Fair allocation of cost reductions for a scale-based product family in a hierarchically structured firm

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Abstract: Size ranges are a widespread type of product family design, which are used to combine high product variety and cost reductions. For the purposes of motivation and remuneration, these cost reductions should be shared in a fair way between the employees. The nature of cooperative cost reductions is super-additive – a fact which impedes a fair assignment. For this reason, the purpose of this paper is to present an approach for calculating and fairly allocating a cost reduction in a hierarchically structured firm. The process of designing and producing size ranges is modelled as a cost-reduction game within a hierarchically structured organisation. Assuming a hierarchical organisation, this study presents a mechanism which distributes this cost reduction across all contributing employees in the firm. The research shows that cost growth laws are suitable for calculating the costs of all members of a product family based on the basic draft using physical characteristics.

Keywords: product family design; affinity laws; cost growth laws; size range; cost-reduction game; hierarchical structure; permission tree.


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1 Introduction

A very common design strategy for reducing costs and maintaining or increasing product variety is the design of a product family. One way of designing a product family is the stretch-based (scale-based) design (Du et al., 2014; Fujita, 2002, p.958; Fujita, 2006, pp.202–203; Jang et al., 2002, p.49). In this case, the family is created by stretching and/or shrinking a product platform in one or more dimensions (Tucker, 2014,
A special form of such a design strategy is the design of a size range, which is used very often in the engine-building industry. Members of the size range are technically identical products and vary only with regard to the size. Different sizes lead to different costs and performances (Aggidis et al., 2010). The crucial aim of size ranges is generating economies of scope in the development stage and economies of scale in the production. Despite the high relevance of size ranges in engineering design they have received only scarce attention in the economic literature. Developing and producing size ranges is realised within a firm by different departments on different hierarchical levels, which cooperate to generate economies of scale and of scope. Incentivising these departments for cooperation requires a fair sharing of the jointly generated effect through the whole organisation. From this follows the necessity of calculating the sum of the cost reduction of the whole size range and the need for distributing this result between all members which have – directly and indirectly – contributed (Radner, 1992).

The paper is grounded on the following three pillars: one crucial fundamental is the calculation of the cost reduction of the size range based on the similarity of the products. For this purpose we present and employ cost growth laws. In addition, several contributions have indicated game-like characteristics of product development and design, mainly focussing on non-cooperative games (Liang et al., 2009; Oruç and Cunningham, 2014; Xiao et al., 2015). Incorporating cooperative game theory into the design process has been discussed with respect to service families (Aggarwal et al., 2013; Moon et al., 2011). We pick up these thoughts and specify them by modelling the design of size ranges as a cooperative cost-reduction game. The third pillar of our contribution is the presentation of a mechanism which assigns the gained cost reduction to all involved departments on different levels. This is done by using the lens of a cooperative game theory too.

Based on these lines of argumentation the aim of the paper is twofold: first, to identify economies of scope and economies of scale in designing and producing a size range; second, to present a mechanism which distributes the cost reduction within the firm throughout different hierarchical levels.

### 2 Characteristics and cost calculation of size ranges

#### 2.1 Physical and economic nature of size ranges

A size range consists of technical entities, but in different sizes, which:

- fulfil the same function,
- are based on the same technological concept, and
- are produced by means of the same materials and production techniques

Design and production of size ranges is very common, not only in the engine-building industry (e.g. gearing, engine, turbochargers, aircraft, pumps, electricity, tool industry, hydro turbines, and wind turbines) but also in other industries (e.g. production of bikes or clothes).

The design of size ranges starts with the basic draft’s detailed design, from which all other sizes, the follow-up sizes, are derived. These follow-up sizes use the same concept, layout and materials as the basic design and only differ with regard to the size. In the
light of the basic draft’s detailed design the technical–physical relations between the basic draft and the follow-up drafts are examined. Basic draft and follow-up drafts are similar, if the relationship of at least one physical quantity is constant. Basic similarities are defined on the basis of fundamental technical–physical values, e.g. length, time, force, speed or temperature (Kuehl and Geue, 2009; Kuneš, 2012, pp.124–126). For an excellent overview over similarity theory and its applications, see Coutinho et al. (2016). Based on these values ‘similarity laws’ (also known as ‘laws of similitude’ and ‘affinity laws’) are derived, which describe the relationships of the scaled technical entities (Aggidis and Židonis, 2014, pp.434–435; Guan et al., 2014, pp.372–375; Moreno et al., 2009, pp.97–98; Papanikolaou, 2014, p.56; Zhou and Sachdeva, 2010). These technical–physical relations between different sizes are called laws as they are amenable to the laws of physics. The analysis of similarity relationships is a basic requirement in the design of size ranges of complex products (e.g. turbochargers, gearings, couplers, motors, drilling and milling machines, wind and hydro turbines, ships, aircraft) (Bogos and Stroe, 2012; Brand, 2002, pp.67–71; Braun and Lindemann, 2007; Jang et al., 2002; Gasch, 2012; Mueller, 2011, p.472; Pahl et al., 2007, pp.488–494; Raymer, 2006, pp.457–461; Tarodiya and Gandhi, 2017; Tirovolis and Serghides, 2005, pp.1381–1385; Zhou and Sachdeva, 2010).

Technical entities are considered to be geometrically similar if the ratio of all lengths of any follow-up draft to all lengths of the basic draft is constant (Sheng et al., 2014, pp.30–31; Tesaf, 2016, p.246). This invariant is determined by the step size \( \phi \), which results from the relation of the characteristic length of the follow-up draft \( l_1 \) to the characteristic length of the basic draft \( l_0 \) as follows (Mueller, 2011, p.470):

\[
\phi = \frac{l_1}{l_0}
\]

Geometric similarity may be limited by the complexity of products, so that only quasi-similar entities result. Quasi-similarity is characterised by the fact that several lengths have different invariants and therefore different step sizes (Tesaf, 2016).

Against this background it can be concluded that the basic draft of a size range serves as the product platform and is scaled to generate different variants (Hirshburg and Siddique, 2014, p.203; Ninan and Siddique, 2014, p.368). Therefore, a size range can be viewed as a special case of a product family. The family is created by shrinking and/or stretching the whole product starting with the basic draft.

The advantages of developing and producing size ranges are the distribution of design costs on a higher product number (economies of scale). Moreover, technical experience and expertise with one size can be applied to other sizes, so that this knowledge is used for several products (economies of scope). In comparison with designing different products separately, this procedure reduces necessary time and cost budget (economies of scope). Designing and producing a size range is an option to minimise variety costs (Zhang and Tseng, 2007, p.133). As the products of a size range are similar – only with differences in size – the production process is the same and the number of the produced units will increase. This fosters learning curve effects and generates economies of scale.
2.2 Cost planning with affinity laws

The cost calculation for the follow-up drafts is based on the detailed design of the basic draft with a special type of model – cost growth laws (Ehrlenspiel et al., 2007, p.443; Mörtl and Schmied, 2015, p.392; Mueller, 2011, p.469; Pahl et al., 2007, pp.547–560). A cost growth law describes the relationship of the basic draft’s costs to follow-up drafts’ costs, with the size step as influencing factor.

The present value of the total costs of designing and producing a size range $TC_0$ consists of the sum of the development costs $DC_0$, of the present value of the material costs $MC_0$, of the present value of the labour costs for production operations $LC_{P,0}$, and of the present value of the labour costs for tooling $LC_{T,0}$ as follows:

$$TC_0 = DC_0 + MC_0 + LC_{P,0} + LC_{T,0} \tag{2}$$

The development costs of the size range $DC_0$ are generated by the design costs of the basic draft and of design costs for the specification of every follow-up draft, which reflects the amount which is necessary for activities of the specification of an additional follow-up design (e.g. deriving the similarity relationships).

Material costs of the follow-up draft $mc_{Foll}$ are estimated from the known material costs of the basic draft $mc_{BasD}$ and the similarity relation between the drafts described by size step $\varphi$, such as (Mueller, 2011, p.474):

$$mc_{Foll} = mc_{BasD} \varphi^\omega \tag{3}$$

The exponent’s $\omega$ value depends on characteristics of the component or product and seems to take the value three if the drafts are geometrically similar, as every component has three dimensions. Nevertheless, in engineering systems inertia forces and elastic forces resulting from the stress-strain relationship have to be considered (Bogos and Stroe, 2012, p.23; Coward, 1957; Mueller, 2011, p.471; Shakoori et al., 2012) so that the exponent’s $\omega$ value is derived on technical-physical analyses which have to be carried out by the designer (see Jenkinson et al., 1999, p.136 for examples in aircraft design). The number of the produced units of the step size $\Theta$ in the period $t$ is denoted with $n_{\Theta t}$. The present value of the material costs – considering that the size range is produced over $\Omega$ periods – results (Mueller, 2011, p.474):

$$MC_0 = \sum_{\Theta=1}^{\Omega} \sum_{t=1}^{\Omega} mc_{BasD} n_{\Theta t} q^t \tag{4}$$

In this equation, $q$ describes the discount factor with $q = 1 + i$, whereas $i$ defines the rate of return.

Labour costs are an additional important type of product cost. They can be distinguished between labour costs of production operations and labour costs of tooling. Labour costs of production operations of the follow-up draft $lc_{P,Foll}$ are calculated from the known labour costs of production operations of the basic draft $lc_{P,BasD}$ as (Mueller, 2011, p.471):

$$lc_{P,Foll} = lc_{P,BasD} \varphi^\theta \tag{5}$$

The exponent’s $\theta$ value depends on production techniques and geometric parameters and ranges from one to three (Pahl et al., 2007, p.549). The present value of the
labour cost for production operations of a size range over Ω periods is derived from the known labour cost for manufacturing the basic draft and equation (5) (Mueller, 2011, p.474):

\[ LC_{P,0} = l_{C,P,BaD} \sum_{\theta=1}^{\Omega} \phi_{\theta} \sum_{i=1}^{\gamma} n_{i,\theta} q^{-i} \]  

(6)

Labour costs decrease, based on learning effects, with an increasing number of produced units. This effect is modelled by the factor λ, which is the percentage on which costs are reduced if the cumulative production is doubled. The cost per unit of the nth unit \( c_n \) results from the costs of the first unit \( c_1 \) under consideration of learning effects with the following equation (Mueller, 2011, p.474):

\[ c_n = \frac{c_1 n^{ln2}}{\lambda} \]  

(7)

For further analyses, learning effects are incorporated by an annual average degression factor for every step size \( d_{t,\theta} \), which is calculated based on equation (7) as follows (Mueller, 2011, p.474):

\[ d_{t,\theta} = \frac{1}{n_{t,\theta}} \left( \sum_{\gamma=1}^{n_{t,\theta}} \frac{\ln 2}{n_{t,\theta}^{ln2}} \right) \]  

(8)

This is included in equation (6), which leads to:

\[ LC_{P,0} = l_{C,P,BaD} \sum_{\theta=1}^{\Omega} \phi_{\theta} \sum_{i=1}^{\gamma} n_{i,\theta} q^{-i} d_{t,\theta} \]  

(9)

Labour costs of tooling the follow-up draft \( l_{C,T,Fol} \) are based on the known labour costs of tooling of the basic draft \( l_{C,T,BaD} \) like (Mueller, 2011, p.471):

\[ l_{C,T,Fol} = l_{C,T,BaD} \phi_{\theta} \]  

(10)

The value of the exponent \( \rho \) is derived from geometric parameters and ranges between 0 and 0.5. The present value of the labour cost for tooling is derived in the same way and results from equation (10) like:

\[ LC_{T,0} = l_{C,T,BaD} \sum_{\theta=1}^{\Omega} \phi_{\theta} \sum_{i=1}^{\gamma} n_{i,\theta} q^{-i} d_{t,\theta} \]  

(11)

The present value of the total costs of designing and producing a size range \( TC_0 \) results from using equations (2), (4), (9) and (11) with:

\[ TC_0 = DC_0 + mc_{BaD} \sum_{\theta=1}^{\Omega} \phi_{\theta} \sum_{i=1}^{\gamma} n_{i,\theta} q^{-i} + l_{C,P,BaD} \sum_{\theta=1}^{\Omega} \phi_{\theta} \sum_{i=1}^{\gamma} n_{i,\theta} q^{-i} d_{t,\theta} + l_{C,T,BaD} \sum_{\theta=1}^{\Omega} \phi_{\theta} \sum_{i=1}^{\gamma} n_{i,\theta} q^{-i} d_{t,\theta} \]  

(12)
With equation (12) the costs for size ranges with different numbers of products – and, therefore, step sizes – can be calculated. In this connection the question may arise, which combination of step sizes and volumes leads to the minimum of overall production costs when a given number is fixed? This question has been answered (Mueller, 2011) and may serve as the first step in the general design process of a size range. Another crucial question is posed by the fact that the cost reductions are gained by all several members of the organisation in cooperation, from which the need for a distributing mechanism stems.

3 Allocation and distribution of cost reductions

3.1 Product family design as cost-reduction game

The design and development of a service and/or product family can be interpreted as a cooperative game (Aggarwal et al., 2013, p.830; Moon et al., 2011, p.158). Assuming constant revenues, different members of the size range are interpreted as players of a cost-reduction game. The cooperative cost-reduction game \( \Gamma \) consists of a set of players \( N = \{1, 2, \ldots, n\} \). A subset \( S \subseteq N \) is referred to as coalition \( S \), whereby \( N \) itself is described as a grand coalition. Very important for further analysis is the fact that the effect of the cost-reduction activities can be divided between all players. Players in our case are different employees/departments of the firm. This is realistic as cost reduction is generated by all members of a firm and may be transferred between them. This transfer allows an interpretation as a transferable utility game.

The function for the cost game \( \Gamma \) is defined by \( \Gamma(N,c) \). From this the characteristic function of the cost-reduction game \( \Gamma(N,v) \) is derived and marks for each coalition \( S \in P(N) \) a value \( v(S) \). The result of an empty coalition is zero, therefore: \( v(\emptyset) = 0 \).

Definition 1: Characteristic function of a game \( \Gamma(N,v) \) with \( v : 2^N \to \mathbb{R} \) is calculated based on the cost function \( \Gamma(N,c) \) by: \( v(S) = \sum_{i \in S} c(\{i\}) - c(S), \forall S \subseteq N \).

The effects of designing and producing a size range are economies of scope and economies of scale, which generate a cost reduction. This is taken into account by the characteristic of the super-additivity of the characteristic function (Fiestras-Janeiro et al., 2011, pp.4–5).

Property 1: A characteristic function \( v \) with transferable utility is a super-additive function, if for all coalitions \( R, S \subseteq N \) out of \( R \cap S = \emptyset \) follows: \( v(R) + v(S) \leq v(R \cup S) \).

In addition, it has to be stated that a greater coalition generates a better result. This is called convexity.

Property 2: Game \( \Gamma(N,v) \) is convex, if for all coalitions \( R, S \subseteq N \) with \( S \subseteq R \) and for every player \( i \in N \), which is neither member of \( S \) nor of \( R \), holds: \( v(S \cup \{i\}) - v(S) \leq v(R \cup \{i\}) - v(R) \).
The following steps of discussion are the calculation of core-allocations and the subsequent distribution of these allocations.

### 3.2 Core-allocations of the game

The space of fair solutions of a cooperative game is restricted by the set of imputations. An imputation describes an allocation, which fulfils two requirements: individual rationality and efficiency (González-Díaz et al., 2010, p.217).

**Definition 2:** An imputation of the game \( \Gamma (N,v) \) is defined by: (a) \( \sum_{i \in N} x_i = v(N) \) as well as (b) \( x_i \geq v(\{i\}) \), \( \forall i \in N \).

For solving a distribution problem only these imputations, which are not dominated by other imputation, are of interest. The set of not-dominated imputations is forming the core (Gillies, 1959; González-Díaz et al., 2010, p.218).

**Definition 3:** The core of a cooperative game \( C(N,v) \) is formed by all imputations \( x \) with: \( C(N,v) = \left\{ x : \sum_{i \in S} x_i \geq v(S) \forall S \subseteq N \right\} \).

The core of a game contains all solutions which are fair and, therefore, stable. It may be small, it may be very large or it may be empty. The core of convex games is never empty (Shapley, 1971, p.24). The core as a solution concept does not deliver a detailed suggestion for sharing synergetic effects, but indicates if there exists one stable solution or even a set of stable solutions. From the existing concepts (for an overview, cf. Fiestras-Janeiro et al., 2011), the Shapley value and the nucleolus will be introduced in the following. In order to determine the Shapley value for a player \( i \) the marginal contribution is noted.

**Definition 4:** The marginal contribution \( \mu_{i,S} \) of the player \( i \) for the coalition \( S \subseteq N \) is determined by: \( \mu_{i,S}(v) = v(S \{i\}) - v(S \{i\}) \).

This marginal contribution \( \mu_{i,S}(v) \) of a player will show a different value for different coalitions. Therefore, all possible marginal contributions are weighted with the probability of forming these coalitions.

**Definition 5:** The weighted marginal contribution of the player \( i \) is described with \( \varepsilon_i(v) \) and results:

\[
\varepsilon_i(v) = \sum_{S \subseteq N-i} \frac{|S|!(n-1-|S|)!}{n!} [v(S \cup \{i\}) - v(S)] .
\]

This solution principle is commonly described as the Shapley value (Shapley, 1953, p.311). The Shapley value concept is widely known and has already been frequently used for cost-allocation problems (Aggarwal et al., 2013; Contreras et al., 2009; Frisk et al., 2010; Moon et al., 2011). For an early and a broad discussion, cf. the volume edited by Roth, 1988, and for a literature review of applications of the Shapley value in the field of
transportation cost allocation, cf. Guajardo and Rönnqvist, 2016, p.380. The Shapley value always exists but has the disadvantage that it only belongs with certainty to the core of convex games (Shapley, 1971, p.29). If the game is not convex it is possible that the Shapley value does not belong to the core.

To overcome this disadvantage another solution concept of cooperative game theory is introduced (Schmeidler, 1969). This concept – called the nucleolus – is searching for a fair distribution by minimising the maximal dissatisfaction of every player. For this, the dissatisfaction of a coalition with a concrete-payoff vector is named in this connection as excess (Kimms and Cetiner, 2012, p.515).

**Definition 6:** The excess of a coalition \( S \) over a payoff vector \( x \) is derived by

\[
ex(S, x) = v(S) - \sum_{i \in S} x_i.
\]

We denote the excess of a coalition \( S_i \) with respect to a payoff vector \( x \) with

\[
ex(S_i, x) = \theta_i(x).
\]

To derive the nucleolus, in the next step the payoff vectors with the highest unhappiness for every player are searched. To do this, these excess values are sorted in non-increasing order.

**Definition 7:** The vector of non-increasingly ordered excess values \( \Xi(x) \) is defined by

\[
\Xi(x) = \{ \theta_1(x), \theta_2(x), ..., \theta_{2^n}(x) \} \text{ with } \theta_i(x) \geq \theta_j \text{ for } 1 \leq i \leq j \leq 2^n.
\]

To compare two payoffs, their vectors of non-increasingly ordered excess values are compared based on the lexicographic order. The vector which is lexicographically smaller than the other is chosen as this vector offers the minimum of the maximal dissatisfaction for all players resulting from the two payoffs. By \( \Xi(x) \preceq \Xi(y) \) we express that either \( \Xi(x) = \Xi(y) \) or \( \Xi(x) \prec \Xi(y) \). With these explanations the nucleolus of a game can be defined as follows (González-Díaz et al., 2010, p.232):

**Definition 8:** In a game \( \Gamma(N, v) \) the nucleolus \( \text{nuc}(N, v) \) is defined with respect to the set of imputations \( I(N, v) \neq \emptyset \) by:

\[
\text{nuc}(v) = \{ x \in I(N, v) : \text{ for each } y \in I(N, v), \Xi(x) \preceq \Xi(y) \}.
\]

The very important advantage of the nucleolus in comparison with the Shapley value is the fact that the nucleolus always belongs to the core, if the core is not empty (González-Díaz et al., 2010, p.233). The nucleolus is well established in the field of game theory (e.g. Kimms and Cetiner, 2012; Mueller, 2016; Frisk et al., 2010. For a literature review of applications of the nucleolus in the field of transportation cost allocation (cf. Guajardo and Rönnqvist, 2016, p.380), but has not drawn as much attention as the Shapley value in the area of management decisions. Concerning the computational effort, the very involved procedure of establishing lexicographic orders of excess vectors for games with many players must be mentioned. There are some mistakes in computing the nucleolus caused by overlooking the possibility that a linear program can have multiple solutions (Guajardo and Jørnsten, 2015).
3.3 Distribution of cost reductions in a hierarchically structured firm

Looking at the process of designing, developing and producing a size range, it has to be stated that different departments cooperate to reach a cost reduction. Usually a firm is characterised by a hierarchy, from which it follows that employees from different hierarchical levels cooperate. Employees on higher levels are managers who organise and coordinate the process of designing, development and producing. In our case these managers decide to form a product family out of different products, and they coordinate the process of designing and developing the new size range. We assume that every employee needs approval or instructions from their direct superior in order to be active, so that a permission tree results (van den Brink, 2008, p.226).

Decisions concerning the design of a product family have a huge influence on a product’s technical and economic future. The majority of a product’s economic properties is specified by the definition of technical characteristics so that the largest impact on product costs can be made in product design (Drury, 2012, p.539). Based on the decision of the managers the production of the size range is realised by employees on a lower hierarchical level. We assume that the production of every size is realised by one department which is led by one responsible person and is classified on the lowest hierarchical level. Moreover, we assume a cost-centre structure, so that the performance of the employees/departments is measured by the costs (or the cost reductions) which are generated.

Design decisions have a huge influence on a product’s technical and economic future. These decisions are made not on the production level, but on a higher level. An incentive for all employees of the organisation consists of receiving a share of the jointly generated cost reduction for their cost centre. Reduced costs lead to a better annual result for the cost centre.

From this the task results in distributing the jointly generated cost reduction – the core-allocation – within the firm’s hierarchy. The problem results from the fact that managers are not directly involved in the realisation of the cost-reduction effects which they had instructed. They contribute only indirectly to the cost reduction.

The firm hierarchy $H : N \rightarrow 2^N$ with $i \notin H(i)$ is a mapping that assigns to every employee $i \in N$ in a firm those employees that are directly subordinate to $i$ (van den Brink, 2008, p.227). By $\hat{H}$ we denote the transitive closure of the firm hierarchy $H$, i.e. $j \in \hat{H}(i)$, if there exists a sequence of employees $(h_1, \ldots, h_t)$ such that $h_k = i, h_{k+1} \in H(h_k)$, for all $1 \leq k \leq t-1$, and $h_t = j$. Employees in $\hat{H}(i)$ are subordinates of employee $i$. Employees in the set $H^{-1}(i) = \{ j \in N \mid i \in H(j) \}$ are direct superiors to $i$.

We assume for the firm a tree structure, which implies that there exists one employee $i_0$ who is superior to all other employees and has no superior. In the hierarchical structure an employee needs permission from his/her superior. All other employees have exactly one superior and are not subordinate of itself. That’s why (van den Brink, 2008, p.228):

- there is exactly one employee $i_0 \in N$ such that $|H^{-1}(i)| = \emptyset$ and $\hat{H}(i_0) = N \setminus \{i_0\}$,
- for every $i \in N \setminus \{i_0\}$ it holds that $|H^{-1}(i)| = 1$, and
- it holds that $i \notin \hat{H}(i) \forall i \in N$. 

Fair allocation of cost reductions
By determining $H$ as a tree structure it can be concluded that there are always employees that have no subordinates. These positions are referred to as workers in $H$, which are directly involved in the production process. The set of workers is denoted by $W_H = \{i \in N : H(i) = \emptyset\}$. Employers with subordinates in this structure are managers who do not operate directly in the production process. The set of managers in $H$ is denoted by $M_H = N \setminus W_H$. For a manager $i \in M_H$ the set $H(i) = H(i) \cap W_H$ is denoted as the set of workers, which are subordinates to manager $i$.

The distribution system is determined based on a function $F$, which assigns a non-negative cost-reduction share $F_i(N, v, H)$ to every employee $i \in N$. With respect to the cost-centre organisation it is important that the mechanism fulfills budget neutrality. The sum of all shares equals the sum of the cost-reduction game, i.e. $\sum_{i \in N} F_i(N, v, H) = v(N)$.

Managers in $M_H$ are denoted as zero players in the game as they do not contribute directly to the cost reduction (van den Brink, 2008, p.229). However, if managers are part of a hierarchy then, although such a player is a zero player in the game, it might be that there are non-zero players that are subordinate to them. In that case it seems reasonable that a zero player gets a non-zero payoff.

The total cost reduction is shared within the firm from the lowest level to the highest level. Such mechanisms are employed frequently in firms with high sales orientation (Casajus et al., 2009, p.930). The shares of the lowest level are calculated by identifying an allocation which belongs to the core of the cooperative game (the so-called core-allocation). This has already been discussed in this paper in the previous sections.

Every employee $j$ with $j \in H^{-1}(g)$ or who belongs to the path $T(i, g)$ receives a share of the core-allocation of the employee $g$ (Casajus et al., 2009, pp.937–938). Employee $i \in N$ receives from the core-allocation of $g$ the share $f_i$ which is defined by:

$$f_i(H, w, g) = \begin{cases} 
[1-w_j] \prod_{l \in H(i)} w_{j, l} & i \in T(i, g) \\
0 & \text{otherwise}
\end{cases}$$

(13)

Value $w_i$ with $0 \leq w_i \leq 1$ represents the weight of employee $i$ in the hierarchy and describes the influence of the superior. This measures the participation of a manager in the results of his subordinates. In the case of size ranges the decision concerning combining different products in one scale-based product family is made at a manager level. The process of developing and designing the size range is carried out at a management level too. This has a huge influence on the result of the production process at the workers level.

By defining $w_i$, the influence and relevance of a manager can be determined. This influence is determined in firms’ daily business in very different ways (Meagher, 2001). The weights may be defined or influenced by the cultural dimension of the firms’ nation, e.g. power distance. A high value of $w_i$ would reflect a high value of power distance (Hofstede, 2011, pp.80–86).

For the root of the hierarchy follows $w_\emptyset = 0$. The structure of the firm is defined by the hierarchy $H$ and the vector $w$. For $g = i$ there exists no $l \in T(i, g)$, which is simultaneously employee of $i$. Therefore, employee $i$ has to transfer the share $w_i$ of its
core-allocation to its direct superior, so that \(i\) receives the share \(1 - w_i\). If the employee \(i\) is no part of the path \(T(i, g)\) he receives no share of the core-allocation of \(g\). The root of the hierarchy \(i = i_0\) does not need to transfer and gets an incentive of \(\prod_{i \in \delta(i), i \neq i_0} w_i\).

The net incentive \(F\) for employee \(i\) is now defined by (Casajus et al., 2009, p.938):

\[
F_i(N, v, H, w) = \sum_{j=1}^{n} f_j(H, w, f) \cdot \text{core-allocation}_j(N, v)
\]

(14)

Combining equations (13) and (14) leads to the following relationship of gross incentive \(F_i(N, v, H, w[i])\) and net incentive:

\[
F_i(N, v, H, w) = (1 - w_i) \cdot F_i(N, v, H, w[i])
\]

(15)

Gross incentive can be calculated based on the gross incentives of its direct subordinates based on:

\[
F_i(N, v, H, w[i]) = \text{core-allocation}_i(N, v) + \sum_{j \in H_i} F_j(N, v, H, w[j])
\]

(16)

This incentive mechanism satisfies efficiency, additivity, weak symmetry, weak inessential employee property and budget neutrality (Casajus et al., 2009, p.940).

3.4 Illustrative example

3.4.1 Initial situation

In the following the situation of a firm which is producing four products (e.g. hydro turbines) is studied. These products were designed separately in the past and are produced separately in the present too. Now, these products have to be redesigned due to market changes. As these products cover the same market segment (small scale turbines) but with different performance parameters, the firm’s designer may develop these products as a size range to generate economies of scope and economies of scale. The company has the opportunity to design:

- every product as a single product,
- a size range which contains two products,
- a size range which contains three products, or
- a size range which contains all four products.

It is planned to produce 250 units per year of every product in the next five years. If every size is designed as a single size range, development costs of \(DC_{0,1}\) will result. The design of two sizes in one size range will require costs of \(DC_{0,2}\). The costs of designing three and four products in one size range are denoted equivalently. Additional information concerning the size range is available in Table 1.
Table 1  Input data for the example

<table>
<thead>
<tr>
<th>mcBasD = 500€ per unit</th>
<th>= 0.95</th>
<th>= 0.5</th>
<th>DC_{0,1} = 150,000 €</th>
</tr>
</thead>
<tbody>
<tr>
<td>lcP; BasD = 100€ per unit</td>
<td>= 5</td>
<td>= 1</td>
<td>DC_{0,2} = 275,000 €</td>
</tr>
<tr>
<td>lcT; BasD = 30€ per unit</td>
<td>= 3</td>
<td>= 1.5</td>
<td>DC_{0,3} = 400,000 €</td>
</tr>
<tr>
<td>n_{1,}, n_{2,}, n_{3,}, n_{4,} = 250</td>
<td>= 2</td>
<td>= 2</td>
<td>DC_{0,4} = 545,000 €</td>
</tr>
</tbody>
</table>

To calculate the costs of the 15 possible design alternatives of the size ranges equation (12) is used and specified for every coalition. The results are visible in Table 2.

Table 2  Costs of different design alternatives

<table>
<thead>
<tr>
<th>S</th>
<th>c(S) [Thousand €]</th>
<th>S</th>
<th>c(S) [Thousand €]</th>
</tr>
</thead>
<tbody>
<tr>
<td>{∅}</td>
<td>0</td>
<td>{2,3}</td>
<td>2900</td>
</tr>
<tr>
<td>{1}</td>
<td>250</td>
<td>{2,4}</td>
<td>5522</td>
</tr>
<tr>
<td>{2}</td>
<td>781</td>
<td>{3,4}</td>
<td>6894</td>
</tr>
<tr>
<td>{3}</td>
<td>2158</td>
<td>{1,2,3}</td>
<td>3115</td>
</tr>
<tr>
<td>{4}</td>
<td>4786</td>
<td>{1,2,4}</td>
<td>5733</td>
</tr>
<tr>
<td>{1,2}</td>
<td>1000</td>
<td>{1,3,4}</td>
<td>7103</td>
</tr>
<tr>
<td>{1,3}</td>
<td>2372</td>
<td>{2,3,4}</td>
<td>7630</td>
</tr>
<tr>
<td>{1,4}</td>
<td>4994</td>
<td>{1,2,3,4}</td>
<td>7860</td>
</tr>
</tbody>
</table>

The analysis of the cost information in Table 2 indicates the design and production of all four products in one size range as the best alternative, as it minimises the total costs. One reason is decreasing development costs when designing all four sizes together. Another important role plays the learning effect. The learning effect results from producing 1000 units per year if all sizes are produced in one size range. If only one product is produced in one size range, only 250 units per year will generate the learning effect as the other products have a different design and, therefore, different production schemes.

To model the distribution of these cost shares we assume a tree hierarchy with a set of employees $N = \{1, \ldots, 7\}$ and hierarchy with $H(1) = H(2) = H(3) = H(4) = ∅$, $H(5) = \{1, 2\}$, $H(6) = \{3, 4\}$ and $H(7) = \{5, 6\}$ (cf. Figure 1).

Figure 1  Firm’s hierarchy in the example
Production of the size ranges is carried out by the set of production departments which we denote as workers \( W_H = \{1,2,3,4\} \) at the lowest hierarchical level. Design and development decisions are made by managers of two research and development departments \( M_H = \{5,6\} \) at a higher level. These managers are superiors to the workers and have a huge influence on the design of the products and on the efficiency of the production process. Both development departments are subordinate to the head of the business unit. The set of managers is constituted \( M_H = \{5,6,7\} \). The set of the direct subordinates of employee 6 is the set \( \hat{H}(6) = \{3,4\} \). The set of all superiors to employee 1 is denoted by \( H^{-1} = \{5,7\} \) and the path between employee 1 and employee 7 is defined by \( T(7,1) = \{7,5,1\} \).

### 3.4.2 Fair cost sharing on the lowest level

Now, the problem of cost allocation on the level of the production of the size range has to be solved. As every production unit contributes in a different way to synergetic effects, it is necessary to identify this contribution. With these cost functions the characteristic function of the game – as a function of the cost reductions – is calculated based on Definition 1 and is shown in Table 3. Checking the cost-reduction data in Table 3 regarding to Property 1 and Property 2, it can be reasoned that this game is super-additive but not convex.

<table>
<thead>
<tr>
<th>( S )</th>
<th>( v(S) ) [Thousand €]</th>
<th>( S )</th>
<th>( v(S) ) [Thousand €]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( {\emptyset} )</td>
<td>0</td>
<td>( {2,3} )</td>
<td>39</td>
</tr>
<tr>
<td>( {1} )</td>
<td>0</td>
<td>( {2,4} )</td>
<td>45</td>
</tr>
<tr>
<td>( {2} )</td>
<td>0</td>
<td>( {3,4} )</td>
<td>50</td>
</tr>
<tr>
<td>( {3} )</td>
<td>0</td>
<td>( {1,2,3} )</td>
<td>74</td>
</tr>
<tr>
<td>( {4} )</td>
<td>0</td>
<td>( {1,2,4} )</td>
<td>84</td>
</tr>
<tr>
<td>( {1,2} )</td>
<td>31</td>
<td>( {1,3,4} )</td>
<td>91</td>
</tr>
<tr>
<td>( {1,3} )</td>
<td>36</td>
<td>( {2,3,4} )</td>
<td>95</td>
</tr>
<tr>
<td>( {1,4} )</td>
<td>42</td>
<td>( {1,2,3,4} )</td>
<td>115</td>
</tr>
</tbody>
</table>

To identify concrete shares for every member of the product family the Shapley values are calculated at first based on Definition 5. The results are (values in thousand €):

\[
e_1 = \frac{71}{3}, \quad e_2 = \frac{78}{3}, \quad e_3 = \frac{90}{3}, \quad e_4 = \frac{106}{3}
\]

Unfortunately, this solution is not a part of the core. If we investigate the coalition \( \{2,3,4\} \) which is – according to Table 3 – able to generate a cost reduction of 95 €, it has to be questioned why these players should agree with a share according to the Shapley value of only 91.33. That means that for players \( \{2\}, \{3\} \) and player \( \{4\} \) there is no incentive to form the grand coalition due to the non-convexity of the game.
Therefore, the nucleolus of the game is calculated based on Definition 8 which leads to the following results (values in thousand €):  

\[ nuc_1 = \frac{79}{4}, \quad nuc_2 = \frac{95}{4}, \quad nuc_3 = \frac{123}{4}, \quad nuc_4 = \frac{163}{4}. \]

Supplementary investigation of coalition \{2,3,4\} shows that the sum of the shares according to the nucleolus is 95.25, which is higher than the value of the three-company coalition. Therefore, this solution offers an incentive for these players. The nucleolus is a part of the non-empty core of this game and represents therefore a stable solution.

### 3.4.3 Distribution of the cost reduction

Fair sharing of cost reduction on the lowest level has been demonstrated in the previous section. To incentivise not only workers but also managers of this business unit it is necessary to distribute the values of the fair shared cost reduction (nucleolus values from previous section) within the hierarchy as it is obvious that workers do not reach the cost reduction on their own decision. We assume \( w_1 = w_2 = w_3 = \frac{1}{4}, \quad w_4 = w_5 = \frac{1}{8} \) and \( w_6 = 0 \) for further discussion. Using equation (14) leads to the following results:

\[
F_1(N,v,H,w) = \left(1 - \frac{1}{4}\right) \cdot nuc_1 = \frac{237}{16},
\]

\[
F_2(N,v,H,w) = \left(1 - \frac{1}{4}\right) \cdot nuc_2 = \frac{285}{16},
\]

\[
F_3(N,v,H,w) = \left(1 - \frac{1}{4}\right) \cdot nuc_3 = \frac{369}{16},
\]

\[
F_4(N,v,H,w) = \left(1 - \frac{1}{4}\right) \cdot nuc_4 = \frac{489}{16},
\]

\[
F_s(N,v,H,w) = \left(1 - \frac{1}{8}\right) \cdot \left[ F_1(N,v,H,w[1]) + F_2(N,v,H,w[2]) \right] = \frac{7}{8} \cdot \frac{1}{4} \left( \frac{79}{4} + \frac{95}{4} \right) = 609 \cdot \frac{64}{64} = 1001.
\]

\[
F_s(N,v,H,w) = \left(1 - \frac{1}{8}\right) \cdot \left[ F_3(N,v,H,w[3]) + F_4(N,v,H,w[4]) \right] = \frac{7}{8} \cdot \frac{1}{4} \left( \frac{123}{4} + \frac{163}{4} \right) = 1001 \cdot \frac{64}{64} = 1001.
\]

\[
F_s(N,v,H,w) = \frac{1}{8} \cdot F_3(N,v,H,w[5]) + \frac{1}{8} \cdot F_4(N,v,H,w[6]) = \frac{1}{8} \cdot \frac{1840}{64} = \frac{230}{64}.
\]
This result is budget balanced as $\sum_{i \in N} F_i(N, v, H, w) = v(N) = 115$. The jointly generated cost reduction is distributed between the departments on the lowest level as well as between all levels of the firm.

4 Criticisms

Necessary criticisms may be separated into three parts: cost calculation model; game-theoretic concepts and the core-allocations; and the mechanism of hierarchical distribution.

The cost calculation model delivers values which may be reached if the production process is realised efficiently. Thus, the calculated values are theoretical target values. In practice these values will hardly be reached and it has to be discussed how to deal with deviations from the target values.

Transferable unit games are characterised by the assumption that the jointly generated result can be transferred between all players. The concept of side payments is realised in our model by distributing the shares of the cost reduction within the firm. The fact that different schemes deliver different core-allocations for identical input data may be astonishing but this has to be interpreted against the background of the different underlying arguements. It becomes apparent that rationality is not an absolute but rather a relative measure. The Shapley value is based on the expected marginal contribution of each player from its merely theoretically realisable participation in the sum of all possible – which means imaginable – coalitions. Nevertheless, it is the most common allocation scheme. The Shapley value of a convex game is unique and always part of the non-empty core. If the game is not convex, the Shapley value does not need to be a part of the core. The nucleolus is always unique and if the core is non-empty, the nucleolus is part of the core. The main disadvantage of the nucleolus is the high calculation effort for large numbers of players.

With respect to the modelled hierarchy of the firm, it has to be pointed out that we used a general model. It is possible to model another structure instead of our example structure. In our hierarchical structure, an employee needs permission from all their superiors to be active in the production process. This assumption is fulfilled in small firms but in large firms it is questionable. The presented distribution mechanism satisfies efficiency, additivity, weak symmetry, and weak inessential employee property.

5 Conclusion

Product families incorporate a very common design strategy for reducing cost and increasing product variety. Size ranges are a special type of product families which are very common in engineering design but which have received the same scarce attention in economic literature as cost growth laws and similarity laws. Owing to their high relevance in design theory and practice we introduce these instruments of cost planning for product design and development and specify them with respect to hierarchical structure in the firm. Recognising firm’s activities as a cooperative effort in which all employees contribute to the generation of a collective result, we can model firms’ activities as a cooperative game. By assuming a cost-centre organisation we analyse the
efforts for cost reduction of a size range. As the major contribution we present a mechanism that incentivises employees in a firm based on its share on the core-allocations of the cost-reduction game. Managers are interpreted in our game as zero players as they do not contribute directly to the result of the firm. Their indirect contribution is rewarded by distributing cost-reduction shares upstream from the lowest level. Thus, the shares of the jointly generated cost reduction of employees on different levels are determined. The presented mechanism satisfies the most important desirable properties which are placed on fair cost-centre allocations such as efficiency, additivity, weak symmetry, and weak inessential employee property and budget neutrality.

We have indicated that such mechanisms are employed frequently in firms with high sales orientation. With respect to the practical application of our model it has to be stated that only time will tell if this will work for internal cost accounting too. Moreover, the assumption of a permission tree seems to be fulfilled very well in small firms, but in these firms the limitations concerning the understanding of the model appear to be much tighter than in bigger firms.

Future work has to focus on some open questions of the presented approach. One item is the definition of the weights in the hierarchy, another one is the influence of the hierarchy on the value of the cost reduction.

References


Fair allocation of cost reductions


Fair allocation of cost reductions

Nomenclature

c_1  costs per unit for the first produced unit

c_n  costs per unit for the nth produced unit

C(N, v)  core of the cooperative cost-reduction game \( \Gamma(N, v) \)

DC_0  present value of the development costs

d_{\Theta t}  average degression rate of costs for size \( \Theta \) in the period \( t \) caused by learning effects

ex(S, x)  excess of coalition \( S \) over payoff vector \( x \)

F_i  cost-reduction share of employee \( i \)

\( H \)  hierarchy of the firm

\( \hat{H}(i) \)  set of subordinates to employee \( i \)

\( H^{-1}(i) \)  set of direct superiors to \( i \)

\( I(N, v) \)  set of imputations of the cooperative cost-reduction game \( \Gamma(N, v) \)

LC_{P,0}  present value of the labour costs for production operations of the whole size range

lc_{P, BasD}  labour costs of production operations of the basic draft

lc_{P, Foll}  labour costs of production operations per unit of the follow-up draft

LC_{T,0}  present value of the labour costs for tooling of the whole size range

lc_{T, BasD}  labour costs of tooling of the basic draft

lc_{T, Foll}  labour costs of tooling per unit of the follow-up draft

\( M_H \)  set of managers in \( H \)

MC_0  present value of the material costs of the whole size range

mc_{BasD}  material costs per unit of the basic draft

mc_{Foll}  material costs per unit of the follow-up draft

\( N \)  number of all players in a cooperative game \( \Gamma(N, v) \)

n_{\Theta t}  number of produced units of the size \( \Theta \) in the period \( t \)

nuc(N, v)  nucleolus of the cooperative cost-reduction game \( \Gamma(N, v) \)

\( q \)  discount rate with \( q = 1 + i \), whereas \( i \) describes the rate of return

\( R \)  coalition as a subset of players in a cooperative game

\( S \)  coalition as a subset of players in a cooperative game

\( t \)  time index, with \( t = \{1, \ldots, \Psi\} \)
20  \textit{D. Mueller}

- $TC_0$: present value of the total costs of the size range
- $\nu$: value of the cooperative game
- $w_i$: weight of the employee $i$ in the hierarchy
- $W_H$: set of workers in $H$
- $y_{i,\Theta}$: lot size for the size $\Theta$ in the period $t$
- $\Gamma(N,c)$: cooperative cost game
- $\Gamma(N,v)$: cooperative cost-reduction game
- $\varepsilon_i(\nu)$: weighted marginal contribution of the player $i$
- $g$: value of the similarity relationship of labour costs for production operations
- $\lambda$: rate of learning
- $\mu_{i,S}$: marginal contribution of the player $i$ for the coalition $S$
- $\rho$: value of the similarity relationship of labour costs for tooling
- $\phi_i$: geometric step size
- $\sigma$: value of the similarity relationship of material costs
- $\theta_i(x)$: excess of a coalition $S_i$ with respect to a payoff vector $x$
- $\Theta$: index for the number of step sizes, with $\Theta = (1, \ldots, \Omega)$
- $\Xi(x)$: vector of non-increasingly ordered excess values
- $\Psi$: total periods in which the product is being produced
- $\Omega$: number of step sizes produced by the company