A rule-based approach for dynamic analytic hierarchy process decision-making

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Abstract: The analytic hierarchy process (AHP) is widely used in many multi-criteria decision-making problems and has been successfully applied to many practical cases. However, the AHP is time-consuming and the decision model is not agile enough for fast changing environment. To overcome this weakness, we develop a rule-based approach for dynamic AHP decision-making in changing environment. We analyse critical factors in the AHP decision process under uncertainty and propose to encode expert knowledge for change handling using event-condition-action rules. We propose a theorem and associated method to determine the change in ordering of decision alternatives based on event-condition-action rule-induced weight updates. We demonstrate the effectiveness of our approach using a case study of the supplier selection decision-making task of the steel and iron industry in Taiwan. The study shows that our mechanism can effectively reach the same level of decision quality as expert decision maker(s).

Keywords: dynamic rule-based AHP; two criteria analysis; steel and iron industry; comparison matrix.

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1 Introduction

Decision-making under fast changing environment is becoming more and more challenging due to the need for quick response and right judgement based on complex and dynamic weighting of multiple decision criteria. Effective decision makers are capable of making decisions under different types of decision-making environments which can be certain, uncertain or at risk (Damghani et al., 2009). For example, a supplier selection problem is a multi-criteria decision-making (MCDM) problem involving both quantitative and qualitative criteria that are highly sensitive to the economy and market status (Zeydan et al., 2011). Right decision at the right time can significantly reduce the purchasing risk, maximise profit and minimise cost.

There are lots of methods available in the literature for the decision-making problems including techniques for ordering preferences by similarity to an ideal solution (TOPSIS) (Shih et al., 2007; Singh et al., 2012; Srikrishna et al., 2014; Zavadskas et al., 2016), fuzzy TOPSIS (Jolai et al., 2011; Zouggari and Benyoucef, 2012), AHP (Peng, 2012; Deng et al., 2014; Salman, 2017), fuzzy AHP (Kahraman et al., 2003; Che et al., 2010; Kilincci and Onal, 2011; Chen and Chao, 2012; Rezaei and Ortt, 2013; Sasirekha et al., 2017), the analytic network process (ANP) (Bahurmoz, 2006; Lin et al., 2010), fuzzy ANP (Dargi et al., 2014), fuzzy logic (Malay, 2011), AHP/ANP (Saaty 2004, 2005), case-based reasoning (CBR) (Wang et al., 2008), ELECTRE (Birgün and Cihan, 2010), data envelopment analysis (DEA) (Liu et al., 2005; Zhang et al., 2005; Che et al., 2010; Gavgani et al., 2017), the multi attribute utility technique (MAUT) (Nadeem et al., 2014), mathematical programming (MP) (Hadi-Vencheh and Moghadam, 2011), grey systems theory (Memon et al., 2015) and preference ranking organisation method of enrichment evaluation (PROMETHEE) (Frikha et al., 2017) etc.

AHP is one of the major techniques in dealing with the MCDM problems that was originally developed by Professor Thomas L. Saaty in 1976. It is based upon making pair-wise comparisons of relative importance of decision criteria in order to achieve the proper ranking of the decision alternatives. It is helpful for decision makers to structure the problems, conduct analysis, and rank the alternatives (Shih et al., 2007; Jiang et al., 2011). Due to its logicality, rationality, and computational simplicity, AHP has been widely applied to the research of evaluation and alternative selection as well as risk analysis problems (Kahraman et al., 2003; Jiang et al., 2011). Additionally, because of its flexibility, it can be integrated with other methods, e.g., quality function deployment (Bhattacharya et al., 2005; Vaidya and Kumar, 2006; Rajesh and Malliga, 2013), data envelopment analysis (Zhang et al., 2005), meta-heuristics (Rad and Abedi, 2008) and SWOT analysis (Kurttila et al., 2000; Shrestha et al., 2004), etc (William 2008; Diamantopoulos et al., 2012). This enables the users to benefit from the advantages of combined methods, and hence, achieve the desired goal in a better way (Vaidya and Kumar, 2006).

However, AHP has some shortcomings. For example, if the decision-maker(s) change any values in the pairwise comparison matrix, the matrix need to be re-evaluated which can significantly affect the entire decision structure. Moreover, the original AHP is time consuming and therefore not adaptive to the dynamics of its context. Without proper mechanisms for handing changes, AHP is unsuitable for fast changing business environment such as the iron and steel industry in our case study.

In this paper, we propose a framework for dynamic AHP decision-making based on the integration of event-condition-action (ECA) rules and traditional AHP decision structure. More specifically, we analyse critical factors in AHP decision-making under uncertainty and develop a theorem which completely characterises the effect of the change in relative weights of a pair of criteria on the final ranking of alternatives. The ECA rules are then used to represent decision maker's knowledge on how the relative weights of a pair of decision criteria should be adjusted under various conditions. Given the ECA rule-induced weight updates, our theorem and associated decision methods can immediately determine the effect of updates on the final ranking of alternatives. We demonstrate the effectiveness of our approach using a case study of the supplier selection decision-making problem of the steel and iron industry in Taiwan. In particular, we compare the decision made by human experts with the suggestion made by our technique. The study shows that our mechanism can effectively reach the same level of decision quality as expert decision makers.

The rest of the paper is organised as follows. Section 2 reviews the AHP literature and related work for the proposed methodology. In Section 3, we analyse the AHP decision model and propose a new ECA rule-based framework for dynamic AHP decision-making. Section 4 presents the details of the theorem for impact analysis of two criteria weight updates on the final ranking of alternatives. The theorem lays the foundation for our dynamic AHP approach. In Section 5, we conduct a case study and a comparison of the decision quality of our approach with that of expert decision maker on supplier selection for a steel bar manufacturer in Taiwan. Section 6 concludes the paper.

2 Related work

AHP is a process of conducting pair-wise comparison of decision criteria, evaluating their consistency, and measuring the combined effect to achieve proper ranking of the decision alternatives. Due to its distinctive characteristics of using intuitively appealing two-criteria weight judgments for complex decision ordering, AHP has been widely applied to situation evaluation, risk analysis and alternative selection problems.

AHP is based on seven basic steps in Wu and Tsai (2012), and Vaidya and Kumar (2006) are described as follows:

1 State the problem and determine its goal.

2 Broaden the objectives of the problem by considering all actions, objectives, and outcomes.

3 Identify the criteria and/or sub-criteria.

4 Structure the problem hierarchically by considering the goal, criteria, sub-criteria, and a set of alternatives.

5 Data are collected from decision maker(s) corresponding to the hierarchical structure. And construct a set of pair-wise comparison matrices.

6 Perform computations to find the maximum eigenvalue, consistency index, consistency ratio (CR), and normalised values for criteria and/or sub-criteria and alternative.

7 Use the normalised values to make decisions if CR is satisfactory with a value less than 0.1. If CR ratio greater than 0.1, the judgment matrix is inconsistent.

Although the AHP method is a simple yet powerful tool for MCDM problems, some researchers have pointed out several weakness of the AHP method. One of the most controversial issues is the rank reversal phenomenon. Belton and Gear (1983), and Saaty (1987b) showed that the ranking of a set of three alternatives changes when a new alternative is added or an existing one is deleted to the set. Harker and Vargas (1987) claim that this problem can be overcome by constructing a network rather than considering the system as a hierarchy. Barzilai and Golani (1994) resolve this problem by extending the axiomatic framework. With AHP the decision problem is decomposed into a number of subsystems, within and between which a substantial number of pair-wise comparisons need to be completed. This approach has the disadvantage that the number of pair-wise comparisons may become very large (n (n – 1)/2) and thus a lengthy task. Moreover, if more than one person is working on this, things can get even more complicated (Saaty, 2003; Omar 2011).

The original AHP has two assumptions (Saaty, 2000; Kou et al., 2014): the independence of higher level elements from the lower level elements and the independence of the elements within a level. However, many decision problems cannot be structured hierarchically due to the complexity and dynamic nature of the problems. Not only does the importance of the criteria determine the importance of the alternatives in a hierarchy, but also the importance of the alternatives themselves determines the importance of the criteria.

Today many researchers and practitioners are working on dynamic decision-making related topics. Searcy (2004) proposed an approach to integrate AHP and BSC for the purpose of estimating the performance of enterprises. Chiang (2005) proposed a dynamic decision approach for long-term vendor selection based on AHP and the balanced-score-card (BSC) for the purpose of choosing the sellers. Lin et al. (2008) proposed an adaptive AHP approach (A^3) that uses a soft computing scheme, Genetic Algorithms, to recover the real weightings of the various criteria in AHP and provide a function for automatically improving the consistency ratio of pairwise comparison matrix. Benítez et al., (2012) proposed a framework that allows users to provide partial and/or incomplete preference data at multiple times. Duleba et al., (2012) proposed an algorithm for scoring so that the missing data of the matrices could be estimated. Donga and Cooper (2016) proposed a group consensus reaching model to facilitate a peer to peer opinion exchange process which relieves the need for a moderator by using an automatic feedback mechanism.

The dynamic analytic hierarchy process (DAHP) is a method that considers the factor of time in the AHP model such that the AHP structure can represent a range of decisions reflecting different trends with time (Gonzalez-Prida et al., 2012). Another way to deal with dynamic decisions is to model the judgment matrices with time dependent functions, namely dynamic judgment matrices (Saaty, 1987a; Li et al., 1997; Gao et al., 2011).

Saaty (1987a, 1987b) provides several functions for the dynamic judgment matrices and discusses the corresponding solutions. However, to find analytical solution in such case is very difficult. So far, a few methods have been proposed for solving this problem, including the least perturbation method (Xu and Wei, 2004), least square method (Gong, 2008) and goal programming method (Wang et al., 2005). Saaty (2003) expressed that there are essentially two ways to study dynamic decisions: structural, by including scenarios and time periods as elements in the structure that represents a decision, and functional, by explicitly involving time in the judgment process. A possible third way would be a hybrid of these two.

Only rarely have researchers investigated the integration of human knowledge in change handling with sensitivity analysis of AHP model for dynamic environments. Our research is to explore alternatives and propose enhancements to existing AHP theory that take into account how decision maker(s) respond to extreme forms of change. To verify our hypothesis, we present the implementation of our dynamic AHP approach with the case studies of supplier selection for a steel bar manufacturer in Taiwan.

In this paper, we propose a dynamic AHP decision framework to encode expert knowledge for change handling using ECA rules. As well as the first study for two criteria change impact analysis in the AHP decision process under uncertainty, then using a case study of the supplier selection decision-making task of the steel and iron industry in Taiwan. The results show that our mechanism can effectively reach the same level of decision quality as expert decision makers.

3 A framework for dynamic AHP decision-making

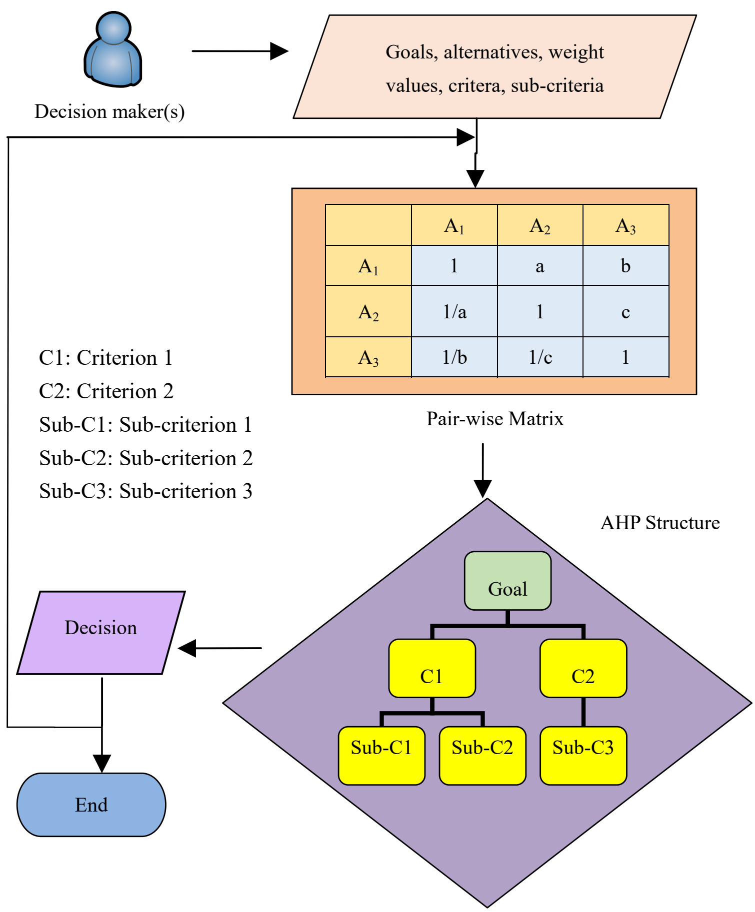
3.1 The general AHP decision-making process

Decision-making is one of the most central processes in organisations and a basic task of management at all levels. According to Wu and Tsai (2012), decision-making is a process of identifying a problem, evaluating alternatives, and selecting best alternative. As Figure 1 shows, the original AHP decision-making process is to select a course of action that satisfies some objectives in an environment. The first step is to identify the goal, criteria, sub-criteria and alternatives. The second step is to define the relative importance of the criteria and fill out relative weights into a pair-wise matrix. The third step is constructing a hierarchical structure that clearly reveals the goals, the criteria and the alternatives of the problem. The final step of AHP is to use the hierarchical structure to help decision makers in focusing on their preferred solutions and arriving at a solution of their choice. In the generalised model, we only collect related information from decision maker(s) when constructing the corresponding decision model based on AHP. If the inputs to a system resulting in similar pattern of activity to occur repeatedly, we can use the same decision model without bothering decision maker(s) again.

3.2 Dynamic AHP decision-making process

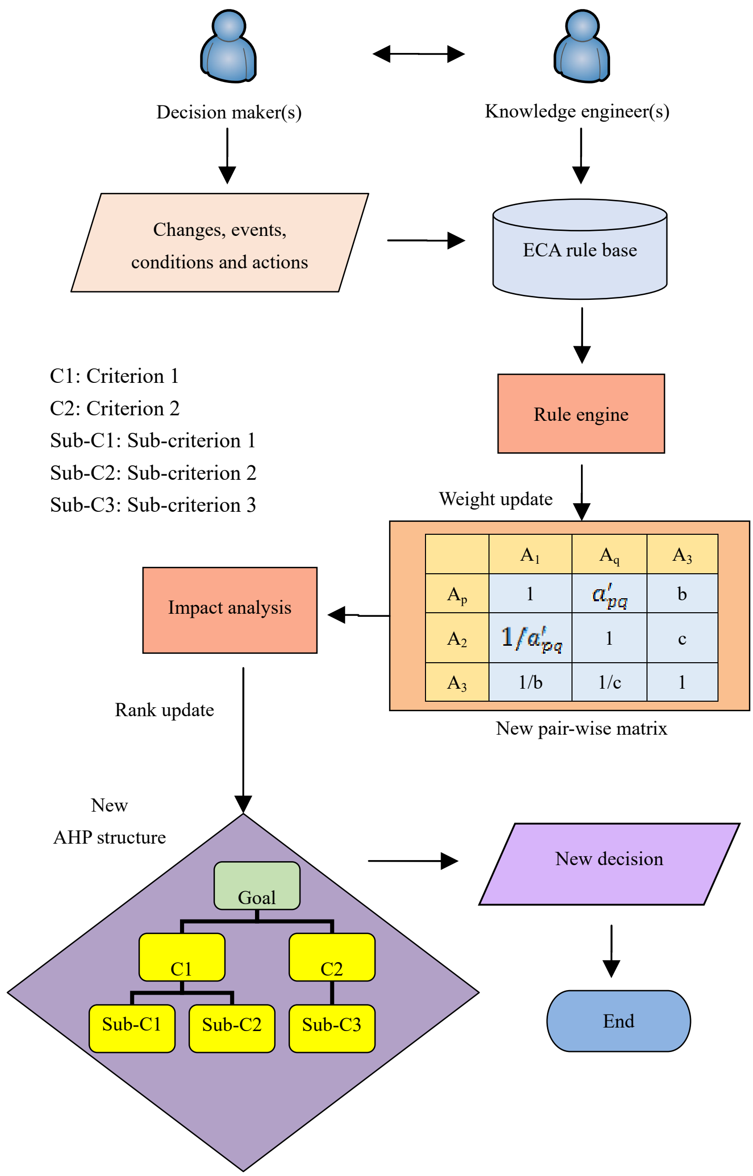
Even though the general AHP method is effective and popular, the problems with the process are that it is time-consuming and human intensive, especially on expert interviews to determine the pair-wise relative weight matrix of decision criteria. When sudden changes occur in the decision environment, it is very unlikely that we can locate the human experts immediately to re-evaluate the pair-wise matrix and re-construct the new AHP structure. Therefore traditional AHP method is not well-suited for fast changing environment. To cope with the problem, we propose a new rule-based dynamic AHP decision process by employing an ECA rule base with rules acquired from human experts to monitor changes in the decision environment and, when appropriate, automatically initiate correction actions on the pairwise comparison matrix of AHP. Then an impact analysis is conducted based on the revised matrix to derive the proper changes in alternative ranking of AHP and therefore reaching a new decision structure in response to the environment changes. The process is illustrated in Figure 2. We note that once the ECA rule base is constructed, the entire process can be carried out completely automatically. Furthermore, the process can easily deal with fast changing environment since the ECA rule base is designed to be triggered on all significant events.

Figure 1 The general AHP decision-making process (see online version for colours)



More specifically, when a sudden change in the business environment takes place, it can usually be characterised by events and conditions. On detecting significant events, the ECA rule base evaluates the status by checking associated conditions. If the conditions are met, proper actions are executed to handle the situation. This architecture provides an effective mechanism for achieving the dynamic AHP decision-making process. The ECA rule base is used to revise the pair-wise comparison matrix of AHP to reflect the new relative weighting of decision criteria. Based on the revised weights, a new AHP model can be derived using our impact analysis method. This leads to a new decision structure which can fully cope with the changing situation.

Figure 2 Dynamic AHP decision-making process (see online version for colours)



As a summary, the decision model built using the traditional AHP algorithm cannot deal with variation in alternative ordering due to changes of the relative weights of primary criteria. To cope with the weight changes, the entire AHP must be carried out all over again. Our dynamic AHP decision-making process, on the other hand, resolves the problem by augmenting traditional AHP with two critical components:

1 An ECA rule base acquired from decision experts on when and how the relative weights of decision criteria should be modified under various changes.

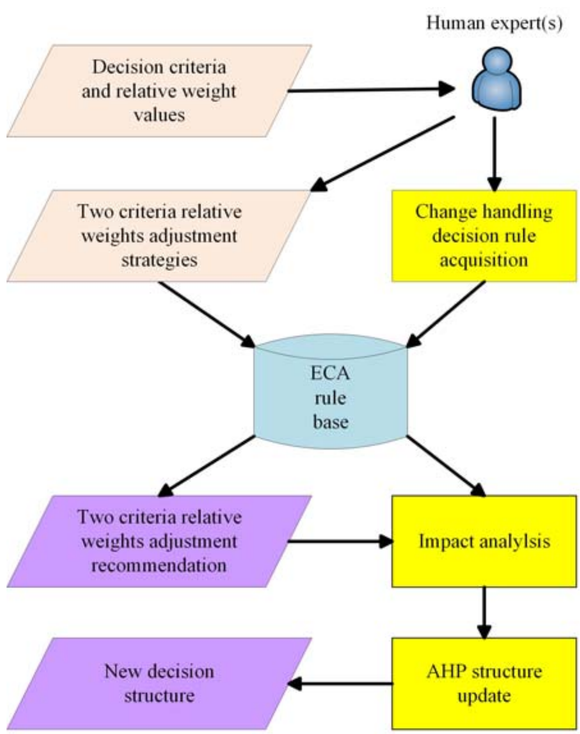
2 An impact analysis method on how should the old decision structure be reorganised given the updated weights derived by the ECA rule base.

Details of the two components are described in the following sections.

3.3 A rule based dynamic decision framework

Most existing decision support models are suitable for static environments only. If some unexpected events occur, traditional methods are usually not able to conduct proper decision-making under uncertain conditions caused by the unexpected events. For the AHP method, most changes in the decision environment cause the need to update the relative weights of two criteria on the pair-wise matrix and human expert(s) can determine the appropriate range of updates to the weight values in response to the changes. Based on these ideas, we propose a new dynamic AHP decision framework to solve the problem of dealing with fast changing environment. The framework includes a theorem for two criteria change impact analysis and a new AHP decision model with ECA rules.

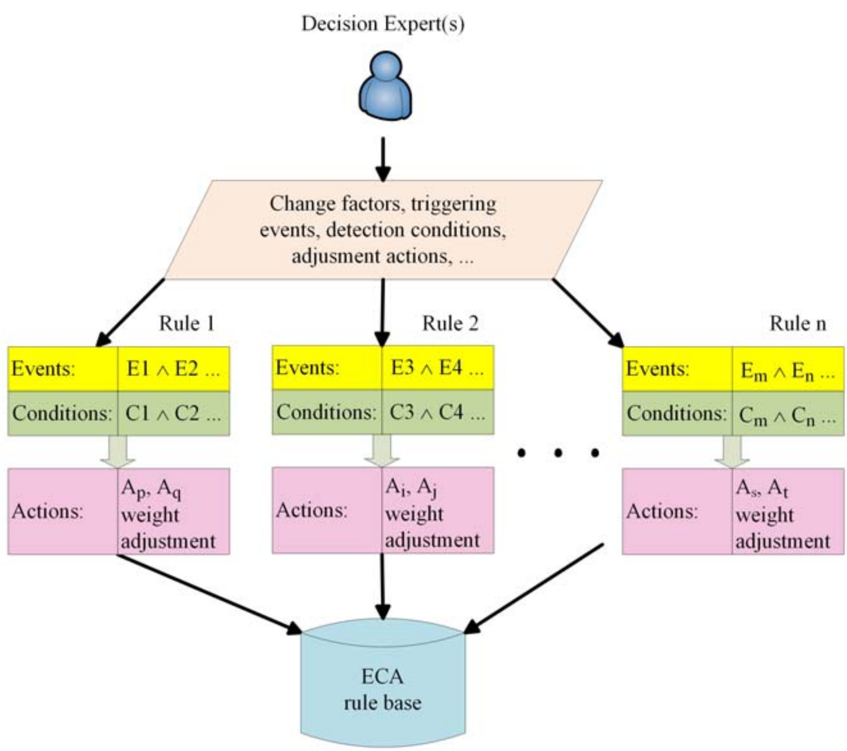
Figure 3 A rule-based dynamic AHP decision framework (see online version for colours)



3.4 Change handling rule base construction process

Many of today's rule-based system use certainty factors to reason with uncertain information. Rule-based systems are designed to emulate the behaviour of a human being who is expert at solving problems within a specialised area. We focus on building and applying the ECA rule base to specific problems and by extracting knowledge from one or more human domain experts (as shown in Figure 4). The ECA rule base is responsible for the event detection and assists the decision maker(s) with the automatic discovery of pair-wise matrix element changes. It also re-calculates the priority vector to aid decision-making by the decision maker(s). As Figure 4 shows, we extend the basic general AHP decision-making process with the ECA rules. The first step is identifying the change criteria, impact of events, and the trend. The second step is to determine what criteria the decision maker(s) use and how many criteria will change if there were changes in the matrix or the input data. We define events using the Boolean operators and, or, not, implies, etc; such as that Bill (Bill is an American.) is planning to purchase a new car. After preliminary analysis of the makes and models available, Bill has a list of decision alternatives limited to three cars, which we will refer to as Toyota, Ford and March. And the criteria thought to be most relevant in the selection are manufacturers, cost, size and styles. Bill constructs the AHP matrix to evaluate the candidates (as shown in Table 1).

Figure 4 Change handling rule base construction process (see online version for colours)



The Ford is Bill's favourite car. But the exchange rate from the US Dollar to the Japanese Yen is unstable (E_1), and one rule checks that US Dollar-Japanese Yen rate has not increased to 2%. If the exchange rate is higher, and Toyota dealer is launching promotions (E_2), then Bill can change the relative weight of cost and manufacturer from 1 to 1/5.

Table 1 is the original evaluation matrix and criteria ranking. If we change the relative weight of cost and manufacturer from 1 to 5, the new matrix and ranking as obtained by applying traditional the AHP method as displayed in Table 2. We can observe that the ranking of 1 and 2 has changed. Then the following events and conditions are considered (as shown in Table 3). Our methods can help Bill to make better decisions under different conditions.

Table 1 Original evaluation matrix and criteria ranking

	Cost	*Manufacturer*	*Size*	*Style*	*Weight*	*Ranking*
Cost	1	1	3	5	1.968	1
Manufacturer	1	1	2	4	1.682	2
Size	$\frac{1}{3}$	$\frac{1}{2}$	1	5	0.955	3
Style	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{5}$	1	0.316	4

Note: C.I. = 0.0478 and C.R. = 0.053.

Table 2 New evaluation matrix and criteria ranking

	Cost	*Manufacturer*	*Size*	*Style*	*Weight*	*Ranking*
Cost	1	$\boxed{3}$	3	5	2.59	1
Manufacturer	$\boxed{\frac{1}{3}}$	1	2	4	1.278	2
Size	$\frac{1}{3}$	$\frac{1}{2}$	1	5	0.955	3
Style	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{5}$	1	0.316	4

Note: C.I. = 0.0698 and C.R. = 0.078.

Table 3 An example of the ECA rules

Event	$E_1 \wedge E_2$
Condition	If (the exchange rate has increased to over 2%) and (the dealer is launching marketing promotions to attract new customers).
Action	1 Adjustment criterion: cost and manufacturer (Bill prefers cost rather than manufacturer.) 2 Adjustment range: +2 $\begin{pmatrix} a_{pq}^{new} = a_{pq} + \alpha = a_{12} + \alpha = 1 + \alpha \\ \Rightarrow \alpha = 3 - 1 = 2 \end{pmatrix}$

When both E_1 and E_2 occur then Bill's selection remains the same (as shown in Table 3). In other words, we use rules to integrate decision-making processes and automatically respond to events created by workflows. In addition, rules that are managed in the ECA rule base keep pace with changing business conditions and are always up-to-date. Event(s) response is flexible; rules can trigger actions in the ECA rule base.

4 Impact analysis of two criteria weight change for dynamic AHP

The AHP can be used to determine optimal strategies when a decision maker is faced with several decision alternatives and complex interrelated criteria. Such a decision process needs to consider how well a system can replicate current operations, and how similar the target conditions are to those described in the historical records. However, the original AHP is time consuming and the decision model is static. Therefore, it is worthwhile to explore the transformation of variations in system input variables into corresponding variations in system outputs when conditions differ from those reflected by the original system inputs. Our research uses a rule base acquired by interviewing one or more human experts to solve the problem of handling changes. We propose an efficient algorithm that can determine the priorities of alternatives from the adjusted relative weights.

We focus on the impact from a change in two criteria on a pair-wise matrix. When one or more events or an unexpected change occurs, it is highly likely to cause the adjustment of relative weight between major criteria which may often result in the change of ranking order between alternatives. On the other hand, it is also possible that small changes in the matrix do not result in any change in the ranking order such that the old decision model remains valid and sufficient. Therefore, we need a way to analyse the impact of weight adjustment on decision ordering. In this section, we propose a method to exactly characterise the conditions for such a transition to occur.

Let $\mathbf{A} = [a_{ij}]_{n\times n}$ $\forall i, j = 1, 2, 3, \ldots, n$ denote a square pair-wise comparison matrix [shown in equation (1)], where a_{ij} gives the importance of element a_i relative to element a_j. Each element in the upper triangle matrix $\mathbf{A}$ is positive, while the lower triangle uses the reciprocal values of the upper diagonal and the main diagonal is always 1. When we try to change the values in two criteria on a pair-wise matrix, let p and q are the indices of the criteria. From decision maker's review, the variables with indices p and q have to be adjusted according to their relative weight in accordance with the current situation. Let a_{pq} and $\frac{1}{a_{pq}}$ be the decision makers' original judgments on the relative importance of the pair of criteria in terms of their contribution to the achievement of the overall goal. The formula $(a_{pq} + \alpha)$ denotes the adjustment of relative weight in the new matrix $\mathbf{A'}$ [shown in equation (2)]. In other words, we assume that the new event/unexpected changes can be sufficiently reflected by adjusting a_{pq} into $(a_{pq} + \alpha)$, i.e., the change in the relative weight between criteria p and q is α, then $a_{qp}^{new} = \frac{1}{a_{pq}^{new}} = \frac{1}{(a_{pq} + \alpha)}$.

$$A = \begin{bmatrix} 1 & a_{12} & \dots & a_{1n} \\ a_{21} & 1 & \boxed{a_{pq}} & a_{2n} \\ \dots & \boxed{\dfrac{1}{a_{pq}}} & 1 & \dots \\ a_{n1} & a_{n2} & \dots & 1 \end{bmatrix} \quad (1)$$

$$\mathbf{A}' = \begin{bmatrix} 1 & a_{12} & \dots & a_{1n} \\ a_{21} & 1 & \boxed{(a_{pq}+\alpha)} & a_{2n} \\ \dots & \boxed{\dfrac{1}{(a_{pq}+\alpha)}} & 1 & \dots \\ a_{n1} & a_{n2} & \dots & 1 \end{bmatrix} \quad (2)$$

Saaty (1987b, 1994) suggested four methods for estimating relative weight and finding the eigenvector: ANC (the average of the normalised columns), NRA (the normalisation of the row average), NCR (the normalisation of the columns and the reciprocal) and NGM (the normalisation of the geometric mean of the rows). Our paper uses NGM, which is the most commonly used among the methods, as in equation (3). The NGM method provides a good estimate of the priorities if accuracy is not of extreme importance. In particular, the AHP matrix was reciprocal and the property is assumed even for a single n-tuple, and moreover the geometric mean satisfied the Pareto Principle (unanimity condition) and homogeneity condition (Aczel, 1989).

$$w = \frac{\left[\prod_{j=1}^{n} a_{ij}\right]^{\frac{1}{n}}}{\sum_{j=1}^{n}\left[\prod_{j=1}^{n} a_{ij}\right]^{\frac{1}{n}}} \quad i, j = 1, 2, ...n \quad (3)$$

The following cases are used for the impact analysis in the proposed model:

Table 4 Explanation of all conditions used for analysis

	Case
1	$\forall k$, if $R(k)$, $R(k+1) \neq p, q$
2	$R(k) \neq p$, $R(k+1) = q$
3	$R(k) \neq q$, $R(k+1) = p$
4	$R(k) = q$, $R(k+1) \neq p$
5	$R(k) = p$, $R(k+1) \neq q$
6	$R(k) = q$, $R(k+1) = p$
7	$R(k) = p$, $R(k+1) = q$

As shown in Table 4, depending on the conditions of p and q, we can divide the adjustment patterns into seven cases. R is the ranking index function, while k is the criterion index. $R(k)$ is the index of the k-th ranking criterion and $w_{R(k)}$ is the weight of the k-th ranking criterion index in the original matrix. Let $w_{R(k)}^{new}$ be the weight of the new k-th ranking criterion index after adjustment. We also assume that the matrix remains consistent and $R(1)$ is the index of the highest ranking criterion. In the following paragraphs, we conduct impact analysis along each case with mathematical proof of its correctness. After theoretical analysis, we provide an illustrative example to demonstrate the validity of our theory. Then, we provide a practical case to demonstrate the real world application of our models.

Before we start to solve the problem, we first assume the following terms:

1 α is the change level in the a_{pq}^{new}. In other words, $a_{pq}^{new} = a_{pq} + \alpha$ and $a_{pq} + \alpha > 0$

2 $a_{qp}^{new} = \dfrac{1}{a_{pq}^{new}} = \dfrac{1}{a_{pq} + \alpha}$

3 Other elements' relative weights remain the same.

4 Par-wise matrixes remains consistent (i.e. C.R. ≤ 0.1).

5 $w_{R(n)} < w_{R(n-1)} < w_{R(n-2)} < \ldots < w_{R(1)}$

Next, we conduct the mathematical analysis of how the decision might change given a change in relative weight of two criteria. Using the conditions of p and q, an analytical equation can be derived for each case and formalised into a corollary. Then, a theorem summarises all results. The notations used in the discussion are summarised in Table 5.

Table 5 Notation table

Parameter	*Description*
i	Criterion index of the original matrix
p	Index of the first affected criterion
q	Index of the second affected criterion (without loss of generality, we assume that $p < q$)
n	Number of criteria
R	Ranking index function
α	Change level between criterion p and q, i.e., $a_{pq}^{new} = a_{pq} + \alpha$ (by the characteristic of the pair-wise comparison matrix, $a_{qp}^{new} = \dfrac{1}{a_{pq}^{new}} = \dfrac{1}{\left(a_{pg} + \alpha\right)}$)
a_{ij}	Relative importance between criterion a_i and a_j
a_{pq}	Relative importance between criterion a_p and a_q
$R(k)$	Index of k-th ranking criterion.
$w_{R(k)}$	Weight of the k-th ranking criterion in the original matrix
$w_{R(k)}^{new}$	Weight of k-th ranking criterion in the new matrix

We conduct impact analysis of the weight change on the new ranking order and derive analytical formula to exactly characterise the condition that will result in order change.

The proofs of cases are illustrated in appendices. In Table 6, we summarise the change conditions and analytical results obtained from Corollary 1–7. The table fully characterises the impact of relative weight change of two decision criteria on the decision ranking as stated in the following theorem.

Theorem 1: Given the change of relative weights of two decision criteria satisfying the assumptions stated above, the corresponding change in decision weight ordering is fully characterised by Table 6.

Proof: The theorem follows naturally from Corollary 1–7. Q.E.D.

Table 6 Summary of the effect on criteria ranking when $a_{pq}^{new} = a_{pq} + \alpha$

Case ID	*Case description*	*Effect on Change*
1	$R(k), R(k+1) \neq p, q$	The relative ranking of criteria $R(k)$ and $R(k+1)$ remains the same. That is, $w_{R(k+1)}^{new} < w_{R(k)}^{new}$.
2	$R(k) \neq p, R(k+1) = q$	The relative ranking of criteria $R(k)$ and $R(k+1)$ changed if $\alpha < a_{pq}\left(\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^n - 1\right)$.
3	$R(k) \neq q, R(k+1 = p)$	The relative ranking of criteria $R(k)$ and $R(k+1)$ changed if $\alpha > a_{pq}\left(\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n - 1\right)$.
4	$R(k) =q, R(k+1) \neq p$	The relative ranking of criteria $R(k)$ and $R(k+1)$ changed if $\alpha > a_{pq}\left(\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n - 1\right)$.
5	$R(k) =q, R(k+1) \neq p$	The relative ranking of criteria $R(k)$ and $R(k+1)$ changed if $\alpha < a_{pq}\left(\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^n - 1\right)$.
6	$R(k) =q, R(k+1) = p$	The relative ranking of criteria $R(k)$ and $R(k+1)$ changed if $\alpha > -a_{pq}\left(1 - \sqrt{\left(\frac{\omega_{R(k)}}{\omega_{R(k+1)}}\right)^n}\right)$.
7	$R(k) =p, R(k+1) = q$	The relative ranking of criteria $R(k)$ and $R(k+1)$ changed if $-a_{pq} < \alpha < a_{pq}\left(1 - \sqrt{\left(\frac{\omega_{R(k+1)}}{\omega_{R(k)}}\right)^n}\right)$.

We illustrate the application of Theorem 1 by the following example. Table 7 is the original matrix and criteria ranking. By changing the relative weights of criteria 3 and 4 from 5 to 1/2, the new matrix and ranking as obtained by applying the traditional AHP method is displayed in Table 8. We can observe changes in the ranking of criteria 3 and 4.

The same conclusion can also be obtained by applying *Theorem 1* as follows. Since the relative weight of criterion 3 and criterion 4 (i.e., a_{34}) has changed, therefore

$p = 3$ and $q = 4$. Let $k = 3$, we can see from Table 7 that $R(k) = R(3) = 3 = p$ and $R(k+1) = 4 = q$ which satisfies case 7 in Table 6.

Table 7 Original matrix and criteria ranking

	Criterion 1	*Criterion 2*	*Criterion 3*	*Criterion 4*	*Weight*	*Ranking*
Criterion 1	1	1	3	5	1.967	1
Criterion 2	1	1	2	4	1.682	2
Criterion 3	$\frac{1}{3}$	$\frac{1}{2}$	1	5	0.955	3
Criterion 4	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{5}$	1	0.316	4

Note: C.I. = 0.0478 and C.R. = 0.053.

Table 8 New matrix and criteria ranking

	Criterion 1	*Criterion 2*	*Criterion 3*	*Criterion 4*	*Weight*	*Ranking*
Criterion 1	1	1	3	5	1.967	1
Criterion 2	1	1	2	4	1.682	2
Criterion 3	$\frac{1}{3}$	$\frac{1}{2}$	1	$\boxed{\frac{1}{2}}$	0.537	4
Criterion 4	$\frac{1}{5}$	$\frac{1}{4}$	$\boxed{2}$	1	0.562	3

Note: C.I. = 0.076 and C.R. = 0.084.

$$-a_{pq} < \alpha < -a_{pq}\left(1 - \sqrt[n]{\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)}\right)$$

$$\Rightarrow -5 < \alpha < -5\left(1 - \sqrt[4]{\left(\frac{0.316}{0.955}\right)}\right)$$

$$\Rightarrow -5 < \alpha < -4.452$$

Since $\alpha = 1/2 - 5 = -4.5$, we can conclude from *Theorem 1* that the ranking of criterion 3 and criterion 4 will change. This coincides with the result obtained from the traditional AHP method.

5 Case studies of supplier selection decision-making for steel bar manufacturing industry in Taiwan

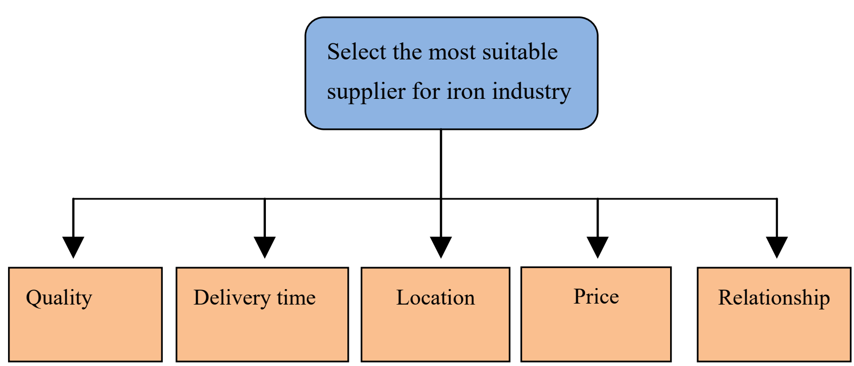
The iron and steel industry is often referred to as the mother of all industries. The industry correlation lies within the upstream steel making industry, the midstream carbon steel industry, and downstream electrical and electronics industry, construction, computer industry, and automotive industries. Because steel is integral to the international flow of goods, its consumption is closely related to a country's industrial structure and economic activity. In addition, the domestic market is often affected by global supply and demand conditions as well as fluctuations. Among them, steel reinforcement is the largest domestic steel industry product. The main raw material of reinforced steel is small 'steel billets' that make up a high proportion of imports of the total consumption. For example, the total imports of steel ingots and other semi-finished products were 5.197 million tons in the year 2008. Among them, 4.042 million tons of steel slabs and 1.019 million tons of small steel billets were imported, which accounted for the largest items. Thus, these two products are deeply impacted by the international steel market. Taiwan's small steel billets are mainly derived from China, Japan, Russia, Brazil, Ukraine and other countries. If we can get a stable source of semi-finished materials from those countries, it will be a great help in stabilising finished products in the domestic steel market. Moreover, the steel industry needs to pay large amounts of capital and human resources, and requires a higher technical level, but its earning returns are very slow. It is therefore a high investment but low return industry.

5.1 A case study of decision-making under changes

Our case study involve interviewing a steel bar manufacturer in the steel and iron industry. The steel bar rolling process starts by inserting billets continuously into a re-heating furnace, heating them up to 1,100°C and discharging the hot billets to the rolling mill train, which rolls and reduces the billets through a rolling stand until they reach the required size. Since the purchase price of billets accounts for over 85% of the total raw materials cost, a decision is often made by comparing the costs and benefits of the available alternatives under various conditions.

Under such circumstances, the import of small steel billets will be significantly affected by major international emergencies in addition to the domestic and international economy. This can create a high purchase cost and an unbalanced supply and demand market. Figure 5 summarises the five decision criteria for the iron industry. The major criteria for supplier selection are quality, delivery time, location, price and relationship. At the president's request, we can not expose some of the private information in the interview records.

Figure 5 Decision criteria used in the case study for the steel and iron industry (see online version for colours)



Too much inventory incurs extra holding and capital cost. Not having enough inventory impacts the ability to manufacture goods or provide customers with product. Based on this situation, if alternative suppliers offer better prices than the current supplier or their location is better than the current one, the firm may consider adding or switching supplier. In our case study, the company has had a similar situation occur before. In May 2010, the company was forced to change supplier from supplier C to supplier D due to the material shortage caused by personnel job rotation (event E_1) and the subsequent inventory mismanagement (event E_2). The purchase cost of inventory has increased dramatically due to internal communication error. These events are expressed below as $E_1 \wedge E_2$.

Table 9 Original matrix and criteria ranking

	Quality	*Delivery*	*Location*	*Relationship*	*Price*	*Weight*	*Ranking*
Quality	1	2	4	2	2	2	1
Delivery	$\frac{1}{2}$	1	3	1	2	1.246	2
Location	$\frac{1}{4}$	$\frac{1}{3}$	1	$\frac{1}{2}$	$\frac{1}{3}$	0.425	5
Relationship	$\frac{1}{2}$	1	2	1	1	1	3
Price	$\frac{1}{2}$	$\frac{1}{2}$	3	1	1	0.944	4

Note: C.I. = 0.02117 and C.R. = 0.0189.

Table 10 New priority matrix and criteria ranking

	Quality	*Delivery*	*Location*	*Relationship*	*Price*	*Weight*	*Ranking*
Quality	1	2	4	2	2	2	1
Delivery	$\frac{1}{2}$	1	3	1	2	1.246	2
Location	$\frac{1}{4}$	$\frac{1}{3}$	1	$\frac{1}{2}$	$\frac{1}{3}$	0.425	5
Relationship	$\frac{1}{2}$	1	2	1	$\boxed{\frac{1}{2}}$	0.871	4
Price	$\frac{1}{2}$	$\frac{1}{2}$	3	$\boxed{2}$	1	1.084	3

Note: C.I. = 0.04 and C.R. = 0.0358.

To follow the proposed framework in Section 3, the first step we use geometric mean to compare suppliers with different criteria. The Criteria used in this study for a steel bars manufacturer and the criteria's weight ranking listed in the Table 9. Step 2 is to compute $\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}}$ and $\left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}}$. In Step 3, examine

$\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}}$ for a change level in a_{pq}^{new}. The results are summarised in Table 10.

The same conclusion can also be obtained by applying Theorem 1 as follows. Since the relative weight of price and relationship (i.e., a_{45}) has changed, therefore $p = 4$ and $q = 5$. Let $k = 3$; we can see from Table 9 that $R(k) = R(3) = 4$ and $R(k + 1) = 5$ which satisfies case 7 in Table 6. By substituting relative parameters with case values above, we obtain the boundary of change as below.

$$-a_{pq} < \alpha < -a_{pq}\left(1-\sqrt{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^{n}}\right)$$

$$\Rightarrow -1 < \alpha < -1\left(1-\sqrt{\left(\frac{0.944}{1}\right)^{5}}\right)$$

$$\Rightarrow -1 < \alpha < -0.12$$

Since $\alpha = 1/2 - 1 = -0.5$ which falls within the range above, we can conclude from *Theorem 1* that the ranking of Price and Relationship should change accordingly. This coincides with the result obtained from the traditional AHP method. In our dynamic AHP decision model, this expert reasoning process can be captured by an ECA rule as demonstrated in Table 11.

Table 11 An ECA rule for change handling

Item	*Description*	
Event	$E_1 \wedge E_2$	
Condition	if (the stock level < the safety stock level) and (received rush orders)	
Action	1	Adjustment criterion: price and relationship
		(Decision-maker prefer price than relationship.)
	2	Weight adjustment: –0.5

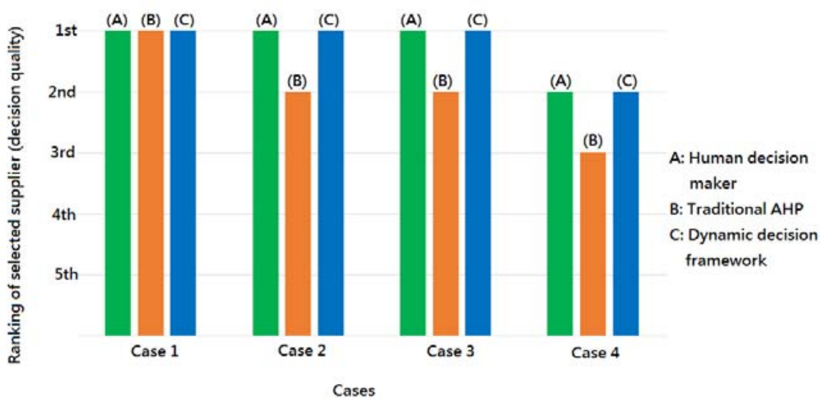
We demonstrate the correctness of our mechanism on the supplier selection decision-making problem with a real case study of the iron and steel industry in Taiwan. The case study involves an actual supplier selection decision case under rapid change. Based on this case study, we have demonstrated that our rule-based mechanism is expressive and powerful enough to handle real-life dynamic decision problems.

5.2 Decision quality comparison between human decision maker, traditional AHP and dynamic AHP methods

To verify the analytical quality of our approach, we compare the decision made by our method (dynamic AHP) with traditional AHP and human decision maker, as shown in Figure 6. We use four historical cases from the same company as in previous section to test the accuracy and quality of all methods. For a specific case and a method, the selected supplier must be the 1st ranking supplier based on that method. The decision is considered accurate if the selected supplier is indeed the true 1st ranking supplier based on manual post-examination of the case. Otherwise, it is considered as reaching a decision with lower quality. Case 1 is a general decision case with common characteristics shared by most cases. In general, when a purchase requirement has an aggregate total cost of NT$1,000,000 or less, the purchase may follow normal decision model without detail evaluation of all decision criteria. In this case, all three methods successfully select the best supplier among all alternatives.

Case 2 was an example where the decision condition changes during the absence of the main decision maker. The unexpected event happened when the decision maker was on a business trip. Even with less experience, the deputy manager was well trained enough to choose proper decision pattern to react to the emergency. In 2005, due to shortage of raw material in Japan, the deputy manager changed the relative weight of price and relationship, causing the switch of preference order between top two suppliers on the supplier list. Due to the ability of handling changes, our method successfully derived the new preference order and reached the same level of decision quality as human decision maker. The traditional AHP, however, was static and therefore retained the original order resulting in suboptimal decision quality.

Figure 6 Comparison of decision quality on different cases (see online version for colours)



Case 3 was an example of external environment change that triggered the altering of preference order. During the financial crisis of 2008, the terms of trade changed in many countries. In particular, the small billet prices deteriorated significantly. Under this situation, some companies delayed the sales and shipment of some of their products on purpose to wait for higher prices or reduce possible lost. To deal with such undesirable conditions, the manager decided to change the relative weight of cost and compliance with due time, causing the switch of preference order between top two suppliers on the supplier list. Both the human decision maker and our method were able to track the changes of external factors over time and adjust the relative weights accordingly. Traditional AHP, on the other hand, failed to recognise such factors and insisted on the original preference order which would definitely result in significant delay on subsequent production of iron steel bar and therefore the cost of performance bond and penalty.

Case 4 was an example involving complex changes of relative weights among multiple decision criteria. Many real-world problems are usually much more complex than that assumed in a decision model such as AHP. In May 2010, due to the material shortage caused by personnel job rotation and subsequent inventory mismanagement, an emergent purchase was in order. Moreover, the exchange rates of Japanese Yen have fallen sharply at the same time. Under this situation, the manager decided to reorder the relative weights of price, relationship and compliance with due time which still lead to suboptimal decision. The example demonstrated that even for a human decision maker, it was hard to make the best decision within a limited time in a case involving multi-criteria assessment and complex interference. Since our method employed expert rules from human decision maker for change handling, it reached the same suboptimal decision as well. The traditional AHP, however, remained static and selected the original best supplier which was actually third under the complex changes. This example also demonstrated that the decision knowledge in the rule base is of crucial importance to the effectiveness of our method.

5.3 Concluding remarks on case studies

As mentioned previously, there is no any mechanism for tracking context and weight changes in traditional AHP. It cannot respond to changes without reconducting the entire process for the new situation. Large scale or rapid changes might lead to worst decision-making. On the contrary, the dynamic AHP method is able respond to changes immediately and properly based on expert rules and our analytical theorem. Furthermore, our method can reach the same level of decision quality as the human decision maker. Sometime a deputy decision maker with less experience and/or knowledge can lead to a decision that is optimal. Our method can act as a consultant to the deputy manager in such cases. Moreover, the knowledge base in our method can accumulate over time. Once the rule base accumulates enough knowledge in handling many different cases, it will certainly help the decision maker to make a good decision even under fast changing environments.

6 Conclusions and future work

In this paper, we propose a new dynamic rule-based AHP decision-making framework for MCDM and illustrate its usage by case studies of supplier selection in iron and steel industry. Our framework improves the original AHP decision model on two perspectives:

1. *Time effectiveness* – the original AHP is time-consuming and the decision model is static. We improve upon the original AHP decision-making model with an ECA rule based system and change impact analysis, which can be conducted completely and automatically without human intervention. And suboptimal decisions can be semi-optimal decision.
2. *Improved decision quality under changes* – in fast changing business environment, the decision criteria and proper weighting are necessarily dynamic. Less experienced decision makers may not be able to make proper adjustment to normal decision model resulting in sub-optimal decisions. We provide a more effective and responsive method than the original AHP to help the decision makers in making better decisions under complex situations or stress.

The core contributions of this paper are summarised as follows:

1 We have proposed an efficient dynamic rule-based AHP decision-making framework to deal with the complexity of decision problems involving the relative weight change of decision criteria.

2 The dynamic decision made by our framework was consistent with human decision maker(s) in real case studies.

In the future, we plan to develop a decision support tool based on our dynamic rule-based AHP decision-making framework. Furthermore, the potential problem of inconsistent comparison matrix still needs to be resolved. At present, our framework applies successfully to the iron and steel industry. We intend to extend its application domains in the future to see if it can be used for other industries as well. We also plan to extend the two criteria impact analysis method into multi-criteria analysis for handling more complex situations.

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Appendix 1

Proof of Corollary 1

Corollary 1: $\forall k$, if $R(k)$, $R(k+1) \neq p, q$ then $w_{R(k+1)}^{new} < w_{R(k)}^{new}$.

Proof: To analyse the condition that leads $w_{R(k)}^{new} < w_{R(k+1)}^{new}$, we must show that

$$w_{R(k+1)j}^{new} = \frac{\left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}^{new}\right)^{\frac{1}{n}}},\quad w_{R(k)j}^{new} = \frac{\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}^{new}\right)^{\frac{1}{n}}}$$

Because the denominators are the same, we just compare the numerators. Except a_{pq} and $\frac{1}{a_{pq}}$, all other elements remain the same.

$$\because R(k),\, R(k+1) \neq p,\, q$$
$$\Rightarrow \forall k,\ a^{new}_{R(k+1)j} = a_{R(k+1)j}$$
$$\Rightarrow \left(\prod_{j=1}^{n} a^{new}_{R(k+1)j}\right)^{\frac{1}{n}} = \left(a^{new}_{R(k+1)1} \times a^{new}_{R(k+1)2} \times \ldots\ldots \times a^{new}_{R(k+1)n}\right)^{\frac{1}{n}}$$
$$= \left(a_{R(k+1)1} \times a_{R(k+1)2} \times \ldots\ldots \times a_{R(k+1)n}\right)^{\frac{1}{n}} = \left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}$$
$$\Rightarrow \left(\prod_{j=1}^{n} a^{new}_{R(k)j}\right)^{\frac{1}{n}} = \left(a^{new}_{R(k)1} \times a^{new}_{R(k)2} \times \ldots\ldots \times a^{new}_{R(k)n}\right)^{\frac{1}{n}}$$
$$= \left(a_{R(k)1} \times a_{R(k)1} \times \ldots\ldots \times a_{R(k)n}\right)^{\frac{1}{n}} = \left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}}$$

$$w_{R(k+1)j} = \frac{\left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}\right)^{\frac{1}{n}}} < w_{R(k)j} = \frac{\left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}\right)^{\frac{1}{n}}}$$
$$\Rightarrow \left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}}$$
$$\Rightarrow \left(\prod_{j=1}^{n} a^{new}_{R(k+1)j}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a^{new}_{R(k)j}\right)^{\frac{1}{n}}$$
$$\Rightarrow \frac{\left(\prod_{j=1}^{n} a^{new}_{R(k+1)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a^{new}_{ij}\right)^{\frac{1}{n}}} < \frac{\left(\prod_{j=1}^{n} a^{new}_{R(k)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a^{new}_{ij}\right)^{\frac{1}{n}}}$$
$$\Rightarrow w^{new}_{R(k+1)} < w^{new}_{R(k)}$$

□

From Corollary 1, we can conclude that the relative ranking of any pair of criterion $R(k)$ and $R(k+1)$ remains the same if $R(k),\ R(k+1) \neq p,\, q$.

Appendix 2

Proof of Corollary 2

Corollary 2: when $R(k) \neq p$, $R(k+1) = q$ then $w_{R(k)}^{new} < w_{R(k+1)}^{new}$ if $\alpha < a_{pq}\left(\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^n - 1\right)$.

Proof: Note that $\because w_{R(k)j} > w_{R(k+1)j} \therefore \left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^n < 1$ and therefore $a_{pq}^{new} = a_{pq} + \alpha < a_{pq}$.

To analyse the condition that leads to $w_{R(k)}^{new} < w_{R(k+1)}^{new}$, we must show that

$$\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}}$$

$$\therefore R(k) \neq p,\ R(k+1) = q \ \therefore R(k) \neq p, q$$

Based on the condition in $R(k) \neq p$, $R(k+1) = q$ and assumption 3.

Except $a_{pq}^{new} = a_{pq} + \alpha$ and $a_{qp}^{new} = \frac{1}{a_{pq}^{new}} = \frac{1}{a_{pq} + \alpha}$, all other elements remain the same.

$$\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}} = \left(a_{R(k)1}^{new} \times a_{R(k)2}^{new} \ldots\ldots \times a_{R(k)n}^{new}\right)^{\frac{1}{n}} = \left(a_{R(k)1} \times a_{R(k)2} \ldots\ldots \times a_{R(k)n}\right)^{\frac{1}{n}} = \left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}}$$

$$\left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}} = \left(a_{q1}^{new} \times a_{q2}^{new} \ldots\ldots \times a_{q(p-1)}^{new} \times \left(a_{qp}^{new}\right) \times a_{q(p+1)}^{new} \times \ldots\ldots \times a_{qn}^{new}\right)^{\frac{1}{n}}$$

$$= \left(a_{q1} \times a_{q2} \ldots\ldots \times a_{q(p-1)} \times \left(\frac{1}{a_{pq} + \alpha}\right) \times a_{q(p+1)} \times \ldots\ldots \times a_{qn}\right)^{\frac{1}{n}}$$

To satisfy $\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}}$

$$\Rightarrow \left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}} < \left(a_{q1} \times a_{q2} \ldots\ldots \times a_{q(p-1)} \times \left(\frac{1}{a_{pq} + \alpha}\right) \times a_{q(p+1)} \times \ldots\ldots \times a_{qn}\right)^{\frac{1}{n}}$$

$$\Rightarrow \left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}} < \left(a_{q1} \times a_{q2} \ldots\ldots \times a_{q(p-1)} \times \left(\frac{a_{qp}}{a_{qp}\left(a_{pq} + \alpha\right)}\right) \times a_{q(p+1)} \times \ldots\ldots \times a_{qn}\right)^{\frac{1}{n}}$$

$$\Rightarrow \left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}} < \left(\left(\frac{1}{a_{qp}\left(a_{pq} + \alpha\right)}\right) \times a_{q1} \times a_{q2} \ldots\ldots \times a_{qn}\right)^{\frac{1}{n}}$$

$$\Rightarrow \left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}} < \left(\frac{1}{a_{qp}\left(a_{pq}+\alpha\right)}\right)^{\frac{1}{n}} \left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}$$

$$\Rightarrow \frac{\left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}\right)^{\frac{1}{n}}} < \left(\frac{1}{a_{qp}\left(a_{pq}+\alpha\right)}\right)^{\frac{1}{n}} \left(\frac{\left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}\right)^{\frac{1}{n}}}\right)$$

$$\Rightarrow w_{R(k)} < \left(\frac{1}{a_{qp}\left(a_{pq}+\alpha\right)}\right)^{\frac{1}{n}} w_{R(k+1)}$$

$$\Rightarrow \left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^{n} < \left(\frac{1}{a_{qp}\left(a_{pq}+\alpha\right)}\right)$$

$$\Rightarrow \left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^{n} < \frac{a_{pq}}{a_{pq}+\alpha}\left(\because a_{qp} = \frac{1}{a_{pq}}\right)$$

$$\Rightarrow a_{pq}+\alpha < a_{pq}\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^{n}$$

$$\Rightarrow \alpha < a_{pq}\left(\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^{n} - 1\right)$$

□

From Corollary 2, we can conclude that under the condition $R(k) \neq p$, $R(k+1) = q$, the relative ranking of criterion $R(k)$ and $R(k+1)$ will change if $\alpha < a_{pq}\left(\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^{n} - 1\right)$.

Appendix 3

Proof of Corollary 3

Corollary 3: When $R(k) \neq q$, $R(k+1) = p$ then $w_{R(k)}^{new} < w_{R(k+1)}^{new}$ if $\alpha > a_{pq}\left(\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^{n} - 1\right)$.

Proof: To analyse the condition that leads to $w_{R(k)}^{new} < w_{R(k+1)}^{new}$, we must show that

$$\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}}$$

$$\therefore R(k) \neq q,\ R(k+1) = p \ \therefore R(k) \neq p,\ q$$

Based on the condition $R(k) \neq q$, $R(k+1) = p$ and assumption 3.

Except $a_{pq}^{new} = (a_{pq} + \alpha)$ and $a_{qp}^{new} = \frac{1}{a_{pq}^{new}} = \frac{1}{a_{pq} + \alpha}$, all other elements remain the same.

$$\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}} = \left(a_{R(k)1}^{new} \times a_{R(k)2}^{new} \ldots\ldots \times a_{R(k)n}^{new}\right)^{\frac{1}{n}} = \left(a_{R(k)1} \times a_{R(k)2} \ldots\ldots \times a_{R(k)n}\right)^{\frac{1}{n}}$$

$$= \left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}}$$

$$\left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}} = \left(a_{p1}^{new} \times a_{p2}^{new} \ldots\ldots \times a_{p(q-1)}^{new} \times \left(a_{pq}^{new}\right) \times a_{p(q+1)}^{new} \ldots\ldots \times a_{pn}^{new}\right)^{\frac{1}{n}}$$

$$= \left(a_{p1} \times a_{p2} \ldots\ldots \times a_{p(q-1)} \times \left(a_{pq} + \alpha\right) \times a_{p(q+1)} \ldots\ldots \times a_{pn}\right)^{\frac{1}{n}}$$

To satisfy $\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}}$

$$\Rightarrow \left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}} < \left(a_{p1} \times a_{p2} \ldots\ldots \times a_{p(q-1)} \times \left(\frac{a_{pq}\left(a_{pq} + \alpha\right)}{a_{pq}}\right) \times a_{p(q+1)} \ldots\ldots \times a_{pn}\right)^{\frac{1}{n}}$$

$$\Rightarrow \left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}} < \left(\frac{a_{pq} + \alpha}{a_{pq}}\right)^{\frac{1}{n}} \left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}$$

$$\Rightarrow \frac{\left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}\right)^{\frac{1}{n}}} < \left(\frac{a_{pq} + \alpha}{a_{pq}}\right)^{\frac{1}{n}} \left(\frac{\left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}\right)^{\frac{1}{n}}}\right)$$

$$\Rightarrow w_{R(k)} < \left(\frac{a_{pq} + \alpha}{a_{pq}}\right)^{\frac{1}{n}} \omega_{R(k+1)}$$

$$\Rightarrow \left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^{n} < \left(\frac{a_{pq} + \alpha}{a_{pq}}\right)$$

$$\Rightarrow \left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^{n} \times a_{pq} < \left(a_{pq} + \alpha\right)$$

$$\Rightarrow \alpha > a_{pq}\left(\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^{n}-1\right)$$

□

From Corollary 3, we can conclude that under the condition $R(k) \neq q$, $R(k+1) = p$, the relative ranking of criterion $R(k)$ and $R(k+1)$ will change if $\alpha > a_{pq}\left(\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^{n}-1\right)$.

Appendix 4

Proof of Corollary 4

Corollary 4: When $R(k) = q$, $R(k+1) \neq p$ then $w_{R(k)}^{new} < w_{R(k+1)}^{new}$ if $\alpha > a_{pq}\left(\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^{n}-1\right)$.

Proof: To analyse the condition that leads to $w_{R(k)}^{new} < w_{R(k+1)}^{new}$, we must know that

$$w_{R(k)j}^{new} = \frac{\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}^{new}\right)^{\frac{1}{n}}} < w_{R(k+1)j}^{new} = \frac{\left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}^{new}\right)^{\frac{1}{n}}}$$

Because the denominators are the same, therefore we must show that

$$\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}}$$

$$\therefore R(k = q,\ R(k+1) \neq p \ \therefore R(k) \neq p, q$$

Based on condition $R(k) = q$, $R(k+1) \neq p$ and assumption 3.

Except $a_{pq}^{new} = (a_{pq} + \alpha)$ and $a_{qp}^{new} = \frac{1}{a_{pq}^{new}} = \frac{1}{a_{pq} + \alpha}$, all other elements remain the same.

$$\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}} = \left(a_{q1}^{new} \times a_{q2}^{new} \ldots . \times a_{q(p-1)}^{new} \times \left(a_{qp}^{new}\right) \times a_{q(p+1)}^{new} \times \ldots . \times a_{qn}^{new}\right)^{\frac{1}{n}}$$

$$= \left(a_{q1} \times a_{q2} \ldots . \times a_{q(p-1)} \times \left(\frac{1}{a_{pq} + \alpha}\right) \times a_{q(p+1)} \times \ldots . \times a_{qn}\right)^{\frac{1}{n}}$$

$$\left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}} = \left(a_{R(k+1)1}^{new} \times a_{R(k+1)2}^{new} \ldots\ldots \times a_{R(k+1)n}^{new}\right)^{\frac{1}{n}}$$

$$= \left(a_{R(k+1)1} \times a_{R(k+1)2} \ldots\ldots \times a_{R(k+1)n}\right)^{\frac{1}{n}} = \left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}$$

To satisfy $\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}}$

$$\Rightarrow \left(a_{q1} \times a_{q2} \ldots\ldots \times a_{q(p-1)} \times \left(\frac{1}{a_{pq} + \alpha}\right) \times a_{q(p+1)} \times \ldots\ldots \times a_{qn}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}$$

$$\Rightarrow \left(a_{q1} \times a_{q2} \ldots\ldots \times a_{q(p-1)} \times \left(\frac{a_{qp}}{a_{qp}\left(a_{pq} + \alpha\right)}\right) \times a_{q(p+1)} \times \ldots\ldots \times a_{qn}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}$$

$$\Rightarrow \left(\left(\frac{1}{a_{qp}\left(a_{pq} + \alpha\right)}\right)^{\frac{1}{n}} \left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}}\right) < \left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}$$

$$\Rightarrow \left(\frac{1}{a_{qp}\left(a_{pq} + \alpha\right)}\right)^{\frac{1}{n}} \left(\frac{\left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}\right)^{\frac{1}{n}}}\right) < \frac{\left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}\right)^{\frac{1}{n}}}$$

$$\Rightarrow \left(\frac{1}{a_{qp}\left(a_{pq} + \alpha\right)}\right)^{\frac{1}{n}} \times w_{R(k)} < w_{R(k+1)}$$

$$\Rightarrow \left(\frac{1}{a_{qp}\left(a_{pq} + \alpha\right)}\right) < \left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^{n}$$

$$\Rightarrow \left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^{n} > \frac{a_{pq}}{a_{pq} + \alpha} \left(\because a_{qp} = \frac{1}{a_{pq}}\right)$$

$$\Rightarrow \left(a_{pq} + \alpha\right) > a_{pq}\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^{n}$$

$$\Rightarrow \alpha > a_{pq}\left(\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^{n} - 1\right)$$

□

From Corollary 4, we can conclude that under the condition $R(k) = q$, $R(k+1) \neq p$, the relative ranking of criterion $R(k)$ and $R(k+1)$ will change if $\alpha > a_{pq}\left(\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n - 1\right)$.

Appendix 5

Proof of Corollary 5

Corollary 5: When $R(k) = q$, and $R(k+1) = p$ then $w^{new}_{R(k)} < w^{new}_{R(k+1)}$, if $\alpha > -a_{pq}\left(1 - \sqrt{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n}\right)$.

Proof: To analyse the condition that leads to $w^{new}_{R(k)} < w^{new}_{R(k+1)}$, we must show that

$$w^{new}_{R(k)j} = \frac{\left(\prod_{j=1}^{n} a^{new}_{R(k)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a^{new}_{ij}\right)^{\frac{1}{n}}} < w^{new}_{R(k+1)j} = \frac{\left(\prod_{j=1}^{n} a^{new}_{R(k+1)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a^{new}_{ij}\right)^{\frac{1}{n}}}$$

Because the denominator is the same, we must show that

$$\left(\prod_{j=1}^{n} a^{new}_{R(k)j}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a^{new}_{R(k+1)j}\right)^{\frac{1}{n}}$$

Based on the condition in $R(k) = q$, and $R(k+1) = p$ and assumption 3.

Except $a^{new}_{pq} = (a_{pq} + \alpha)$ and $a^{new}_{qp} = \frac{1}{a^{new}_{pq}} = \frac{1}{a_{pq} + \alpha}$, all other elements remain the same.

$$\left(\prod_{j=1}^{n} a^{new}_{R(k)j}\right)^{\frac{1}{n}} = \left(a^{new}_{q1} \times a^{new}_{q2} \ldots\times a^{new}_{q(p-1)} \times \left(a^{new}_{qp}\right) \times a^{new}_{q(p+1)} \times \ldots \times a^{new}_{qn}\right)^{\frac{1}{n}}$$

$$= \left(a_{q1} \times a_{q2} \ldots\times a_{q(p-1)} \times \left(\frac{1}{a_{pq} + \alpha}\right) \times a_{q(p+1)} \times \ldots \times a_{qn}\right)^{\frac{1}{n}}$$

$$\left(\prod_{j=1}^{n} a^{new}_{R(k+1)j}\right)^{\frac{1}{n}} = \left(a^{new}_{p1} \times a^{new}_{p2} \ldots\times a^{new}_{p(q-1)} \times \left(a^{new}_{pq}\right) \times a^{new}_{p(q+1)} \ldots \times a^{new}_{pn}\right)^{\frac{1}{n}}$$

$$= (a_{p1} \times a_{p2} \ldots\times a_{p(q-1)} \times \left(a_{pq} + \alpha\right) \times a_{p(q+1)} \ldots \times a_{pn})^{\frac{1}{n}}$$

To satisfy $\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}}$

$$\Rightarrow \left(a_{q1} \times a_{q2} \ldots \times a_{q(p-1)} \times \left(\frac{1}{a_{pq}+\alpha} \right) \times a_{q(p+1)} \times \ldots \times a_{qn} \right)^{\frac{1}{n}}$$

$$< \left(a_{p1} \times a_{p2} \ldots \times a_{p(q-1)} \times \left(a_{pq}+\alpha \right) \times a_{p(q+1)} \ldots \times a_{pn} \right)^{\frac{1}{n}}$$

$$\Rightarrow \left(a_{q1} \times a_{q2} \ldots \times a_{q(p-1)} \times \left(\frac{a_{qp}}{a_{qp}\left(a_{pq}+\alpha\right)} \right) \times a_{q(p+1)} \times \ldots \times a_{qn} \right)^{\frac{1}{n}}$$

$$< \left(a_{p1} \times a_{p2} \ldots \times a_{p(q-1)} \times \left(\frac{a_{pq}\left(a_{pq}+\alpha\right)}{a_{pq}} \right) \times a_{p(q+1)} \ldots \times a_{pn} \right)^{\frac{1}{n}}$$

$$\Rightarrow \left(\left(\frac{1}{a_{qp}\left(a_{pq}+\alpha\right)} \right) \times a_{q1} \times a_{q2} \ldots \times a_{qn} \right)^{\frac{1}{n}} < \left(\left(\frac{a_{pq}+\alpha}{a_{pq}} \right) \times a_{p1} \times a_{p2} \ldots \times a_{pn} \right)^{\frac{1}{n}}$$

$$\Rightarrow \left(\frac{1}{a_{qp}\left(a_{pq}+\alpha\right)} \right)^{\frac{1}{n}} \left(\prod_{j=1}^{n} a_{R(k)j} \right)^{\frac{1}{n}} < \left(\frac{a_{pq}+\alpha}{a_{pq}} \right)^{\frac{1}{n}} \left(\prod_{j=1}^{n} a_{R(k+1)j} \right)^{\frac{1}{n}}$$

$$\Rightarrow \left(\frac{1}{a_{qp}\left(a_{pq}+\alpha\right)} \right)^{\frac{1}{n}} \left(\frac{\left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}\right)^{\frac{1}{n}}} \right) < \left(\frac{a_{pq}+\alpha}{a_{pq}} \right)^{\frac{1}{n}} \left(\frac{\left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}\right)^{\frac{1}{n}}} \right)$$

$$\Rightarrow \left(\frac{1}{a_{qp}\left(a_{pq}+\alpha\right)} \right)^{\frac{1}{n}} w_{R(k)} < \left(\frac{a_{pq}+\alpha}{a_{pq}} \right)^{\frac{1}{n}} w_{R(k+1)}$$

$$\Rightarrow \left(\frac{a_{pq}}{\left(a_{pq}+\alpha\right)} \right)^{\frac{1}{n}} w_{R(k)} < \left(\frac{a_{pq}+\alpha}{a_{pq}} \right)^{\frac{1}{n}} w_{R(k+1)} \left(\because a_{qp} = \frac{1}{a_{pq}} \right)$$

$$\Rightarrow \left(\frac{1}{a_{pq}+\alpha} \right)\left(a_{pq}\right)\left(w_{R(k)}\right)^{n} < \left(\frac{1}{a_{pq}} \right)\left(a_{pq}+\alpha\right)\left(w_{R(k+1)}\right)^{n}$$

$$\Rightarrow (a_{pq})^2 \left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n < (a_{pq}+\alpha)^2$$

$$\Rightarrow (a_{pq}+\alpha)^2 - (a_{pq})^2 \left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n > 0$$

$$\Rightarrow \alpha^2 + 2a_{pq}\alpha + (a_{pq})^2 \left(1-\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n\right) > 0$$

We use the quadratic formula to solve equations in standard form $\alpha = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$\Rightarrow \alpha = \frac{-2a_{pq} \pm \sqrt{4(a_{pq})^2 - 4(a_{pq})^2 \left(1-\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n\right)}}{2}$$

$$\Rightarrow \alpha = -a_{pq} \pm \sqrt{(a_{pq})^2 - (a_{pq})^2 \left(1-\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n\right)}$$

$$\Rightarrow \alpha = -a_{pq} \pm a_{pq}\sqrt{1-\left(1-\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n\right)}$$

$$\Rightarrow \alpha = -a_{pq}\left(1 \pm \sqrt{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n}\right)$$

Note that: $w_{R(k)} > w_{R(k+1)} \therefore \sqrt{(\frac{w_{R(k)}}{w_{R(k+1)}})^n} > 1$

$$\Rightarrow \alpha > -a_{pq}\left(1 - \sqrt{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n}\right) \text{ or } \alpha < -a_{pq}\left(1 + \sqrt{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n}\right)$$

For $\alpha < -a_{pq}\left(1 + \sqrt{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n}\right)$

$$\therefore \sqrt{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n} > 1$$

$$\Rightarrow \left(1 + \sqrt{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n}\right) > 2$$

$\Rightarrow \alpha < -2a_{pq}$ which violate assumption 1, that is $a_{pq} + \alpha > 0$, therefore $\alpha > -a_{pq}\left(1+\sqrt[n]{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)}\right)$ is invalid.

For $\alpha > -a_{pq}\left(1-\sqrt[n]{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)}\right)$

$$\because \left(1-\sqrt[n]{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)}\right) < 0$$

$$\Rightarrow \alpha > -a_{pq}\left(1-\sqrt[n]{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)}\right) > 0$$

$\Rightarrow \alpha > -a_{pq}\left(1-\sqrt[n]{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)}\right)$ is the only solution.

□

From Corollary 5, we can conclude that under the condition $R(k) = q$, and $R(k + 1) = p$, the relative ranking of criterion $R(k)$ and $R(k + 1)$ will change if $\alpha > -a_{pq}\left(1-\sqrt[n]{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)}\right)$.

Appendix 6

Proof of Corollary 6

Corollary 6: When $R(k) = p$, $R(k + 1) \neq q$ then $w^{new}_{R(k)} < w^{new}_{R(k+1)}$ if $\alpha < a_{pq}\left(\sqrt[n]{\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)} - 1\right)$.

Proof: Note that $\because w_{R(k)j} > w_{R(k+1)j} \therefore \left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^n < 1$, and therefore $a^{new}_{pq} = a_{pq} + \alpha < a_{pq}$.

To analyse the condition that leads to $w^{new}_{R(k)} < w^{new}_{R(k+1)}$, we must show that

$$w^{new}_{R(k)j} = \frac{\left(\prod_{j=1}^{n} a^{new}_{R(k)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a^{new}_{ij}\right)^{\frac{1}{n}}} < w^{new}_{R(k+1)j} = \frac{\left(\prod_{j=1}^{n} a^{new}_{R(k+1)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a^{new}_{ij}\right)^{\frac{1}{n}}}$$

Because the denominator is the same, we must show that

$$\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}}$$

$$\because R(k)=p,\ R(k+1)\neq q \therefore R(k+1)\neq p, q$$

Based on condition $R(k) = p$, $R(k + 1) \neq q$ and assumption 3.

Except $a_{pq}^{new}=(a_{pq}+\alpha)$ and $a_{qp}^{new}=\frac{1}{a_{pq}^{new}}=\frac{1}{a_{pq}+\alpha}$, all other elements remain the same.

$$\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}} = \left(a_{p1}^{new}\times a_{p2}^{new}\ldots\times a_{p(q-1)}^{new}\times\left(a_{pq}^{new}\right)\times a_{p(q+1)}^{new}\ldots\times a_{pn}^{new}\right)^{\frac{1}{n}}$$

$$= \left(a_{p1}\times a_{p2}\ldots\times a_{p(q-1)}\times\left(a_{pq}+\alpha\right)\times a_{p(q+1)}\ldots\times a_{pn}\right)^{\frac{1}{n}}$$

$$\left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}} = \left(a_{R(k+1)1}^{new}\times a_{R(k+1)2}^{new}\ldots\times a_{R(k+1)n}^{new}\right)^{\frac{1}{n}}$$

$$= \left(a_{R(k)1}\times a_{R(k)2}\ldots\times a_{R(k)n}\right)^{\frac{1}{n}} = \left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}$$

To satisfy $\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k+1)j}^{\text{new}}\right)^{\frac{1}{n}}$

$$\Rightarrow \left(a_{p1}\times a_{p2}\ldots.\times a_{p(q-1)}\times\left(a_{pq}+\alpha\right)\times a_{p(q+1)}\ldots\times a_{pn}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}$$

$$\Rightarrow \left(a_{p1}\times a_{p2}..\times a_{p(q-1)}\times\left(\frac{a_{pq}\left(a_{pq}+\alpha\right)}{a_{pq}}\right)\times a_{p(q+1)}\ldots\times a_{pn}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}$$

$$\Rightarrow \left(\left(\frac{a_{pq}+\alpha}{a_{pq}}\right)\times a_{p1}\times a_{p2}\ldots\times a_{pn}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}$$

$$\Rightarrow \left(\left(\frac{a_{pq}+\alpha}{a_{pq}}\right)^{\frac{1}{n}}\left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}}\right) < \left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}$$

$$\Rightarrow\left(\frac{a_{pq}+\alpha}{a_{pq}}\right)^{\frac{1}{n}}\left(\frac{\left(\prod_{j=1}^{n}a_{R(k)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n}a_{ij}\right)^{\frac{1}{n}}}\right)<\frac{\left(\prod_{j=1}^{n}a_{R(k+1)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n}a_{ij}\right)^{\frac{1}{n}}}$$

$$\Rightarrow\left(\frac{a_{pq}+\alpha}{a_{pq}}\right)^{\frac{1}{n}}\omega_{R(k)}<w_{R(k+1)}$$

$$\Rightarrow\left(\frac{a_{pq}+\alpha}{a_{pq}}\right)<\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^{n}$$

$$\Rightarrow\left(a_{pq}+\alpha\right)<\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^{n}\times a_{pq}$$

$$\Rightarrow\alpha<a_{pq}\left(\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^{n}-1\right)$$

From Corollary 6, we can conclude that under the condition $R(k) = p$, $R(k + 1) \neq q$, the relative ranking of criterion $R(k)$ and $R(k + 1)$ will change if $\alpha<a_{pq}\left(\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^{n}-1\right)$.

Appendix 7

Proof of Corollary 7

Corollary 7: When $R(k) = p$, $R(k + 1) = q$ then $\mathrm{w}_{R(k)}^{new}<w_{R(k+1)}^{new}$ if $-a_{pg}<\alpha<a_{pq}\left(1-\sqrt{\frac{w_{R(k+1)}}{w_{R(k)}}}\right)$.

Proof: To analyse the condition that leads to $\mathrm{w}_{R(k)}^{new}<w_{R(k+1)}^{new}$, we must show that

$$w_{R(k)j}^{new}=\frac{\left(\prod_{j=1}^{n}a_{R(k)j}^{new}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n}a_{ij}^{new}\right)^{\frac{1}{n}}}<w_{R(k+1)j}^{new}=\frac{\left(\prod_{j=1}^{n}a_{R(k+1)j}^{new}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n}a_{ij}^{new}\right)^{\frac{1}{n}}}$$

Because the denominators are the same, therefore we must show that

$$\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}}$$

Based on the condition in $R(k) = p$, $R(k + 1) = q$ and assumption 3.

Except $a_{pq}^{new} = (a_{pq} + \alpha)s$ and $a_{qp}^{new} = \frac{1}{a_{pq}^{new}} = \frac{1}{a_{pq} + \alpha}$, all other elements remain the same.

$$\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}} = \left(a_{p1}^{new} \times a_{p2}^{new} \ldots \times a_{p(q-1)}^{new} \times \left(a_{pq}^{new}\right) \times a_{p(q+1)}^{new} \ldots \times a_{pn}^{new}\right)^{\frac{1}{n}}$$

$$= (a_{p1} \times a_{p2} \ldots \times a_{p(q-1)} \times \left(a_{pq} + \alpha\right) \times a_{p(q+1)} \ldots \times a_{pn})^{\frac{1}{n}}$$

$$\left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}} = \left(a_{q1}^{new} \times a_{q2}^{new} \ldots \times a_{q(p-1)}^{new} \times \left(a_{qp}^{new}\right) \times a_{q(p+1)}^{new} \times \ldots \times a_{qn}^{new}\right)^{\frac{1}{n}}$$

$$= \left(a_{q1} \times a_{q2} \ldots \times a_{q(p-1)} \times \left(\frac{1}{a_{pq} + \alpha}\right) \times a_{q(p+1)} \times \ldots \times a_{qn}\right)^{\frac{1}{n}}$$

To satisfy $\left(\prod_{j=1}^{n} a_{R(k)j}^{new}\right)^{\frac{1}{n}} < \left(\prod_{j=1}^{n} a_{R(k+1)j}^{new}\right)^{\frac{1}{n}}$

$$\Rightarrow \left(a_{p1} \times a_{p2} \ldots \times a_{p(q-1)} \times \left(a_{pq} + \alpha\right) \times a_{p(q+1)} \ldots \times a_{pn}\right)^{\frac{1}{n}}$$

$$< \left(a_{q1} \times a_{q2} \ldots \times a_{q(p-1)} \times \left(\frac{1}{a_{pq} + \alpha}\right) \times a_{q(p+1)} \times \ldots. \times a_{qn}\right)^{\frac{1}{n}}$$

$$\Rightarrow \left(a_{p1} \times a_{p2} \ldots \times a_{p(q-1)} \times \left(\frac{a_{pq}\left(a_{pq} + \alpha\right)}{a_{pq}}\right) \times a_{p(q+1)} \ldots \times a_{pn}\right)^{\frac{1}{n}}$$

$$< \left(a_{q1} \times a_{q2} \ldots \times a_{q(p-1)} \times \left(\frac{a_{qp}}{a_{qp}\left(a_{pq} + \alpha\right)}\right) \times a_{q(p+1)} \times .. \times a_{qn}\right)^{\frac{1}{n}}$$

$$\Rightarrow \left(\left(\frac{a_{pq} + \alpha}{a_{pq}}\right) \times a_{p1} \times a_{p2} \ldots \times a_{pn}\right)^{\frac{1}{n}} < \left(\left(\frac{1}{a_{qp}\left(a_{pq} + \alpha\right)}\right) \times a_{q1} \times a_{q2} \ldots \times a_{qn}\right)^{\frac{1}{n}}$$

$$\Rightarrow \left(\frac{a_{pq} + \alpha}{a_{pq}}\right)^{\frac{1}{n}} \left(\prod_{j=1}^{n} a_{R(k)j}\right)^{\frac{1}{n}} < \left(\frac{1}{a_{qp}\left(a_{pq} + \alpha\right)}\right)^{\frac{1}{n}} \left(\prod_{j=1}^{n} a_{R(k+1)j}\right)^{\frac{1}{n}}$$

$$\Rightarrow\left(\frac{a_{pq}+\alpha}{a_{pq}}\right)^{\frac{1}{n}}\left(\frac{\left(\prod_{j=1}^{n}a_{R(k)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n}a_{ij}\right)^{\frac{1}{n}}}\right)<\left(\frac{1}{a_{qp}\left(a_{pq}+\alpha\right)}\right)^{\frac{1}{n}}\left(\frac{\left(\prod_{j=1}^{n}a_{R(k+1)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n}a_{ij}\right)^{\frac{1}{n}}}\right)$$

$$\Rightarrow\left(\frac{a_{pq}+\alpha}{a_{pq}}\right)^{\frac{1}{n}}w_{R(k)}<\left(\frac{1}{a_{qp}\left(a_{pq}+\alpha\right)}\right)^{\frac{1}{n}}w_{R(k+1)}$$

$$\Rightarrow\left(\frac{a_{pq}+\alpha}{a_{pq}}\right)^{\frac{1}{n}}\left(\prod_{j=1}^{n}a_{R(k)j}\right)^{\frac{1}{n}}<\left(\frac{1}{a_{qp}\left(a_{pq}+\alpha\right)}\right)^{\frac{1}{n}}\left(\prod_{j=1}^{n}a_{R(k+1)j}\right)^{\frac{1}{n}}$$

$$\Rightarrow\left(\frac{a_{pq}+\alpha}{a_{pq}}\right)^{\frac{1}{n}}\left(\frac{\left(\prod_{j=1}^{n}a_{R(k)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n}a_{ij}\right)^{\frac{1}{n}}}\right)<\left(\frac{1}{a_{qp}\left(a_{pq}+\alpha\right)}\right)^{\frac{1}{n}}\left(\frac{\left(\prod_{j=1}^{n}a_{R(k+1)j}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n}a_{ij}\right)^{\frac{1}{n}}}\right)$$

$$\Rightarrow\left(\frac{a_{pq}+\alpha}{a_{pq}}\right)^{\frac{1}{n}}w_{R(k)}<\left(\frac{1}{a_{qp}\left(a_{pq}+\alpha\right)}\right)^{\frac{1}{n}}w_{R(k+1)}$$

$$\Rightarrow\left(\frac{1}{a_{pq}}\right)\left(a_{pq}+\alpha\right)\left(\omega_{R(k)}\right)^{n}<\left(\frac{1}{a_{pq}+\alpha}\right)\left(a_{pq}\right)\left(w_{R(k+1)}\right)^{n}\left(\because a_{qp}=\frac{1}{a_{pq}}\right)$$

$$\Rightarrow\left(a_{pq}+\alpha\right)^{2}<\left(a_{pq}\right)^{2}\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^{n}$$

$$\Rightarrow\left(a_{pq}+\alpha\right)^{2}-\left(a_{pq}\right)^{2}\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^{n}<0$$

$$\Rightarrow\alpha^{2}+2a_{pq}\alpha+\left(a_{pq}\right)^{2}\left(1-\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^{n}\right)<0$$

We use the quadratic formula to solve equations in standard form $\alpha=\frac{-b\pm\sqrt{b^{2}-4ac}}{2a}$

$$\Rightarrow\alpha=\frac{-2a_{pq}\pm\sqrt{4\left(a_{pq}\right)^{2}-4\left(a_{pq}\right)^{2}\left(1-\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^{n}\right)}}{2}$$

$$\Rightarrow\alpha=-a_{pq}\pm\sqrt{\left(a_{pq}\right)^{2}-\left(a_{pq}\right)^{2}\left(1-\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^{n}\right)}$$

$$\Rightarrow \alpha = -a_{pq} \pm a_{pq}\sqrt{1-\left(1-\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^n\right)}$$

$$\Rightarrow \alpha = -a_{pq}\left(1 \pm \sqrt{\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^n}\right)$$

$$\Rightarrow -a_{pq}\left(1+\sqrt{\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^n}\right) < \alpha < -a_{pq}\left(1-\sqrt{\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^n}\right)$$

For $\alpha > -a_{pq}\left(1+\sqrt{\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^n}\right)$. Since $w_{R(k)} < w_{R(k+1)}$ therefore $\sqrt{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n} > 0$ and $\left(1+\sqrt{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n}\right) > 1$ which implies $\alpha > -a_{pq}$, which violate assumption 1, that is $a_{pq} + \alpha > 0$, therefore $\alpha > -a_{pq}\left(1+\sqrt{\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^n}\right)$ is invalid.

$$\therefore \sqrt{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n} > 0$$

$$\Rightarrow \left(1+\sqrt{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n}\right) > 1 \Rightarrow \alpha > -a_{pq}$$

For $\alpha < -a_{pq}\left(1-\sqrt{\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^n}\right)$

$$\because \omega_{R(k)} < \omega_{R(k+1)} \therefore \sqrt{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n} < 1$$

$$\Rightarrow \left(1-\sqrt{\left(\frac{w_{R(k)}}{w_{R(k+1)}}\right)^n}\right) < 1$$

$$\Rightarrow -a_{pq} < \alpha < -a_{pq}\left(1-\sqrt{\left(\frac{w_{R(k+1)}}{w_{R(k)}}\right)^n}\right)$$

□

From Corollary 7, we can conclude that under the condition $R(k) = p$, $R(k + 1)$ q, the relative ranking of criterion $R(k)$ and $R(k + 1)$ will change if $-a_{pq} < \alpha < -a_{pq}\left(1-\left(\frac{w_{r(k+1)}}{w_{R(k)}}\right)^n\right)$.