MAOAM: multiple-aspect outstandingness appraisement methodology

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Abstract: In this contribution, we devise and apply a method for ranking multi-attribute alternatives. The suggested method is founded, inter alia, on the so-called positive and negative outstandingness predicates. This method, herein called multiple-aspect outstandingness appraisement method (MAOAM), employs in the aggregation phase a family of multivariate generalised weighted Heronian means having as special cases the Beliakov et al.'s (2008) generalised Heronian means as well as that of Chu and Liu (2015). With positive and negative outstandingness predicates and multivariate generalised weighted Heronian means, we are better equipped to rank rationally pre-specified alternatives. The feasibility and accuracy of the proposed method are shown in a practical application.

Keywords: multi-criteria decision making; MCDM; positive outstandingness predicate; negative outstandingness predicate; weighted generalised Heronian mean operator.

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1 Introduction

Multi-criteria decision making (MCDM) enables to select the most appropriate alternative among predetermined alternatives by evaluating them in terms of many criteria (Papathanasiou and Ploskas, 2018). According to recent literature (Kaya et al., 2019; Xing et al., 2018; Xu et al., 2018; Mao et al., 2018; Zhang et al., 2017; Zhu et al., 2019), there are different types of fuzzy MCDM methods and their applications in the literature such as:

a   pairwise comparison-based methods (e.g., analytic hierarchy process) where the decision makers can systematically assess the alternatives by making pair-wise comparisons for each of the chosen criteria (Saaty, 1980)

b   outranking methods [e.g., PROMETHEE (Brans and Mareschal, 2005) and SIR (Xu, 2001)] in which – based on the deviation between the evaluations of every pair of alternatives in each criterion – aggregated preference indices and positive/negative or superiority/inferiority outranking flows are computed

c   distance based methods [e.g., VIKOR and TOPSIS (Opricovic and Tzeng, 2007, 2004; San Cristóbal, 2011; Yazdani and Graeml, 2014)] in which alternatives are evaluated according to their distance to ideal solutions.

Nevertheless, expressing decision-makers' fuzzy decision information such as his preference information in an expressive and flexible manner remain a huge challenge. This is mainly due to the fact that subjective factors related to the decision-makers' own experience, his cognitive ability, and intuition hugely influence the decision information and hence the decision process and outcome.

Recently, tools and methods that can describe uncertain information and quite a few influential theories have been proposed. According to De Luca and Termini (1972), Weber (1984), Czogała et al. (1982), Farr and Brown (1971) and Ying (1990), there are three types of uncertainty when dealing with problems in fuzzy decision making:

1   *randomness* that can be captured by a probabilistic measure

2   *fuzziness* that can be measured using entropy or energy measures

3   *alternativity*, the opposite of outstandingness, that can be measured using the outstandingness measure introduced in Ying (1990).

Two types of outstandingness fuzzy binary predicates have been proposed in Ying (1990): a positive fuzzy predicate expressing the degree of positive outstandingness of one alternative over another whereas the negative fuzzy predicate expresses the degree of negative outstandingness of one alternative over another. A critical issue may, however, emerge from the process of subject judgments concerning appraisal behaviours when the MCDM approaches ignore one side of the evaluation. Indeed, the fuzzy set theory gives rise to type I bipolarity called the bipolar univariate model (Osgood, 1957). Fuzzy sets only give a membership degree to each element and the non-membership degree is always equivalent to the complementary. Therefore, a decrease in the negative degree (non-membership function) goes hand in hand with an increase of the positive degree (membership function). But, according to recent development in psychological studies, the positive and negative degrees are not necessarily complementary to each other (Cacioppo and Berntson, 1994; Cacioppo et al., 1997; Cacioppo and Berntson, 1999; Norris et al., 2010). An interesting property of the positive and negative outstandingness predicates is that they may be defined in such a way that they are *antagonistic* and completely *independent* of each other. Hence, such outstandingness predicates may exhibit a good ability to describe and express decision-makers' fuzzy decision information in an expressive and flexible manner that takes into consideration the latest development in psychological studies.

Another challenge in the design of MCDM methods is that attributes in most real problems are correlated. This led to a number of works in the literature to consider the

application of the Bonferroni (1950) and Heronian means (Beliakov et al., 2008; Chu and Liu, 2015; Janous, 2001; Yu and Wu, 2012) in the aggregation phase. According to a recent work Pramanik et al. (2020), however, the Heronian mean operator is shown to be better equipped to embody the interaction between attributes to avoid unreasonable situations in information aggregation. In Wang et al. (2017), the Bonferroni mean aggregation operator has been shown to neglect the relationship between each attribute and other attributes repeatedly when compared to the Hernonian mean. Furthermore, the Bonferroni aggregation operator has a larger computational complexity compared to the Heronian mean.

The main objective of this paper is to rank rationally pre-specified alternatives starting, from pairwise comparisons of alternative evaluations on each criterion that measure outstandingness. Towards this end, we make use of, among others, the fuzzy positive and negative outstandingness predicates which measure the degree of outsandingness of an alternative against another alternative in both the positive and negative senses (Ying, 1990). We give a number of various outstandingness predicates and study the properties of such predicate pairs. The proposed method, herein called multiple-aspect outstandingness appraisement method (MAOAM), employs in the aggregation phase, a family of multivariate generalised weighted Heronian means having as special cases the Beliakov et al.'s (2008) generalised Heronian means as well as that of Chu and Liu (2015). With positive and negative outstandingness predicates and multivariate generalised weighted Heronian means, we are better equipped to rank rationally pre-specified alternatives based on outstandingness that allow for

1   expressing the positive outstandingness degrees in an independent manner from the negative one

2   aggregating information that handles the possible interaction between attributes.

The rest of the paper is organised as follow. In Section 2, we define the problem and our assumptions. In Section 3, we introduce the concepts of positive and negative outstandingness predicates as well as the Heronian mean and its generalisations which constitute the mathematical tools for the MAOAM method. Next, the description of the MAOAM method as a ranking method well adapted to problems where a finite number of alternatives are to be ranked considering several criteria is presented in Section 4. Finally, before concluding in Section 6, we present an illustrative exampel using a real-world investment bank dataset in Section 5.

2   Problem definition

To begin, the problem formulation can be set out as follows:
    Given

- A set of m feasible alternatives $A = \{A_i | i \in \{1, ..., m\}\}$.

- A set of n aspects $C = \{C_j | j \in \{1, .., n\}\}$.

- A $m \times n$ achievement (evaluation) matrix a where a_{ij} denotes the achievement value of alternative A_i with respect to aspect C_j. Each line corresponds to an alternative and each column corresponds to an aspect. Each alternative has a vector of achievement (or degrees of satisfaction) $\in [0, 1]^n$ unit hypercube.

$$a = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{pmatrix} C_1 & C_2 & \ldots & C_n \\ a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{pmatrix}$$

Note that $(a_{i,1}, a_{i,2}, ..., a_{i,n})$ is the achievement vector associated with alternative A_i whereas $(a_{1,j}, a_{2,j}, ..., a_{m,j})$ is the achievement vector associated with aspect C_j.

- A weight vector $w = (w_1, w2, ..., w_n)$ which represents weights of relative importance of the different aspects. These weights are non-negative numbers independent from the measurement units of the aspect. The higher the weight, the more important is the aspect. Each aspect weight w_j is in (0, 1) and

$$1 \leq \sum_j w_j \leq n$$

The objective is to rank the available alternatives using their positive and negative outstandingness predicates whose formulas will be set hereafter (Subsection 3.1)

3   Development of MAOAM

We introduce the concepts of positive and negative outstandingness predicates (Ying, 1990) as well as the Heronian mean and its generalisations which constitute the mathematical tools for the MAOAM method.

3.1   Positive and negative outstandingness predicates

Ying (1990) defined two fuzzy binary predicates on the unit interval [0, 1]. One expresses the positive outstandingness of an alternative x over alternative y, denoted by $\mathcal{P}(x, y)$, and the other for expressing the negative outstandingness, denoted by $\mathcal{N}(x, y)$.

Definition 3.1: A mapping $\mathcal{P}: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a positive outstandingness predicate if and only if it satisfies the following properties.

1    $\mathcal{P}(1, 0) = 1$

2   if $x \leq y$, then $\mathcal{P}(x, y) = 0$

3a   if $y < x_1 < x_2$, then $\mathcal{P}(x_1, y) \leq \mathcal{P}(x_2, y)$

3b   if $x_2 < x_1 < y$, then $\mathcal{P}(y, x_1) \leq \mathcal{P}(y, x_2)$.

The following predicates satisfy all four properties stated above and, hence, are all positive outstandingness predicates.

$$\mathcal{P}_1(x, y) = \max(x - y, 0)$$

$$\mathcal{P}_2(x, y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{P}_3(x, y) = \begin{cases} x & \text{if } x > y \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{P}_4(x, y) = \begin{cases} \frac{x-y}{x} & \text{if } x > y \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{P}_5(x, y) = \begin{cases} \frac{x}{x+y} & \text{if } x > y \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{P}_6(x, y) = \begin{cases} \frac{x-y}{x+y} & \text{if } x > y \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{P}_7(x, y) = \begin{cases} \frac{x-y}{1+\frac{y}{x}} & \text{if } x > y \\ 0 & \text{otherwise.} \end{cases}$$

It is important to mention that the generalised preference predicates defined as $P(d)$ where $d = x - y$ introduced in PROMETHEE (Brans and Mareschal, 2005) can also be viewed as positive outstandingness predicates for all constants $p, q, s \in [0, 1]$:

$$\mathcal{P}_2(d) = \mathcal{P}_2(x, y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{P}_8(x, y) = \begin{cases} 1 & \text{if } x - y > q \\ 0 & x - y \leq q \end{cases}$$

$$\mathcal{P}_9(x, y) = \begin{cases} 1 & \text{if } x - y > p \\ \frac{x-y}{p} & 0 \leq x - y \leq p \\ 0 & x - y \leq 0 \end{cases}$$

$$\mathcal{P}_{10}(x, y) = \begin{cases} 1 & \text{if } x - y > p \\ \frac{1}{2} & q \leq x - y \leq p \\ 0 & x - y \leq q \end{cases}$$

$$\mathcal{P}_{11}(x, y) = \begin{cases} 1 & \text{if } x - y > p \\ \frac{x-y-q}{p-q} & q \leq x - y \leq p \\ 0 & x - y \leq p \end{cases}$$

$$\mathcal{P}_{12}(x, y) = \begin{cases} 1 - e^{-\frac{(x-y)^2}{2s^2}} & \text{if } x - y > 0 \\ 0 & x - y \leq 0 \end{cases}$$

Definition 3.2: A mapping $\mathcal{N}$: [0, 1] × [0, 1] → [0, 1] is a negative outstandingness predicate if and only if it satisfies the following properties.

1    $\mathcal{N}(0, 1) = 1$

2   if $x \leq y$, then $\mathcal{N}(y, x) = 0$

3a if $y < x_1 < x_2$, then $\mathcal{N}(y, x_1) \leq \mathcal{N}(y, x_2)$

3b if $x_2 < x_1 < y$, then $\mathcal{N}(x_1, y) \leq \mathcal{N}(x_2, y)$.

The following predicates satisfy all four properties stated above and, hence, are all negative outstandingness predicates.

$$\mathcal{N}_1(y, x) = \max(x - y, 0)$$

$$\mathcal{N}_2(y, x) = \begin{cases} 0 & \text{if } y > x \\ 1 & \text{otherwise.} \end{cases}$$

$$\mathcal{N}_3(y, x) = \begin{cases} 0 & \text{if } y > x \\ 1 - y & \text{otherwise.} \end{cases}$$

$$\mathcal{N}_4(y, x) = \begin{cases} 0 & \text{if } y \geq x \\ \frac{x-y}{x} & \text{otherwise.} \end{cases}$$

$$\mathcal{N}_5(y, x) = \begin{cases} 0 & \text{if } y \geq x \\ \frac{x}{x+y} & \text{otherwise.} \end{cases}$$

$$\mathcal{N}_6(y, x) = \begin{cases} 0 & \text{if } y \geq x \\ \frac{x-y}{x+y} & \text{otherwise.} \end{cases}$$

$$\mathcal{N}_7(y, x) = \begin{cases} 0 & \text{if } y \geq x \\ \frac{x-y}{1+\frac{y}{x}} & \text{otherwise.} \end{cases}$$

Next, the following proposition states the compliment of a positive outstandingness predicate (Ying, 1990).

Proposition 1: If $\mathcal{P}$ is a positive outstandingness predicate, then $\mathcal{P}^c$ defined by

$$\mathcal{P}^c(x, y) = \mathcal{P}(1 - x, 1 - y) \quad \forall x, y \in [0, 1]$$

is a negative outstandingness predicate.

Similarly, we can derive the compliment of a negative outstandingness predicate as follows (Ying, 1990):

Proposition 2: If $\mathcal{N}$ is a negative outstandingness predicate, then $\mathcal{N}^c$ defined by

$$\mathcal{N}^c(x, y) = \mathcal{N}(1 - x, 1 - y) \quad \forall x, y \in [0, 1]$$

is a positive outstandingness predicate.

Definition 3.3: Let $\mathcal{P}$ be a positive outstandingness predicate and $\mathcal{N}$ is a negative outstandingness predicate such that:

$$\mathcal{N}(x, y) = \mathcal{P}(1 - x, 1 - y) \quad \forall x, y \in [0, 1]$$

and

$$\mathcal{P}(x, y) = \mathcal{N}(1 - x, 1 - y) \quad \forall x, y \in [0, 1]$$

Then, $(\mathcal{P}, \mathcal{N})$ are said to be *double* pair predicates denoted D-pair.

In addition to the D-pair, we introduce four more pairs of the outstandingness predicates.

Definition 3.4: If $\mathcal{P}(x,y) = \mathcal{N}(y,x) \forall x, y \in [0,1]$ then $(\mathcal{P}, \mathcal{N})$ are called *unbiased* pair predicates denoted by U-pair.

For instance, $(\mathcal{P}_1, \mathcal{N}_1)$ is a U-pair. Consider the generalised preference predicates $\mathcal{P}_2$, $\mathcal{P}_8$, $\mathcal{P}_9$, $\mathcal{P}_{10}$, $\mathcal{P}_{11}$, and $\mathcal{P}_{12}$ introduced in PROMETHEE (Brans and Mareschal, 2005) to compute the preference of alternative x over alternative y. To compute the preference of alternative y over x instead (from which negative outranking flow are computed), the PROMETHEE method simply swaps the inputs of $\mathcal{P}(x,y)$, i.e., it uses $\mathcal{P}(y,x)$. Since $\mathcal{P}(x,y)$ is a positive outstandingness predicate, it can easily be shown that $\mathcal{P}(y,x)$ is indeed a negative outstandingness predicate. Thus, in the PROMETHEE method it turns out that *only U-pairs* of the form $(\mathcal{P}(x,y), \mathcal{P}(y,x))$ are indeed used to compute the poditive/negative preference index between pairs of alternatives in each criterion.

It is important to note that U-pairs are not independent of each other and do provide complimentary evaluations. Next, we introduce pairs whose positive and negative evaluations are independent.

Definition 3.5: If $\mathcal{P}(x,y) \leq \mathcal{N}(y,x) \forall x, y \in [0,1]$ then $(\mathcal{P}, \mathcal{N})$ are called *negativity biais* pair predicates denoted by NB-pair.

For instance, $(\mathcal{P}_5, \mathcal{N}_2)$ is a NB-pair.

Definition 3.6: If $\mathcal{P}(x,y) \geq \mathcal{N}(y,x) \forall x, y \in [0,1]$ then $(\mathcal{P}, \mathcal{N})$ are called *positivity biais* pair predicates denoted by PB-pair.

For instance, $(\mathcal{P}_4, \mathcal{N}_1)$ is a PB-pair.

Definition 3.7: If all the previous conditions are not satisfied, then $(\mathcal{P}, \mathcal{N})$ are called *arbitrary* pair predicates denoted by A-pair.

For instance, $(\mathcal{P}_3, \mathcal{N}_3)$ is an A-pair.

3.2   *The Heronian mean and its generalisations*

Let a and b be two non-negative real numbers. The Heronian mean of a and b is defined in equation (1) as follows (Beliakov et al., 2008):

$$He(a,b) = \frac{a + \sqrt{ab} + b}{3} \tag{1}$$

Recently, Janous (2001) defined further the generalised Heronian mean as shown in equation (2):

$$H_\omega(a,b) = \begin{cases} \frac{a + \omega\sqrt{ab} + b}{\omega+2} & 0 \leq \omega \leq +\infty \\ \sqrt{ab} & \omega = +\infty \end{cases} \tag{2}$$

Note that if $\omega = 1$, it is clear that $H_1(a, b) = He(a, b)$. If $\omega = 0$, then $H_0(a, b) = \frac{a+b}{2} = A(a, b)$ where $A(a, b)$ is the arithmetic mean. If $\omega = \infty$, then $H_\infty(a, b) = \sqrt{ab} = G(a, b)$ where $G(a, b)$ is the geometric mean.

$$G(a, b) \leq H_\omega(a, b) \leq A(a, b) \tag{3}$$

Note that the above Heronian mean and its generalisations are those of two real numbers. Next, we will define the generalised Heronian mean in n variables:

Let $x = (x_1, x_2, ..., x_n)$; $x_i \geq 0$, $i = 1, 2, ..., n$. The generalised Heronian mean in n variables $x_1, x_2, ..., x_n$ is defined in equation (4).

$$H_\omega(x) \begin{cases} \sum_i x_i + \omega \sqrt[n]{x_1 x_2 ... x_n} & 0 \leq \omega < +\infty \\ \sqrt[n]{x_1 x_2 ... x_n} & \omega = +\infty \end{cases} \tag{4}$$

Note that if $n = 2$, $H_\omega(x) = H_\omega(a, b)$.

Beliakov et al. (2008) has proposed a different generalisation of the Hernian mean in x which does not allow for $\sqrt[n]{x_1 x_2 ... x_n}$ to evaluate to zero if any of the x_i is zero and is shown in equation (5).

$$HR(x) = \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{x_i x_j} \tag{5}$$

Note that for $n = 2$, we have $HR(a, b) = He(a, b)$.

We might as well extend Beliakov's definition to introduce ω as in Janous (2001) and the extended form is shown in equation (6):

$$HR(x, \omega) = \frac{\sum_{k=1}^{n} x_k + \omega \sum_{k=1}^{n} \sum_{l=k+1}^{n} \sqrt{x_i x_j}}{n + \omega \frac{n(n-1)}{2}} \tag{6}$$

Note that $HR(x, 1) = HR(x)$.

Unfortunately, the Heronian mean $HR(x)$ and the generalised Heronian mean $H_\omega(x)$ operators ignore the weights of the criteria which has been incorporated in the generalisation of Chu and Liu (2015) called the weighted generalised Heronian mean ($WGHM$). Let $W = (w_1, w_2, ..., w_n)$ be the relative importance weights of $x = (x_1, x_2, ..., x_n)$ such that $\sum_{i=1}^{n} w_i = 1$, respectively. Let $p, q \geq 0$. The $WGHM^{p,q}(x, W)$ defined in Chu and Liu (2015) in shown in equation (7):

$$WGHM^{p,q}(x, W) = \frac{\left(\sum_{k=1}^{n} \sum_{l=1}^{n} (w_k x_k)^p (w_l x_l)^q \right)^{\frac{1}{p+q}}}{\left(\sum_{k=1}^{n} \sum_{l=k}^{n} w_k^p w_l^q \right)^{\frac{1}{p+q}}} \tag{7}$$

In the following, we extend the $WGHM$ operator by taking into account ω (as in Walher Janous) as well as weights.

Let $W = (w_1, w_2, ..., w_n)$ the relative importance weights of $x = (x_1, x_2, ..., x_n)$, respectively. Let $p, q \geq 0$. Let $|W| = \sum_{k=1}^{n} w_k$ be the magnitude of W. We propose a generalisation of the Hernian mean, named $WGHM$, in n variables that takes into consideration the relative importance weights as follows.

When $0 \leq \omega < +\infty$, we have equation (8).

$$WGHM_\omega^{p,q}(x,W)$$
$$= \frac{(\sum_{k=1}^n (w_k x_k)^{p+q} + \omega \sum_{k=1}^n \sum_{l=k+1}^n (w_k x_k)^p (w_l x_l)^q)^{\frac{1}{p+q}}}{(\sum_{k=1}^n w_k^{p+q} + \omega \sum_{k=1}^n \sum_{l=k+1}^n w_k^p w_l^q)^{\frac{1}{p+q}}} \tag{8}$$

When $\omega = +\infty$, we have equation (9).

$$WGHM_\omega^{p,q}(x,W) = \frac{(\sum_{k=1}^n \sum_{l=k+1}^n (w_k x_k)^p (w_l x_l)^q)^{\frac{1}{p+q}}}{(\sum_{k=1}^n \sum_{l=k+1}^n w_k^p w_l^q)^{\frac{1}{p+q}}} \tag{9}$$

If the sum of the weights is one and $\omega = 1$, we have:

$$WGHM_\omega^{p,q}(x,W) = WGHM^{p,q}(x,W)$$

If the relative importance weights are equal and $p = q = \frac{1}{2}$, we have:

$$WGHM_\omega^{\frac{1}{2},\frac{1}{2}}(x,W) = HR(x,\omega)$$
$$WGHM_{\omega=1}^{\frac{1}{2},\frac{1}{2}}(x,W) = HR(x)$$
$$WGHM_\omega^{\frac{1}{2},\frac{1}{2}}((x_1,x_2),W) = H_\omega(x_1,x_2)$$

In the following, we prove that $WGHM_\omega^{p,q}(x,W)$ has the properties of idempotency, monotonicity, and internality.

Theorem 3: $WGHM_\omega^{p,q}(x,W)$ is idempotent.

Proof: Let $x_i = a$ for all $i = 1, 2, ..., n$.

$$WGHM_\omega^{p,q}(x,W) = \frac{(\sum_{k=1}^n w_k^{p+q} a^{p+q} + \omega \sum_{k=1}^n \sum_{l=k+1}^n w_k^p w_l^q a^{p+q})^{\frac{1}{p+q}}}{(\sum_{k=1}^n w_k^{p+q} + \omega \sum_{k=1}^n \sum_{l=k+1}^n w_k^p w_l^q)^{\frac{1}{p+q}}}$$
$$= \frac{(a^{p+q}(\sum_{k=1}^n w_k^{p+q} + \omega \sum_{k=1}^n \sum_{l=k+1}^n w_k^p w_l^q))^{\frac{1}{p+q}}}{(\sum_{k=1}^n w_i^{p+q} + \omega \sum_{k=1}^n \sum_{l=k+1}^n w_k^p w_l^q)^{\frac{1}{p+q}}}$$
$$= a \frac{(\sum_{k=1}^n w_k^{p+q} + \omega \sum_{k=1}^n \sum_{l=k+1}^n w_k^p w_l^q)^{\frac{1}{p+q}}}{(\sum_{k=1}^n w_k^{p+q} + \omega \sum_{k=1}^n \sum_{l=k+1}^n w_k^p w_l^q)^{\frac{1}{p+q}}}$$
$$= a \frac{(\sum_{k=1}^n w_k^{p+q} + \omega \sum_{k=1}^n \sum_{l=k+1}^n w_k^p w_l^q)^{\frac{1}{p+q}}}{(\sum_{k=1}^n w_k^{p+q} + \omega \sum_{k=1}^n \sum_{l=k+1}^n w_k^p w_l^q)^{\frac{1}{p+q}}}$$
$$= a$$

$\square$

Theorem 4: $WGHM_\omega^{p,q}(x,W)$ is monotone.

Proof: Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ such that $x_k \geq y_k$ for all k. We need to show that $WGHM_\omega^{p,q}(x, W) \geq WGHM_\omega^{p,q}(y, W)$.

Consider

$$WGHM_\omega^{p,q}(x, W) = \frac{(\sum_{k=1}^n (w_k x_k)^{p+q} + \omega \sum_{k=1}^n \sum_{l=k+1}^n (w_k x_k)^p (w_l x_l)^q)^{\frac{1}{p+q}}}{(\sum_{k=1}^n w_k^{p+q} + \omega \sum_{k=1}^n \sum_{l=k+1}^n w_k^p w_l^q)^{\frac{1}{p+q}}}$$

Since, $p, q \geq 0$, $0 \leq \omega \leq +\infty$, and $x_k \geq y_k$, we have $(w_k x_k)^{p+q} \geq (w_k y_k)^{p+q}$. Thus,

$$WGHM_\omega^{p,q}(x, W) \geq \frac{(\sum_{k=1}^n (w_k y_k)^{p+q} + \omega \sum_{k=1}^n \sum_{l=k+1}^n (w_k x_k)^p (w_l x_l)^q)^{\frac{1}{p+q}}}{(\sum_{k=1}^n w_k^{p+q} + \omega \sum_{k=1}^n \sum_{l=k+1}^n w_k^p w_l^q)^{\frac{1}{p+q}}}$$

Similarly, because $(w_l x_l)^p \geq (w_k y_k)^p$ and $(w_l x_l)^q \geq (w_l y_l)^q$ we have

$$WGHM_\omega^{p,q}(x, W) \geq \frac{(\sum_{k=1}^n (w_k y_k)^{p+q} + \omega \sum_{k=1}^n \sum_{l=k+1}^n (w_k y_k)^p (w_l y_l)^q)^{\frac{1}{p+q}}}{(\sum_{k=1}^n w_k^{p+q} + \omega \sum_{k=1}^n \sum_{l=k+1}^n w_k^p w_l^q)^{\frac{1}{p+q}}}$$

Thus, $WGHM_\omega^{p,q}(x, W) \geq WGHM_\omega^{p,q}(y, W)$. $\quad\square$

Theorem 5: $WGHM_\omega^{p,q}(x, W)$ satisfies the internality property.

Proof: Let $x^- = \min(x_1, x_2, ..., x_n)$ and $x^+ = \max(x_1, x_2, ..., x_n)$. According to Theorem 1, we have

$$WGHM_\omega^{p,q}((x^-, x^-, ..., x^-), W) = x^-$$
$$WGHM_\omega^{p,q}((x^+, x^+, ..., x^+), W) = x^+$$

Since, $x^- \leq x_k \leq x^+$ for all k, we have

$$WGHM_\omega^{p,q}((x^-, x^-, ..., x^-), W)$$
$$\leq WGHM_\omega^{p,q}(x, W)$$
$$\leq WGHM_\omega^{p,q}((x^+, x^+, ..., x^+), W)$$

Thus,

$$x^- \leq WGHM_\omega^{p,q}(x, W) \leq x^+$$

$\quad\square$

In this section, we have introduced two generalisation of the Heronian mean. The first one denoted $HR(x, \omega)$ extends Beliakov's definition by introducing the parameter ω as in Janous (2001) and the extended form is shown in equation (6). The introduction of ω provides more flexibility to the decision-maker since when ω is set to zero, $HR(x, \omega)$ becomes the arithmetic mean, and as ω gets larger and larger ($\omega \rightsquigarrow \infty$), it approaches the geometric mean. For the other values of ω, we have a wide range of Heronian means. The second generalisation extends the $WGHM$ operator by taking into account ω (as in Walher Janous) as well as weights [denoted by $WGHM_\omega^{p,q}(x, W)$ and shown in equations (8) and (9)]. The advantage of this generalisation, like $HR(x, \omega)$, is the introduction of the paramter ω which allows it express various means (arithmetic, geometric, and Heronian). Furthermore, it also takes into consideration the relative weights when aggregating the information. These features allow it to be an expressive and flexible aggregation operator.

4   The MAOAM method

MAOAM is a ranking method well adapted to problems where a finite number of alternatives are to be ranked considering several criteria. Recall that we are given,

- A set of m feasible alternatives $A = \{A_i | i \in \{1, ..., m\}\}$.

- A set of n aspects $C = \{C_j | j \in \{1, ..., n\}\}$.

- A $m \times n$ achievement (evaluation) matrix a where a_{ij} denotes the achievement value of alternative A_i with respect to aspect C_j. Each line corresponds to an alternative and each column corresponds to an aspect. Each alternative has a vector of achievement (or degrees of satisfaction) $\in [0, 1]^n$ unit hypercube.

- A weight vector $w = (w_1, w2, ..., w_n)$ which represents weights of relative importance of the different aspects. These weights are non-negative numbers independent from the measurement units of the aspect. The higher the weight, the more important is the aspect. Each aspect weight w_j is in (0, 1) and

$$1 \leq \sum_j w_j \leq n$$

The objective is to rank the available alternatives using their positive and negative outstandingness predicates. The procedural steps of the parameterised MAOAM method are enlisted below:

In our first step, we make pairwise comparisons between all the alternatives for each aspect using positive outstandigness predicates (POP) and negative outstandigness predicates (NOP). Assume we have a POP $\mathcal{P}$ (resp. NOP $\mathcal{N}$). For each aspect C_j and for each alternative A_k, we compute the positive (resp. negative) outstandingness of A_k against every other alternative A_l ($\neq A_k$) denoted by $PO_{k,l}^j$ (resp. $NO_{k,l}^j$) as follows:

$$PO_{k,l}^j = \mathcal{P}(a_{k,j}, a_{l,j}) \forall j \in \{1, ..., n\}, \forall k \in \{1, ..., m\}, \forall l \in \{1, ..., m | l \neq k\}$$

$$NO_{k,l}^j = \mathcal{N}(a_{k,j}, a_{l,j}) \forall j \in \{1, ..., n\}, \forall k \in \{1, ..., m\}, \forall l \in \{1, ..., m | l \neq k\}$$

For each aspect j and alternative k, let

$$PO_k^j = (PO_{k,l}^1, ..., PO_{k,m}^j)$$

where $\forall l \in \{1, ..., m | l \neq k\}$.

Similarity, let

$$NO_k^j = (NO_{k,l}^1, ..., NO_{k,m}^j)$$

Our second step is to compute the partial positive outstandingness $^+\Omega_k^j$ using the extended Hernonian mean of Beliakov et al. (2008) introduced in Section 3:

$$^+\Omega_k^j = HR(PO_k^j, \omega)$$

Similarly, we compute the partial negative outstandingness $^-\Omega_k^j$ as follows:

$$^-\Omega_k^j = HR(NO_k^j, \omega)$$

In the third step, we compute $^{+}\Omega_k$ as the global positive outstandingness index of each alternative k by using our generalised hernonian mean as follows:

$$^{+}\Omega_k = WGHM_{\omega}^{p,q}((^{+}\Omega_k^1, ..., ^{+}\Omega_k^j, ..., ^{+}\Omega_k^n), W)$$

Similarly, we compute $^{-}\Omega_k$ as the global negative outstandingness index of each alternative k as follows:

$$^{-}\Omega_k = WGHM_{\omega}^{p,q}((^{-}\Omega_k^1, ..., ^{-}\Omega_k^j, ..., ^{-}\Omega_k^n), W)$$

In the fourth step, we calculate the bounded difference overall outstandigness for each alternative k as follows:

$$\Omega_k = \max\left(0, 1 - \frac{^{-}\Omega_k}{^{+}\Omega_k}\right)$$

Or, alternatively, we compute the net outstandingness for each alternative k as follows:

$$\Omega_k = ^{+}\Omega_k - ^{-}\Omega_k$$

Finally, in step five, we determine the ranking of all alternatives depending on the values of Ω_k. The higher value of Ω_k, the more outstanding is the alternative.

5 Illustrative example

In this section, we validate the properties and evaluate the sensitivity of the MAOAM method using a real-life use case by mainly varying the:

- *Outstandingness predicates:* We consider four pairs of oustandingness predicates: an unbiased U-pair $((P1, N1))$, a negatively biased NB-pair $((P5, N2))$, a positively biased U-pair $((P4, N1))$, and an arbitrary A-pair $((P3, N3))$. This variation will allow us to validate the properties of these different outstandingness predicates.

- *Heronian means parameters:* In the MAOAM method, we introduced two Heronian means in the second and third step where each operator is parameterised with ω. Here, we use ω_1 instead of ω in the aggregation operator in the second step and ω_2 instead of ω in the third step to remove any ambiguity since each aggregation in each step can use different values for ω. We then consider values $\omega_1, \omega_2 \in \{0, 1,000\}$ that allows us to consider extreme cases when the aggregation is compensatory (when $\omega_1 = \omega_2 = 0$) and when the aggregation is non-compensatory (when $\omega_1 = \omega_2 = 1,000$) and combinations of these two extreme cases. This variation will allow us to study the sensitivity of the MAOAM method to different instantiations of the proposed Heronian means.

5.1 Context

The managers of a Tunsian investment bank receive applications of ivestment loans from various businessmen. The criteria alongside their weights used to evaluate the alternatives are shown in Table 1.

There are 32 candidates and their respective evaluations on the 18 criteria are shown in Table 3 where the objective is to rank the candidates according to their 'credibility' on a set of 18 weighted criteria.

Table 1   The criteria and their corresponding weights

	Criterion	Weight
C_1	Quality of the project proposal	0.075
C_2	Applicant's responsiveness	0.05
C_3	Applicant's risk aversion	0.075
C_4	Administrative and financial management	0.12
C_5	Technical knowledge	0.12
C_6	Commercial position	0.06
C_7	Infrastructure and logistics risk	0.025
C_8	Technology risk	0.05
C_9	Supply risk	0.05
C_{10}	Regulation risk	0.025
C_{11}	Demand risk	0.05
C_{12}	Offer risk	0.025
C_{13}	Quality risk	0.05
C_{14}	Distribution risk	0.025
C_{15}	Financial risk	0.05
C_{16}	Customer risk	0.05
C_{17}	Profitability risk	0.05
C_{18}	Financial exploitation risk	0.05

Table 2   Values of the measure A

	U-pair (P1, N1)	NB-pair (P5, N2)	PB-pair (P4, N1)	A-pair (P3, N3)
$\omega_1 = \omega_2 = 0$	16	2	20	16
$\omega_1 = 0, \omega_2 = 1,000$	16	2	20	16
$\omega_1 = 1,000, \omega_2 = 0$	16	2	17	16
$\omega_1 = 1,000, \omega_2 = 1,000$	16	2	16	16

5.2   Results

The results of are shown in Table 4 ($\omega_1 = \omega_2 = 0$), Table 5 ($\omega_1 = 0, \omega_2 = 1,000$), Table 6 ($\omega_1 = 1,000, \omega_2 = 0$), and Table 7 ($\omega_1 = \omega_2= 1,000$) where we report, for each alternative k and for each oustandingness predicate pair, the net oustandingess value Ω_k and the alternative's rank. Note that, we set $p = q = \frac{1}{2}$ for all experiments.

To validate the properties of the various fuzzy outstandingness predicate pairs, we introduce the following measure, denoted *measure A*: which is the number of alternatives with positive net outstandingness divided by the total number of alternatives. We summarise the results of *measure A* in Table 2 which confirm that the positively biased pair weighs most the positive outstandingness of an alternative wrt the negative one, followed by the U-pair and the A-pair and the negatively biased pair weighs most the negative outstandingness of an alternative instead. Furthermore, on this example, the only *measure A* that is not stable across different configurations of the ω's is the PB-pair $(P4, N1)$.

Table 3 Input data

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12	C13	C14	C15	C16	C17	C18
1	1	1	0.6	0.8	0.8	0.8	0.8	1	0.8	1	1	1	0.8	1	0.8	1	1	0.6
2	0.75	1	0.4	1	1	1	0.8	1	0.6	1	1	1	0.8	1	0.4	1	1	0.6
3	1	1	0.2	1	1	1	0.6	1	0.6	1	0.6	0.6	0.6	0.8	0.4	0.6	1	0.8
4	1	1	0.4	0.8	0.8	0.4	1	1	0.4	1	0.6	0.8	0.6	1	0.6	0.4	1	0.6
5	1	1	0.8	0.6	0.6	0.4	1	1	0.6	1	0.6	0.8	0.6	0.8	0.4	0.4	1	0.6
6	1	1	0.2	1	0.4	1	1	1	0.6	1	0.6	1	0.6	0.6	0.4	0.6	1	0.6
7	0.75	1	0.6	0.6	0.6	0.4	0.6	0.4	0.9	1	0.8	0.8	0.8	0.8	0.8	0.4	0.6	0.6
8	0.75	1	0.8	0.8	0.8	0.6	0.6	0.4	0.8	1	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8
9	0.5	1	0.2	0.4	0.4	0.8	0.4	0.4	0.4	1	0.4	0.4	0.4	0.4	0.2	0.4	0.4	0.4
10	0.75	1	0.2	0.2	0.4	0.4	0.2	0.2	0.6	0.33	0.4	0.6	0.6	0.4	0.4	0.6	0.4	0.4
11	0.75	1	0.2	0.2	0.4	0.4	0.2	0.2	0.6	0.33	0.4	0.6	0.6	0.4	0.4	0.6	0.4	0.4
12	0.75	1	0.6	0.8	0.8	0.8	0.6	0.4	0.8	1	0.8	0.8	0.6	0.8	0.8	0.6	0.6	0.8
13	0.75	1	0.6	0.8	0.8	0.8	0.6	0.4	0.8	1	0.8	0.8	0.6	0.8	0.8	0.6	0.6	0.8
14	0.75	1	0.4	1	0.8	0.6	0.8	1	0.8	1	0.8	0.8	1	0.8	0.8	0.6	0.8	0.8
15	0.75	1	0.6	0.8	0.8	0.8	0.8	1	0.6	1	0.8	0.6	0.6	0.8	0.6	0.8	0.8	0.8
16	0.75	1	0.6	0.8	0.8	0.8	0.8	1	0.6	1	0.8	0.6	0.6	0.8	0.6	0.8	0.8	0.8
17	0.75	1	0.6	0.8	0.8	0.8	0.8	1	0.6	1	0.8	0.6	0.6	0.8	0.6	0.8	0.8	0.8
18	0.5	0.33	0.4	0.4	0.4	0.4	0.6	0.4	0.4	0.33	0.2	0.2	0.4	0.4	0.2	0.2	0.4	0.4
19	0.75	1	0.2	0.6	0.6	0.4	0.8	1	0.6	1	0.4	0.4	0.4	0.4	0.2	0.4	0.6	0.6
20	0.5	1	0.4	0.2	0.2	0.4	0.4	0.4	0.6	0.33	0.4	0.6	0.4	0.4	0.4	0.6	0.4	0.4
21	0.75	1	0.2	0.6	0.4	0.4	0.6	1	0.4	1	0.4	0.4	0.6	0.4	0.2	0.6	0.4	0.4
22	0.5	1	0.4	0.6	0.8	0.8	0.6	1	0.6	1	0.6	0.6	0.6	0.6	0.6	0.4	0.2	0.4
23	0.5	1	0.4	0.2	0.2	0.2	0.6	0.4	0.6	1	0.6	0.6	0.6	0.2	0.2	0.6	0.4	0.4
24	0.75	1	0.6	0.8	0.4	0.8	0.8	1	0.8	1	0.8	1	0.8	0.8	0.8	0.6	0.8	1
25	0.75	1	0.6	0.8	0.4	0.8	0.8	1	0.8	1	0.8	1	0.8	0.8	0.8	0.6	0.8	1
26	0.75	1	0.4	0.4	0.8	0.6	0.6	1	0.4	1	0.6	0.4	0.2	0.4	0.6	0.4	0.4	0.4
27	0.75	1	0.4	0.4	0.6	0.4	0.6	1	0.4	1	0.4	0.4	0.4	0.4	0.6	0.6	0.4	0.4
28	0.75	1	0.8	0.6	0.4	0.6	0.6	1	0.4	1	0.6	0.6	0.6	0.6	0.4	0.6	0.6	0.4
29	0.75	1	0.8	0.6	0.4	0.6	0.6	1	0.4	1	0.6	0.6	0.6	0.6	0.4	0.6	0.6	0.4
30	0.75	1	0.8	0.6	0.39	0.6	0.6	1	0.4	1	0.6	0.6	0.6	0.6	0.4	0.6	0.6	0.4
31	0.75	1	0.2	0.6	0.2	0.6	0.6	1	0.4	1	0.4	0.4	0.4	0.4	0.6	0.6	0.4	0.4
32	0.75	1	1	1	0.4	0.4	0.6	0.2	0.4	1	0.6	0.6	0.6	0.6	0.6	0.6	0.2	0.4

Table 4 Results for $\omega_1 = \omega_2 = 0$

	MAOAM: net outstandingness, $p = q = \frac{1}{2}$; $\omega_1 = 0$; $\omega_2 = 0$							
Alternative	U-pair: (P1, N1)		NB-pair: (P5, N2)		PB-pair: (P4, N1)		A-pair: (P3, N3)	
	Value	Rank	Value	Rank	Value	Rank	Value	Rank
1	0.218	1	0.061	1	0.256	1	0.502	1
2	0.214	2	0.02	2	0.233	2	0.472	2
3	0.156	4	–0.129	5	0.173	8	0.326	4
4	0.085	14	–0.296	14	0.117	13	0.155	14
5	0.061	15	–0.321	15	0.098	15	0.117	15
6	0.087	13	–0.265	13	0.106	14	0.178	13
7	0.021	16	–0.357	16	0.068	16	0.058	16
8	0.135	5	–0.117	4	0.183	4	0.313	5
9	–0.198	27	–0.853	29	–0.182	27	–0.46	27
10	–0.215	28	–0.847	27	–0.199	28	–0.471	28
11	–0.215	28	–0.847	27	–0.199	28	–0.471	28
12	0.101	11	–0.24	11	0.149	11	0.204	11
13	0.101	11	–0.24	11	0.149	11	0.204	11
14	0.165	3	–0.11	3	0.202	3	0.345	3
15	0.132	6	–0.209	8	0.179	5	0.252	8
16	0.132	6	–0.209	8	0.179	5	0.252	8
17	0.132	6	–0.209	8	0.179	5	0.252	8
18	–0.28	32	–0.891	32	–0.262	32	–0.562	32
19	–0.082	23	–0.656	23	–0.053	23	–0.226	23
20	–0.232	30	–0.879	31	–0.215	30	–0.508	31
21	–0.14	26	–0.809	26	–0.122	26	–0.376	26
22	–0.011	17	–0.531	18	0.027	17	–0.083	18
23	–0.233	31	–0.858	30	–0.217	31	–0.496	30
24	0.124	9	–0.197	6	0.163	9	0.252	6
25	0.124	9	–0.197	6	0.163	9	0.252	6
26	–0.059	22	–0.623	22	–0.027	22	–0.186	22
27	–0.107	24	–0.724	24	–0.078	24	–0.293	24
28	–0.022	19	–0.582	19	0.013	18	–0.125	19
29	–0.022	19	–0.582	19	0.013	18	–0.125	19
30	–0.023	21	–0.588	21	0.011	20	–0.13	21
31	–0.124	25	–0.77	25	–0.104	25	–0.338	25
32	–0.021	18	–0.511	17	–0.003	21	–0.081	17

Furthermore, we observe that for all ω configurations, the various outstandingess pairs exhibit some variation in rankings despite that they often agree more than disagree overall.

In what concerns varying the ω values, we observe some variation in rankings for different outstandingness prdeicates pairs across the same ω configuration. But, across configurations, the rankings often agree more than disagree in this particular case. It seems that the effect of the Hernian means for large ω values is not as great as one expects. This led us to question the level of correlation among the attributes in our input data. We indeed measured the correlation between each pair of attributes and

averaged them. The avarage mean correlation among attributes is indeed low being 0.564. This findings partly explains the observed lack of significant variation among different rankings across various ω configurations.

Table 5 Results for $\omega_1 = 0, \omega_2 = 1,000$

| | *MAOAM: net outstandingness*, $p = q = \frac{1}{2}$; $\omega_1 = 0$; $\omega_2 = 1,000$ | | | | | | | |
| *Alternative* | *U-pair: (P1, N1)* | | *NB-pair: (P5, N2)* | | *PB-pair: (P4, N1)* | | *A-pair: (P3, N3)* | |
	Value	*Rank*	*Value*	*Rank*	*Value*	*Rank*	*Value*	*Rank*
1	0.216	1	0.095	1	0.251	1	0.5	1
2	0.189	2	0.041	2	0.217	2	0.434	2
3	0.119	10	–0.185	7	0.144	10	0.253	7
4	0.079	14	–0.303	14	0.11	14	0.153	14
5	0.07	15	–0.31	15	0.107	15	0.14	15
6	0.09	13	–0.247	11	0.117	13	0.198	11
7	0.039	16	–0.33	16	0.081	16	0.072	16
8	0.13	6	–0.13	4	0.174	6	0.284	6
9	–0.177	27	–0.88	30	–0.167	27	–0.419	27
10	–0.179	29	–0.867	27	–0.169	28	–0.435	30
11	–0.179	29	–0.867	27	–0.169	28	–0.435	30
12	0.097	11	–0.265	12	0.141	11	0.181	12
13	0.097	11	–0.265	12	0.141	11	0.181	12
14	0.152	3	–0.1	3	0.191	3	0.326	3
15	0.12	7	–0.228	8	0.163	7	0.24	8
16	0.12	7	–0.228	8	0.163	7	0.24	8
17	0.12	7	–0.228	8	0.163	7	0.24	8
18	–0.283	32	–0.925	32	–0.276	32	–0.583	32
19	–0.071	23	–0.7	23	–0.051	23	–0.199	23
20	–0.183	31	–0.886	31	–0.172	31	–0.431	29
21	–0.128	26	–0.848	26	–0.116	26	–0.351	26
22	–0.009	17	–0.586	18	0.02	17	–0.083	17
23	–0.178	28	–0.871	29	–0.17	30	–0.419	28
24	0.137	4	–0.149	5	0.176	4	0.29	4
25	0.137	4	–0.149	5	0.176	4	0.29	4
26	–0.052	22	–0.685	22	–0.028	22	–0.179	22
27	–0.093	24	–0.775	24	–0.072	24	–0.265	24
28	–0.015	18	–0.604	19	0.014	18	–0.097	18
29	–0.015	18	–0.604	19	0.014	18	–0.097	18
30	–0.016	20	–0.608	21	0.014	20	–0.1	20
31	–0.097	25	–0.801	25	–0.084	25	–0.28	25
32	–0.033	21	–0.55	17	–0.015	21	–0.12	21

6 Concluding remarks

In this paper, we introduced the MAOAM framework founded on two key concepts:

- Outstandingness predicates which are fuzzy predicates that measure the degree of outsandingness of an alternative against another alternative in both the positive

and negative senses. We proposed different types of oustandingness pairs: U-pair, NB-pair, PB-pair, and A-pair and validated empirically their properties.

Table 6 Results for $\omega_1 = 1,000, \omega_2 = 0$

Alternative	MAOAM: net outstandingness, $p = q = \frac{1}{2}$; $\omega_1 = 1,000$; $\omega_2 = 0$							
	U-pair: (P1, N1)		NB-pair: (P5, N2)		PB-pair: (P4, N1)		A-pair: (P3, N3)	
	Value	Rank	Value	Rank	Value	Rank	Value	Rank
1	0.16	2	0.123	1	0.182	2	0.392	2
2	0.18	1	0.063	2	0.187	1	0.406	1
3	0.128	3	−0.085	5	0.133	4	0.253	3
4	0.059	14	−0.239	14	0.072	14	0.075	14
5	0.046	15	−0.262	15	0.062	15	0.052	15
6	0.075	11	−0.216	13	0.08	13	0.116	11
7	0.018	16	−0.298	16	0.04	16	0.004	16
8	0.094	5	−0.05	4	0.123	5	0.21	5
9	−0.147	27	−0.832	29	−0.145	27	−0.464	27
10	−0.165	28	−0.819	27	−0.162	28	−0.479	28
11	−0.165	28	−0.819	27	−0.162	28	−0.479	28
12	0.064	12	−0.169	11	0.089	11	0.107	12
13	0.064	12	−0.169	11	0.089	11	0.107	12
14	0.118	4	−0.047	3	0.137	3	0.247	4
15	0.081	8	−0.134	8	0.106	8	0.142	8
16	0.081	8	−0.134	8	0.106	8	0.142	8
17	0.081	8	−0.134	8	0.106	8	0.142	8
18	−0.223	32	−0.871	32	−0.221	32	−0.551	32
19	−0.067	23	−0.609	23	−0.057	23	−0.268	23
20	−0.182	30	−0.859	31	−0.179	30	−0.507	31
21	−0.102	26	−0.777	26	−0.099	26	−0.4	26
22	−0.022	21	−0.474	17	−0.008	21	−0.142	18
23	−0.193	31	−0.834	30	−0.189	31	−0.504	30
24	0.087	6	−0.133	6	0.107	6	0.158	6
25	0.087	6	−0.133	6	0.107	6	0.158	6
26	−0.055	22	−0.574	22	−0.044	22	−0.229	22
27	−0.081	24	−0.685	24	−0.073	24	−0.316	24
28	−0.013	18	−0.529	19	−0.001	18	−0.168	19
29	−0.013	18	−0.529	19	−0.001	18	−0.168	19
30	−0.016	20	−0.536	21	−0.005	20	−0.173	21
31	−0.101	25	−0.736	25	−0.096	25	−0.371	25
32	−0.005	17	−0.474	18	0	17	−0.113	17

- Multi-variate (weighted) generalised Heronian means employed in the aggregation phase which are a family of multivariate generalised weighted Heronian means having as special cases the Beliakov et al.'s (2008) generalised Heronian means as well as that of Chu and Liu (2015). These types of means are especially useful when there is a significant correlation among attributes. We, however, through the proper parametrisation of these two means through ω_1 and ω_2 provide the

flexibility to balance the proper level of compensation one should expect fromthese means. The application of these means on a real use case that lacks significant correlation among its attributes did not allow us to observe the results of such a diversity.

Table 7 Results for $\omega_1 = 1,000, \omega_2 = 1,000$

	MAOAM: net outstandingness, $p = q = \frac{1}{2}$; $\omega_1 = 1,000$; $\omega_2 = 1,000$							
Alternative	*U-pair: (P1, N1)*		*NB-pair: (P5, N2)*		*PB-pair: (P4, N1)*		*A-pair: (P3, N3)*	
	Value	*Rank*	*Value*	*Rank*	*Value*	*Rank*	*Value*	*Rank*
1	0.141	1	0.153	1	0.163	1	0.349	1
2	0.123	2	0.084	2	0.136	2	0.296	2
3	0.069	7	−0.115	7	0.079	10	0.135	7
4	0.042	14	−0.214	14	0.053	14	0.058	14
5	0.034	15	−0.231	15	0.047	15	0.042	15
6	0.05	13	−0.17	11	0.059	13	0.089	11
7	0.019	16	−0.261	16	0.034	16	0.002	16
8	0.079	4	−0.042	4	0.102	4	0.17	4
9	−0.126	29	−0.834	30	−0.126	30	−0.41	28
10	−0.116	27	−0.809	27	−0.115	27	−0.419	30
11	−0.116	27	−0.809	27	−0.115	27	−0.419	30
12	0.052	11	−0.172	12	0.071	11	0.078	12
13	0.052	11	−0.172	12	0.071	11	0.078	12
14	0.087	3	−0.036	3	0.107	3	0.189	3
15	0.063	8	−0.136	8	0.083	7	0.115	8
16	0.063	8	−0.136	8	0.083	7	0.115	8
17	0.063	8	−0.136	8	0.083	7	0.115	8
18	−0.211	32	−0.886	32	−0.21	32	−0.56	32
19	−0.051	23	−0.615	23	−0.047	23	−0.215	23
20	−0.13	31	−0.84	31	−0.129	31	−0.417	29
21	−0.081	26	−0.79	26	−0.08	26	−0.351	26
22	−0.01	17	−0.486	17	−0.001	17	−0.119	17
23	−0.127	30	−0.81	29	−0.125	29	−0.402	27
24	0.078	5	−0.069	5	0.096	5	0.163	5
25	0.078	5	−0.069	5	0.096	5	0.163	5
26	−0.037	22	−0.599	22	−0.032	22	−0.198	22
27	−0.064	24	−0.709	24	−0.062	24	−0.272	24
28	−0.013	18	−0.523	19	−0.006	18	−0.137	18
29	−0.013	18	−0.523	19	−0.006	18	−0.137	18
30	−0.014	20	−0.527	21	−0.007	20	−0.14	20
31	−0.065	25	−0.73	25	−0.063	25	−0.285	25
32	−0.024	21	−0.492	18	−0.019	21	−0.158	21

We have provided various pairs of outstandingness predicates alongside a taxonomy that could be further studied to align them with various decision-maker preferences in specifying how to measure outstandigness among alternatives. This might bring some added value with respect to existing method that focus more on outstandingness. Indeed, an possible future research direction is to deploy a carefully chosen outstandingness pairs for each attribute that describes more the fuzzy needs of the decision-maker on each attribute separately.

In future works, more exhaustive experiments need to be done in order to fully understand the impact of the ω parameters especially when the attributes exhibit a high level of correlation. Finally, it remains an important and challenging task to find the proper combination of outstanding predicate pair as well as proper ω parameters for each specific application that would meet the decison-makers intentions and preferences.

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