial markets. Second, it cannot efficiently make use of information from the historical data. For example, consider two stocks A and B with price relative sequence $x_{t-3} = (0.9, 1.1)^\top$, $x_{t-2} = (0.9, 1.1)^\top$ and $x_{t-1} = (1.1, 1.1)^\top$ of recent three periods. Intuitively, according to the mean reversion property, stock B is more likely to revert than A because the price of B has increased for a long time with a large amount. However, PAMR just uses $x_{t-1}$ and ignores the information revealed in previous periods, and thus treats these two stocks in the same way. To make use of this information, we assume that recently (not just on current period) better-performing stocks will perform worse on the next period and vice versa. Here the key issue is to measure the recent performances of stocks in a proper way. To this end, we propose two metrics which generate performance vector $x_{t-1} = (\bar{x}_t, \bar{x}_{t-1}, \bar{x}_{t-2}, \ldots, \bar{x}_{t-m})^\top$. The first metric, see equation (4), is the weighted geometric mean of historical price relatives. For the second metric, see equation (5), we employ the weighted mean of multi-period price relatives to measure recent performances of stocks. We define $\bar{x}_{t-1}$ as the one-period price relative and $\bar{x}_{t-3,i}$ the two-period price relative of the $i$th stock on the $(t - 1)^{th}$ period and so on. Moreover, the data farther from current period is less important, so our metrics employ decreasing weights sequence $\alpha, (1 - \alpha)\alpha, \ldots$. 


Considering the above example with equation (5), if where \( r \) to represent the number of stocks to be sold at the top of the sorted stock sequence, and that of the last one is most likely to increase intuitively. Thus the first stock should \( x \) which are more likely to revert. For example, the stock which recently performed the
Algorithm 3. To avoid massive turnover, we just transfer the wealth between stocks portfolio for the next trading period. The pseudocode of this step is presented in
recently according to the second metric (also the first metric), so our strategy does not
treat stocks A and B in the same way.

These metrics can explore more information from historical price relatives than PAMR. Considering the above example with equation (5), if \( \alpha = 0.5 \), we obtain \( \bar{x}_{t-1,A} = 1.02025 \) and \( \bar{x}_{t-1,B} = 1.18525 \). Obviously, stock B performed better than stock A recently according to the second metric (also the first metric), so our strategy does not treat stocks A and B in the same way.

After measuring recent performances of stocks, the next step is to determine a portfolio for the next trading period. The pseudocode of this step is presented in Algorithm 3. To avoid massive turnover, we just transfer the wealth between stocks which are more likely to revert. For example, the stock which recently performed the best or the worst is more likely to revert than other stocks. We sort the stocks based on \( \bar{x}_{t-1} \) in descending order, and then the price of the first stock is most likely to decrease and that of the last one is most likely to increase intuitively. Thus the first stock should be sold and the last one should be bought preferentially. Here, we adopt a parameter \( r \) to represent the number of stocks to be sold at the top of the sorted stock sequence, where \( r \) can be set to 0, 1, . . . or \( m - 1 \) in a market with \( m \) stocks. For example, \( r = 2 \) indicates that the first two stocks will be sold. Our strategy transfers all the wealth of these \( r \) stocks to the worst stock. This method appears to be valid in later numerical experiments.

**Algorithm 3**  Portfolio Selection of MRTC: PS \((x_1, x_2, ..., x_{t-1}, b_{t-1, r, \alpha})\)

<table>
<thead>
<tr>
<th>Input:</th>
<th>(x_1, x_2, ..., x_{t-1}): Price relative sequence; (b_{t-1}): Closing price adjusted portfolio; (r): Number of stocks whose wealth will be transferred; (\alpha): Weight.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output:</td>
<td>(b_t): Portfolio of the (t^{th}) trading period.</td>
</tr>
</tbody>
</table>

1. Compute \(\bar{x}_{t-1}\) according to Metric 1 or Metric 2 and obtain \(x_{t-1} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_m)\);
2. Sort \(\bar{x}_{t-1}\) in descending order and obtain \(\bar{x}_t = (\bar{x}_{k_1}, \bar{x}_{k_2}, ..., \bar{x}_{k_m})\);
3. Set \(b_{t,k_i} = \begin{cases} 0, & i = 1, 2, ..., r; \\ \frac{b_{t-1,k_i}}{i = r + 1, r + 2, ..., m - 1}; \\ \sum_{j=1}^{r} b_{t-1,k_j}, & i = m. \end{cases} \)

The amount of transaction costs is determined by the rate of transaction costs and the turnover. A large value of \(r\) leads to a mass of turnover and a small one appears to be conservative. In other words, a too large or too small value of \(r\) would result in less cumulative return. Thus we intend to select proper values of \(r\) to control the turnover.
Reversion strategy for online portfolio selection with transaction costs

so as to achieve more cumulative wealth. For a certain market, we assume that a value of $r$ which achieved high return will keep its superiority in the future. So we select the values of $r$ adaptively with historical cumulative returns they brought about. In historical trading, we run Algorithm 3 with every possible value of $r$ and select the optimal value which brings about the largest amount of accumulative wealth. The selection of $r$ on the $t^{th}$ period can be formulated as

$$r_t = \arg \max_{r \in M} \prod_{\tau=1}^{t-1} b^{r}_{\tau} \cdot x_{\tau},$$

(6)

where $b^{r}_{\tau} = PS(x_1, x_2, \ldots, x_{\tau-1}, \hat{b}_{\tau-1}, r, \alpha)$, and $M = \{0, 1, \ldots, m - 1\}$.

5 Numerical experiments

In this section, we conduct numerical experiments on the proposed algorithm and several state-of-the-art algorithms, and then present comparative results of them.

5.1 Datasets

In these experiments, we employ four datasets used in many literatures (Borodin et al., 2004; Li et al., 2012, 2015), which were collected from several diverse financial markets. Brief information of these datasets is listed in Table 2.

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Market</th>
<th>Region</th>
<th>Time frame</th>
<th>Periods</th>
<th>Assets</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSCI</td>
<td>Index</td>
<td>Global</td>
<td>1 Apr. 2006–31 Mar. 2010</td>
<td>1,043</td>
<td>24</td>
</tr>
</tbody>
</table>

The first dataset NYSE(O) was collected from New York Stock Exchange (NYSE) containing 5,651 daily price relatives of 36 stocks from 3 July 1962 to 31 December 1984. The second dataset NYSE(N) was collected by Li et al. (2012) with price relatives of 6,431 trading days from 1 January 1985 to 30 June 2010. The third dataset TSE was collected by Borodin et al. (2004) from Toronto Stock Exchange (TSE), which consists of 1,259 daily price relatives of 88 stocks ranging from 4 January 1994 to 31 December 1998. The final dataset MSCI is a collection of global equity indices from the MSCI World Index, and includes price relatives of 24 indices of 1,043 periods from 1 April 2006 to 31 March 2010.

5.2 Comparison algorithms

We conduct experiments to compare two realisations of our strategy, MRTC1 and MRTC2 (based on metric 1 and metric 2, respectively), with benchmarks and some classic existing strategies. The parameters of existing strategies are set according to their original studies as listed below.
1 Market: uniform buy-and-hold (BAH) strategy
2 Best: the best stock in the dataset which gains the maximum cumulative return
3 BCRP: best constant rebalanced portfolios strategy in hindsight
4 UP: cover’s universal portfolios implemented according to Kalai and Vempala (2003), where the parameters are set as $\delta_0 = 0.004, \delta = 0.005, m = 100, S = 500$
5 EG: exponential gradient algorithm with the best learning rate $\eta = 0.05$ according to Helmbold et al. (1998)
6 ONS: online Newton step with the parameters suggested by Agarwal (2006), that is, $\eta = 0, \beta = 1, \gamma = 1/8$
7 Anticor: BAH$_{30}$ (Anticor (Anticor)) proposed by Borodin et al. (2004)
8 PAMR: passive aggressive mean reversion algorithm with $\epsilon = 0.5$ as suggested by Li et al. (2012)
9 OLMAR: online moving average reversion (Li et al., 2015) with $\alpha = 0.5$ and $\epsilon = 10$.

For our strategy, the values of $r_t$ ($t = 2, \ldots, n$) are adaptively determined according to equation (6), which is time-consuming in numerical experiments if $m$ is too large. To handle this problem, we replace the set $M$ in equation (6) with its subset $\{\lfloor \theta m \rfloor; \theta = 0, 0.1, \ldots, 0.9\}$ if $m > 10$, where $\lfloor \cdot \rfloor$ denotes the floor function.

There is only one parameter $\alpha$ to be set artificially in the proposed algorithm. For these two realisations, we set $\alpha = 0.3$ in all experiments except parameter sensitivity analysis. We adopt the final cumulative return to measure the performances of strategies.

5.3 Cumulative returns with fixed transaction cost rates

The above algorithms are evaluated with rates of transaction costs 0%, 0.25% and 0.5%, and the cumulative returns are shown in Table 3. As can be seen in Table 3, though guaranteed theoretically to strategy BCRP, the universal strategies (e.g., UP, EG) fall behind BCRP with large gaps. On the contrary, the state-of-the-art strategies based on the mean reversion principle often perform much better than BCRP. However, existing mean reversion strategies (e.g., PAMR, OLMAR) lose their wealth rapidly as the rate of transaction costs increases. Our algorithm considers transaction costs during trading, and thus can efficiently avoid this limitation. For one thing, with zero transaction cost, our strategy can perform as well as the best existing strategy. For another, our algorithm shows its advantage when the rate of transaction costs increases and even beats BCRP strategy on two datasets when the rate reaches 0.5%. Besides, our strategy can still profit in a lost market MSCI even with the rate of 0.5%.
### Table 3
Cumulative returns with different transaction cost rates

<table>
<thead>
<tr>
<th>Algorithms</th>
<th>NYSE(O)</th>
<th>NYSE(N)</th>
<th>TSE</th>
<th>MSCI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0%</td>
<td>0.25%</td>
<td>0.5%</td>
<td>0%</td>
</tr>
<tr>
<td>Market</td>
<td>14.50</td>
<td>14.46</td>
<td>14.42</td>
<td>18.06</td>
</tr>
<tr>
<td>Best</td>
<td>54.14</td>
<td>54.00</td>
<td>53.87</td>
<td>83.51</td>
</tr>
<tr>
<td>BCRP</td>
<td>250.60</td>
<td>182.01</td>
<td>132.20</td>
<td>120.32</td>
</tr>
<tr>
<td></td>
<td>14.50</td>
<td>14.46</td>
<td>14.42</td>
<td>18.06</td>
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<tr>
<td></td>
<td>54.14</td>
<td>54.00</td>
<td>53.87</td>
<td>83.51</td>
</tr>
<tr>
<td></td>
<td>250.60</td>
<td>182.01</td>
<td>132.20</td>
<td>120.32</td>
</tr>
<tr>
<td>UP</td>
<td>26.68</td>
<td>17.59</td>
<td>11.36</td>
<td>31.49</td>
</tr>
<tr>
<td>EG</td>
<td>27.09</td>
<td>23.08</td>
<td>19.66</td>
<td>31.00</td>
</tr>
<tr>
<td>ONS</td>
<td>109.91</td>
<td>51.47</td>
<td>24.26</td>
<td>21.59</td>
</tr>
<tr>
<td>Anticor</td>
<td>2.41E+8</td>
<td>6.37E+4</td>
<td>16.68</td>
<td>6.21E+6</td>
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<tr>
<td>PAMR</td>
<td>5.01E+15</td>
<td>2.04E+5</td>
<td>0.00</td>
<td>1.26E+6</td>
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<td>1.02E+18</td>
<td>2.00E+8</td>
<td>0.04</td>
<td>4.69E+8</td>
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<tr>
<td>MRTC1</td>
<td>2.21E+18</td>
<td>2.84E+9</td>
<td>6.26E+4</td>
<td>7.67E+8</td>
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<tr>
<td>MRTC2</td>
<td>3.69E+18</td>
<td>2.25E+9</td>
<td>1.33E+4</td>
<td>7.13E+8</td>
</tr>
</tbody>
</table>

Note: The top two results in each column, except results of strategies in hindsight, are highlighted in bold.
<table>
<thead>
<tr>
<th>Statistics</th>
<th>NYSE(O)</th>
<th></th>
<th>NYSE(N)</th>
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<th>TSE</th>
<th></th>
<th>MSCI</th>
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<td>0.25%</td>
<td>0.5%</td>
<td></td>
<td>0%</td>
<td>0.25%</td>
<td>0.5%</td>
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<td>MER(MRTC)</td>
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<td>0.0043</td>
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<td>0.0037</td>
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<td>0.0011</td>
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<tr>
<td>MER(Market)</td>
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<td>0.0004</td>
<td>0.0004</td>
<td></td>
<td>0.0004</td>
<td>0.0004</td>
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<td>α</td>
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<td>0.0033</td>
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<td>0.0007</td>
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<td>β</td>
<td>1.3052</td>
<td>1.2775</td>
<td>1.1126</td>
<td></td>
<td>1.1319</td>
<td>1.098</td>
<td>1.097</td>
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<td>5.0912</td>
<td></td>
<td>7.8229</td>
<td>3.4158</td>
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<tr>
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<td>3.4158</td>
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<tr>
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<td>0.0000</td>
<td>0.0006</td>
<td>0.0140</td>
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<tr>
<td>t-statistics</td>
<td>16.68</td>
<td>8.5915</td>
<td>5.0912</td>
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<td>7.8229</td>
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<tr>
<td>p-values</td>
<td>0.0000</td>
<td>0.0000</td>
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<td></td>
<td>0.0000</td>
<td>0.0006</td>
<td>0.0140</td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Statistical t-test of MRTC1
To analyse if the cumulative returns generated by our strategy are obtained with just simple luck, we conduct a statistical test. Daily return $\gamma_p(t)$ of portfolio can be divided into the benchmark component $\beta_p\gamma_b(t)$ and the residual component $\alpha_p + \epsilon_p(t)$ by regressing the portfolio excess returns against the benchmark excess returns, that is, $\gamma_p(t) = \alpha_p + \beta_p\gamma_b(t) + \epsilon_p(t)$. Alpha appears in Figure 1 as the intercept of the regression line with the vertical axis, and beta is the slope of the regression line. The regression analysis provides us with confidence intervals for estimates of $\alpha_p$ and $\beta_p$. The $t$ statistic for the $\alpha_p$ provides a rough test of its statistical significance. A rule of thumb is that a $t$ statistic of 2 or more indicates that the performance of a strategy is due to skill rather than luck. The probability of observing such a large alpha by chance is only 5%, assuming normal distributions (Grinold, 1999).

The statistical $t$-test results of MRTC1 are presented in Table 4. According to $p$-values, performances of MRTC1 on all datasets are not due to luck when ignoring transaction costs. Though still significant on datasets NYSE(O) and NYSE(N) in the case of the rate of 0.5%, alpha decreases as the rate increases. When the rate is greater than 0.25%, alpha is not significant on datasets TSE and MSCI, which is consistent with the cumulative returns in Table 3.

### 5.4 Cumulative returns with varying transaction cost rates

To better illustrate the superiority of our strategy when considering transaction costs, we further employ Figure 2 to compare the changes of cumulative wealth of MRTC, Market, BCRP and two mean reversion strategies, PAMR and OLMAR, which can exploit great wealth without transaction costs. As can be seen from Figure 2, MRTC1 and MRTC2 perform as well as PAMR and OLMAR when transaction cost rate is zero. They obviously beat PAMR and OLMAR when the rate reaches 0.2% on most datasets, and this superiority becomes more distinct as the rate increases. Besides, MRTC can still beat the hindsight strategy BCRP when the rate is 0.2% and beat the market in the case of 0.5%. In summary, MRTC can efficiently handle the issue of rapidly decreasing cumulative wealth of existing mean reversion strategies when the rate increases.
5.5 Average turnovers with varying transaction cost rates

Existing mean reversion-based portfolio selection algorithms tend to aggressively exploit the mean reversion property by moving capital substantially, which leads to vast transaction costs. To solve this problem, our proposed algorithm MRTC adaptively
controls the turnover with a parameter \( r \). Figure 3 presents the comparison of the average turnovers of MRTC1 and MRTC2 with those of PAMR and OLMAR. Average turnovers of PAMR and OLMAR maintain high levels with varying rates of transaction costs, because formulations of them do not take transaction costs into account; while average turnovers of MRTC1 and MRTC2 decrease significantly, which indicates that they can reduce the turnovers when the rate of transaction costs rises.

5.6 Cumulative returns with varying values of parameter \( \alpha \)

When measuring performances of stocks, metric 1 and metric 2, see equations (4) and (5), involve in parameter \( \alpha \), which is set to 0.3 in the above experiments. Now we show the sensitivity of cumulative wealth to \( \alpha \) in Figure 4. As is revealed in Figure 4, when \( \alpha = 0 \) the cumulative returns are the minimum ones on all datasets, because in this case, performances of stocks are always evaluated by their price relatives on the first period. The cumulative returns rise with a large amount when \( \alpha \) changes from 0 to 0.1. When \( \alpha \) varies between 0.1 and 1, the cumulative returns are stable and none of the values is absolutely optimal on all datasets. In general, cumulative returns are not sensitive to the parameter \( \alpha \).

Figure 4  Cumulative returns with varying values of parameter \( \alpha \), (a) NYSE(O) (b) NYSE(N) (c) TSE (d) MSCI (see online version for colours)

5.7 Comparison of adaptive MRTC and MRTC with fixed \( r \)

For our strategy, the values of \( r_t \) (\( t = 2, ..., n \)) are selected according to equation (6) adaptively, instead of artificially. To illustrate the advantage of this process, we compare MRTC with MRTC(\( \theta \)), as presented in Figure 5. Here, MRTC(\( \theta \)) represents the strategy MRTC with fixed \( r = \lfloor \theta m \rfloor \). MRTC with large value of \( r \), e.g., MRTC(0.8), appears to
be aggressive and performs well when the rate of transaction costs is low. However, it loses this superiority when the rate of transaction costs increases and falls behind MRTC with small value of \( r \). On the contrary, MRTC with small \( r \), e.g., MRTC(0.2), is passive and performs well under large rates of transaction costs, while performs poorly under low rates. In summary, different values of \( r \) are suitable for different rates of transaction costs and for a certain transaction cost rate, we do not know which value of \( r \) is suitable at the beginning of the investment. So we choose the values of \( r \) adaptively. Figure 5 illustrates that the adaptive MRTC performs better than the MRTC with fixed \( r \) under most cases, which indicates the effectiveness of our strategy.

Figure 5  Cumulative returns of adaptive MRTC and MRTCs with fixed \( r \), (a) NYSE(O) (b) NYSE(N) (c) TSE (d) MSCI (see online version for colours)

6 Conclusions

In this paper, we present an online portfolio selection strategy which exploits the multi-period mean reversion property and considers the issue of transaction costs. Since the amount of transaction costs is determined by the rate of transaction costs and the turnover, we proposed a method to adaptively adjust the turnover. This method can efficiently overcome the shortage of existing mean reversion strategies that their aggressively transferred wealth leads to abundant transaction costs and less cumulative return. Experiments show that the proposed strategy performs best in most cases of datasets or the rates of transaction costs.

However, there are still some limitations of our proposed strategy. First, we just consider proportional transaction costs, whereas actual transaction cost models may be various. Second, in some datasets, although our strategy beats the market under high rates of transaction costs, but it is not significant as shown by statistical \( t \)-test result. Finally, our strategy achieves attractive final cumulative returns just experimentally.
without theoretical guarantee. The above limitations are worth of further investigations in the future.

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References


Reversion strategy for online portfolio selection with transaction costs


