Modelling, stability analysis and computational aspects of nonlinear fuzzy PID controllers using Mamdani minimum inference

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Abstract: This paper presents two new mathematical models of the simplest fuzzy PID controller which employ two fuzzy sets (negative and positive) on each of the three input variables (error, change in error and double change in error) and four fuzzy sets (−2, −1, +1, +2) on the output variable (incremental control). L-type, Γ-type and Π-type membership functions are considered in fuzzification process of input and output variables. Controller modelling is done via algebraic product AND operation, maximum/bounded sum OR operation, Mamdani minimum inference method, and centre of sums (CoS) defuzzification. The new models obtained in this manner turn out to be nonlinear, and their properties are studied. Since digital controllers are implemented on the digital processors, the computational and memory requirements of the fuzzy controllers and conventional (non-fuzzy) controller are compared. Stability analysis of closed loop systems containing the fuzzy controller models is done using the small gain theorem.

Keywords: fuzzy control; Mamdani type controller; PID controller; variable gain control; nonlinear control; variable structure control; mathematical modelling.


Biographical notes: N.K. Arun received his Bachelors degree in Electrical and Electronics Engineering from Nagarjuna University in 2008, and Masters degree in Electrical Engineering from National Institute of Technology, Calicut in 2011. Presently, he is a research student pursuing his Doctoral degree in the Department of Electrical Engineering at Indian Institute of Technology, Kharagpur. His research interests are mathematical modelling and stability analysis of fuzzy control systems and integral resonant control (IRC) of smart structures.
1 Introduction

Conventional (linear) PID controllers are the most widely used controllers in industry (Kumar et al., 2011) due to their ease of implementation and well known tuning methods. However, they generally do not ensure satisfactory performance for nonlinear systems, higher order linear systems and time-delay systems. To overcome this limitation, various kinds of nonlinear PID controllers such as auto tuned and adaptive PID controllers have been developed. In these cases of nonlinear PID controllers usually their analytical structure \( u = f(x) \), where \( x \) and \( u \) are the inputs and outputs of the controller, is known. So their design involves selection of suitable \( f \) and tuning the parameters to meet the performance specifications. However, these methods require accurate mathematical model of the plant. Alternatively, controllers employing fuzzy logic were also used. Fuzzy logic controllers perform well for complex plants without accurate mathematical models because, to a large extent, the system’s behaviour is captured and represented in the forms of fuzzy sets and fuzzy control rules provided by the human control expert. The drawbacks of this method are

1. lot of trials are required to choose fuzzy sets and control rules and are subject to experience of human control expert
2. analysis and precise understanding of the controller is not possible.

In this context, finding analytical structure \( f \) with various components of fuzzy logic controllers which gave better results previously will have the following advantages:

1. analysis and design can be done with the help of various time-tested nonlinear control methods, thereby minimising trial and error effort
2. the structure can be easily implemented on various digital platforms like field programmable gate arrays (FPGAs), digital computers, microcontrollers, etc., at lower price
3. they can be applied to safety critical industries such as nuclear power plants as stability of the overall control system is guaranteed.
The historical developments in fuzzy PID control technology are presented now. Analytical structure for a fuzzy PID controller and its bounded input bounded output (BIBO) stability analysis have been studied (Misir et al., 1996). A systematic and hierarchical approach to the design of a hybrid fuzzy-PID controller through the application of a learning-based algorithm was described (Homaifar et al., 1997). The stability of fuzzy controllers was analysed with the help of passivity theorem (Sio and Lee, 1998). It was shown that nonlinear PID controller, based on conventional PID controller, could be easily realised by applying the simplified indirect fuzzy inference method (Hayashi et al., 1999). Fuzzy PI and fuzzy PD controllers have been combined in parallel to get a fuzzy PID controller (Kim and Oh, 2000). Some practical recommendations for replacing control by a human operator with a fuzzy controller, and for choosing the structure and parameters of this fuzzy controller were given (Reznik et al., 2000). A flexible complexity reduced design approach for PID-like fuzzy controllers was proposed (Tao and Taur, 2000). Analytical structures of various fuzzy controllers were presented in the book (Ying, 2000). Stability analysis, design and real time implementation of some of the developed analytical structures were nicely discussed in this book.

Several forms of decomposed PID fuzzy logic controllers have been tested and compared (Golob, 2001). A function-based evaluation approach has been proposed (Hu et al., 2001) for a systematic study of fuzzy PID controllers. Structural analysis of fuzzy controllers with nonlinear input fuzzy sets in relation to nonlinear PID control with variable gains was done (Haj-Ali and Ying, 2004). A novel algorithm to produce an analytical solution for three-input fuzzy PID systems was proposed (Mann and Gosine, 2005). An efficient solution algorithm to produce the general fuzzy output solution using a minimum number of nonlinear expressions was provided. It was shown that algebraic product triangular norm-bounded sum triangular co-norm-Larsen product inference method-CoS defuzzification method combination leads to a linear fuzzy PID controller (Mohan and Sinha, 2005). It was shown that the analytical structures of fuzzy PID controllers derived via minimum triangular norm are not suitable for control (Mohan and Sinha, 2008a). An analytical structure for fuzzy PID controller by employing algebraic product triangular norm, bounded sum triangular co-norm, minimum inference method and CoS defuzzification method was presented (Mohan and Sinha, 2008b). Sufficient conditions for stability using the Small Gain theorem were also derived. An error based online rule weight adjustment method for fuzzy PID controller was proposed (Karasakal et al., 2011). A novel fractional order fuzzy PID controller and its optimal time-domain tuning was proposed (Das et al., 2012).

An optimal Type 2 fuzzy PID controller was introduced (Khooban et al., 2013) for the control of a class of nonlinear systems in presence of uncertainties and external disturbances. Stability analysis of parallel fuzzy P plus fuzzy I plus fuzzy D control systems was done with the help of mathematical models of the controllers (Kumar et al. 2013). A method of online tuning of fuzzy PID controllers via rule weighting based on normalised acceleration was proposed (Karasakal et al., 2013). A nonlinear fuzzy PID control algorithm with adjustable membership function was proposed (Zhang et al., 2014). Effectiveness and feasibility of this function were verified in the simulation and experimental results. A fuzzy model-based adaptive PID controller design for nonlinear and uncertain processes was proposed (Savran and Kahraman, 2014). A robust tuning of Internal Model Control (IMC) plus PID for cascade control systems was proposed (Azar and Serrano, 2014). The IMC parameters are adjusted according to the desired
frequency response of each loop with a minimum interaction between the inner and outer PID controllers. The proposed method was simple, flexible and accurate. Several experiments were shown to compare and validate the effectiveness of the proposed tuning procedure over other sequential tuning methods.

A new adaptive configuration of PID type fuzzy logic controller was proposed (Fereidouni et al., 2015). A performance driven approach for tuning of fuzzy PID controllers for multi-input multi-output (MIMO) systems was presented (Gil et al., 2015). Three approaches using frequency domain and state space control techniques were presented to stabilise and control mechanical systems with backlash (Azar and Serrano, 2015a). Modelling and design of several anti wind up controllers for single-input single-output (SISO) and MIMO systems were proposed (Azar and Serrano, 2015b). Design of a proportional derivative plus sliding mode controller for the Furuta pendulum was proposed (Azar and Serrano, 2015c). The proposed controller was shown to be effective for under actuated systems due to chattering avoidance. A novel approach for the deadbeat control of multivariable discrete time systems with variable time delays was proposed (Azar and Serrano, 2015d). The benefits of sliding modes when applied to the field of fault tolerant control were presented (Mekki et al., 2015). Recent advances in computational intelligence for modelling and control were comprehensively presented in the book (Azar and Vaidyanathan, 2015). Very recently, PI loop shaping control design for systems with backlash is developed (Azar and Serrano, 2016). A fuzzy adaptive controller for a class of fractional-order chaotic systems with uncertain dynamics and external disturbances is proposed to realise a practical projective synchronisation (Boulkroune et al., 2016a). An adaptive fuzzy control-based function vector synchronisation between two chaotic systems with both unknown dynamic disturbances and input nonlinearities (dead-zone and sector nonlinearities) is presented (Boulkroune et al., 2016b).

In the literature very few models of three-input fuzzy PID controllers of Mamdani type are available. As the mathematical models of fuzzy logic controller are useful in the analysis, design and implementation, in this paper an attempt is made to derive two nonlinear three-input fuzzy PID controllers using the combination of algebraic product AND-maximum/bounded sum OR-Mamdani minimum inference-CoS defuzzification. The computational aspects of both classes of controllers are addressed and a comparison is made so that the control practitioners can have a fairly good idea about their relative merits and demerits. The sufficient conditions for BIBO stability of closed loop systems containing one of these fuzzy controller models are established.

The paper is organised as follows: The fundamental components of a typical fuzzy PID controller are described in Section 2. Mathematical models of fuzzy PID controllers are presented in Section 3. Properties of the models are discussed in Section 4. In Section 5 computational aspects of fuzzy PID controller models are discussed. In Section 6 BIBO stability analysis of feedback systems involving the simplest fuzzy PID controller is presented. Section 7 concludes the paper.

2 Principal components of a typical fuzzy PID controller

The block diagram of a typical computer controlled system is shown in Figure 1. For brevity we drop $T$ in $kT$ in the following expressions.
The incremental control effort generated by a discrete-time PID controller is given by

$$\Delta u(k) = K_d^P \Delta e(k) + K_d^I e_n(k) + K_d^D \Delta^2 e(k)$$

where $\Delta u(k) = u(k) - u(k-1)$, $\Delta e(k) = e(k) - e(k-1)$, $\Delta^2 e(k) = \Delta e(k) - \Delta e(k-1)$, $e_n(k) = K_d^I e(k)$, $\Delta e_n(k) = K_d^P \Delta e(k)$, $\Delta^2 e_n(k) = K_d^D \Delta^2 e(k)$ and $K_d^P$, $K_d^I$ and $K_d^D$ are respectively the proportional, integral and derivative constants of discrete-time PID controller. The inputs to fuzzy PID controller are shown in Figure 2. The inputs to fuzzy PID controller are shown in Figure 2. The block diagram of a typical fuzzy PID controller is shown in Figure 3 in which $S_e$, $S_{\Delta e}$, $S_{\Delta^2 e}$ and $S_{\Delta u}^{-1}$ represent the scaling factors of the fuzzy controller. The scaled inputs are given by

$$e_s(k) = S_e \cdot e(k),$$
$$\Delta e_s(k) = S_{\Delta e} \cdot \Delta e(k)$$
$$\Delta^2 e_s(k) = S_{\Delta^2 e} \cdot \Delta^2 e(k).$$
The principal components of fuzzy PID controller are fuzzification and defuzzification modules, control rule base, and inference engine which are described in the following.

### 2.1 Fuzzification module

The inputs are fuzzified by $L$-type and $\Gamma$-type membership functions, shown in Figure 4, whose mathematical description is given by

\[
\mu_{N_e}(e_s(k)) = \begin{cases} 
1 & \text{if } -H_e \leq e_s(k) \leq -h_e \\
\frac{-e_s(k) + h_e}{2h_e} & \text{if } -h_e \leq e_s(k) \leq h_e \\
0 & \text{if } h_e \leq e_s(k) \leq H_e 
\end{cases}
\]

\[\text{(3)}\]

\[
\mu_{P_e}(e_s(k)) = \begin{cases} 
0 & \text{if } -H_e \leq e_s(k) \leq -h_e \\
\frac{e_s(k) + h_e}{2h_e} & \text{if } -h_e \leq e_s(k) \leq h_e \\
1 & \text{if } h_e \leq e_s(k) \leq H_e 
\end{cases}
\]

\[\text{(4)}\]

Figure 4  Input membership functions

Similarly, the mathematical descriptions of the other membership functions on two scaled inputs $\Delta e_s(k)$ and $\Delta^2 e_s(k)$ can be defined. Notice that $\mu_{N_e}(e_s(k)) + \mu_{P_e}(e_s(k)) = 1, \forall e_s(k)$. This is true for the other two inputs. The membership functions for the scaled output are shown in Figure 5, where $\Delta u_s(k) = S_{\Delta u} \cdot \Delta u(k)$. 

Figure 5  Reference and inferred output membership functions via Mamdani minimum inference
2.2 Control rule base

Total eight \((2^3)\) rules are required as there are two fuzzy sets defined on each of the three input variables. In this paper rules having the same consequent part are merged into one rule. As there are four fuzzy sets defined on output variable we have the following four control rules:

\[ R_1 : \text{IF } e_s(k) \text{ is } N_e \text{ AND } \Delta e_s(k) \text{ is } N_{\Delta e} \text{ AND } \Delta^2 e_s(k) \text{ is } N_{\Delta^2 e} \text{ THEN } \Delta u_s(k) \text{ is } \Delta U_{-2} \]

\[ R_2 : \text{IF } (e_s(k) \text{ is } N_e \text{ AND } \Delta e_s(k) \text{ is } N_{\Delta e} \text{ AND } \Delta^2 e_s(k) \text{ is } P_{\Delta^2 e}) \text{ OR } (e_s(k) \text{ is } N_e \text{ AND } \Delta e_s(k) \text{ is } P_{\Delta e} \text{ AND } \Delta^2 e_s(k) \text{ is } N_{\Delta^2 e}) \text{ OR } (e_s(k) \text{ is } P_e \text{ AND } \Delta e_s(k) \text{ is } N_{\Delta e}) \text{ THEN } \Delta u_s(k) \text{ is } \Delta U_{-1} \]

\[ R_3 : \text{IF } (e_s(k) \text{ is } N_e \text{ AND } \Delta e_s(k) \text{ is } P_{\Delta e} \text{ AND } \Delta^2 e_s(k) \text{ is } P_{\Delta^2 e}) \text{ OR } (e_s(k) \text{ is } P_e \text{ AND } \Delta e_s(k) \text{ is } N_{\Delta e} \text{ AND } \Delta^2 e_s(k) \text{ is } P_{\Delta^2 e}) \text{ OR } (e_s(k) \text{ is } P_e \text{ AND } \Delta e_s(k) \text{ is } P_{\Delta e} \text{ AND } \Delta^2 e_s(k) \text{ is } N_{\Delta^2 e}) \text{ THEN } \Delta u_s(k) \text{ is } \Delta U_{+1} \]

\[ R_4 : \text{IF } e_s(k) \text{ is } P_e \text{ AND } \Delta e_s(k) \text{ is } P_{\Delta e} \text{ AND } \Delta^2 e_s(k) \text{ is } P_{\Delta^2 e} \text{ THEN } \Delta u_s(k) \text{ is } \Delta U_{+2} \]

Algebraic product t-norm is considered to perform the AND operation in the rule base and is defined as:

\[ t(\mu_A(x), \mu_B(y)) = \mu_A(x) \cdot \mu_B(y) \]

Maximum and bounded sum t-co-norms are considered to perform the OR operation in the rules \(R_2\) and \(R_3\), and are defined as:

- Maximum: \(s(\mu_A(x), \mu_B(y)) = \max\{\mu_A(x), \mu_B(y)\}\)
- Bounded sum: \(s(\mu_X(x), \mu_Y(y)) = \min\{1, \mu_X(x) + \mu_Y(y)\}\)

We consider all possible combinations of three input variables in a 3D space. A point, say \((x_t, y_t, z_t)\), in a 3D space can always be distinctly shown by taking its projections on the \(xy\), \(yz\) and \(xz\) planes. So, as shown in Figure 6, twenty input combinations are considered in each input plane \((e_s(k) \Delta e_s(k), \Delta e_s(k) \Delta^2 e_s(k) \text{ and } \Delta^2 e_s(k))\) so that the input point \((e_s(k), \Delta e_s(k), \Delta^2 e_s(k))\) can be uniquely located in the 3D cell (subspace) represented by the triplet \((n_a, n_b, n_c)\) where \(n_a, n_b, n_c = 1, 2, 3, ..., 20\). For example, the triplet \((9, 11, 20)\) represents the 3D cell with 9 from (a), 11 from (b), and 20 from (c) of Figure 6. A cell is said to be valid if and only if the relations between \(e_s(k)\) and \(\Delta e_s(k)\), and \(\Delta e_s(k)\) and \(\Delta^2 e_s(k)\) produce the relation between \(\Delta^2 e_s(k)\) and \(e_s(k)\). For example, the cell \((13, 14, 16)\) is a valid cell because by combining the relations \(\Delta^2 e_s(k) \geq e_s(k)\) (valid in 13 of Figure 6(a)) and \(\Delta^2 e_s(k) \geq \Delta e_s(k)\) (valid in 14 of Figure 6(b)) will produce the relation \(\Delta^2 e_s(k) \geq \Delta^2 e_s(k)\) which is valid in 16 of Figure 6(c). The control rules \(R_1\) to \(R_4\) are used to evaluate appropriate control law in each valid cell \((n_a, n_b, n_c)\). Let the outcomes of antecedent parts of rules \(R_1\), \(R_2\), \(R_3\) and \(R_4\) be \(\mu_{-2}\), \(\mu_{-1}\), \(\mu_{+1}\) and \(\mu_{+2}\) respectively. The resultant expressions of \(\mu_{-2}\), \(\mu_{-1}\), \(\mu_{+1}\) and \(\mu_{+2}\) are found in each valid cell and are shown in Tables 1 (using maximum OR) and 2 (using bounded sum OR).
<table>
<thead>
<tr>
<th>Cells</th>
<th>$\mu=2$</th>
<th>$\mu=1$</th>
<th>$\mu=3$</th>
<th>$\mu=4$</th>
<th>$S_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 2, 3), (2, 3, 1), (3, 1, 2)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>–</td>
</tr>
<tr>
<td>(1, 3, 4), (3, 4, 1), (4, 1, 3)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>(2, 2, 2)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>–</td>
</tr>
<tr>
<td>(4, 4, 4)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>+</td>
</tr>
</tbody>
</table>

**Case a:** All three inputs lie in the outer cuboid

**Case b:** One input lies in the inner cuboid and two inputs in the outer cuboid

**Case c:** Two inputs lie in the inner cuboid and one input lies in the outer cuboid

**Case d:** All three inputs lie in the inner cuboid

$$-h_{\pm} \leq e_{\pm}(k) \leq h_{\pm} \leq e_{\pm} = -h_{\pm} \leq \Delta e_{\pm}(k) \leq h_{\pm}$$

<table>
<thead>
<tr>
<th>Cells</th>
<th>$\mu=2$</th>
<th>$\mu=1$</th>
<th>$\mu=3$</th>
<th>$\mu=4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 2, 3), (2, 3, 1), (3, 1, 2)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(1, 3, 4), (3, 4, 1), (4, 1, 3)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(2, 2, 2)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(4, 4, 4)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 1** Outcomes of algebraic product AND and maximum OR and attributes of $S_1$ in different cells.
### Table 2  Outcomes of algebraic product AND and bounded sum OR

<table>
<thead>
<tr>
<th>Cells</th>
<th>(\mu = -2)</th>
<th>(\mu = -1)</th>
<th>(\mu = +1)</th>
<th>(\mu = +2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((5, 7, 16), (5, 7, 17), (5, 8, 18), (5, 8, 19), (6, 7, 14), (6, 7, 15), (6, 8, 13), (6, 8, 20))</td>
<td>(N_eN_\Delta^2e)</td>
<td>(N_eP_\Delta^2e + P_eN_\Delta^2e)</td>
<td>(P_eP_\Delta^2e)</td>
<td>0</td>
</tr>
<tr>
<td>((7, 14, 6), (7, 15, 6), (7, 16, 5) (7, 17, 5), (8, 13, 6), (8, 18, 5), (8, 19, 5), (8, 20, 6))</td>
<td>(N_\Delta^2eN_\Delta^2e)</td>
<td>(N_\Delta^2eP_\Delta^2e + P_eN_\Delta^2e)</td>
<td>(P_eP_\Delta^2e)</td>
<td>0</td>
</tr>
<tr>
<td>((9, 11, 13), (9, 11, 20), (9, 12, 14) (9, 12, 15), (10, 11, 18), (10, 11, 19), (10, 12, 16), (10, 12, 17))</td>
<td>0</td>
<td>(N_eN_\Delta^2e)</td>
<td>(N_eP_\Delta^2e)</td>
<td>(P_eP_\Delta^2e)</td>
</tr>
<tr>
<td>((11, 13, 9), (11, 19, 10), (11, 20, 9) (12, 14, 9), (11, 18, 10), (12, 15, 9), (12, 16, 10), (12, 17, 10))</td>
<td>0</td>
<td>(N_\Delta^2eN_\Delta^2e)</td>
<td>(N_\Delta^2eP_\Delta^2e + P_eN_\Delta^2e)</td>
<td>(P_eP_\Delta^2e)</td>
</tr>
<tr>
<td>((13, 6, 8), (14, 6, 7), (15, 7, 6), (16, 5, 7), (17, 5, 7), (18, 5, 8), (19, 5, 8), (20, 6, 8))</td>
<td>(N_eN_\Delta^2e)</td>
<td>(N_eP_\Delta^2e + P_eN_\Delta^2e)</td>
<td>(P_eP_\Delta^2e)</td>
<td>0</td>
</tr>
<tr>
<td>((13, 9, 11), (14, 9, 12), (15, 9, 12) (16, 10, 12), (17, 10, 12), (18, 10, 11), (19, 10, 11), (20, 9, 11))</td>
<td>0</td>
<td>(N_eN_\Delta^2e)</td>
<td>(N_eP_\Delta^2e + P_eN_\Delta^2e)</td>
<td>(P_eP_\Delta^2e)</td>
</tr>
</tbody>
</table>

Case a: All three inputs lie in the outer cuboid

Case b: One input lies in the inner cuboid and two inputs lie in the outer cuboid

Case c: Two inputs lie in the inner cuboid and one input lies in the outer cuboid

Same is in Table 1

Case d: All three inputs lie in the inner cuboid

\(-h_s \leq c_s(k) \leq h_s, -h_\Delta^2e \leq \Delta c_s(k) \leq h_\Delta^2e, -h_\Delta^2e \leq \Delta^2c_s(k) \leq h_\Delta^2e\)

\((13, 14, 16), (13, 15, 16), (13, 15, 17), (13, 16, 18), (13, 16, 19), (13, 17, 19), (14, 14, 15), (14, 15, 14), (14, 15, 15), (14, 16, 13), (14, 16, 20), (14, 17, 20), (15, 14, 15), (15, 14, 15), (15, 15, 14), (15, 16, 13), (15, 17, 13), (15, 17, 20), (16, 13, 14), (16, 13, 15), (16, 18, 13), (16, 18, 20), (16, 19, 13), (16, 20, 14), (17, 13, 15), (17, 18, 20), (17, 19, 13), (17, 19, 20), (17, 20, 14), (17, 20, 15), (18, 13, 16), (18, 18, 19), (18, 19, 18), (18, 19, 19), (18, 20, 16), (18, 20, 17), (19, 13, 16), (19, 13, 17), (19, 18, 18), (19, 18, 19), (19, 19, 18), (19, 20, 17), (20, 14, 16), (20, 14, 17), (20, 15, 17), (20, 16, 18), (20, 17, 18), (20, 17, 19)\)
2.3 Inference mechanism

The expression of $\mu_{-2}$ obtained via algebraic product t-norm is used to modify the reference output fuzzy set $\Delta U_{-2}$ in rule $R_1$ by employing a particular type of inference method. Similarly, $\Delta U_{-1}$, $\Delta U_{+1}$ and $\Delta U_{+2}$ are also modified. The modified membership functions ($\Delta \tilde{U}_{-2}$, $\Delta \tilde{U}_{-1}$, $\Delta \tilde{U}_{+1}$ and $\Delta \tilde{U}_{+2}$) of the scaled output variable ($\Delta u_s(k)$), obtained via Mamdani minimum inference method, are shown with hatching in Figure 5.

2.4 Defuzzification module

Defuzzification is done here using the well known centre of sums (CoS) method (Drainkov et al., 1993). According to this method, the crisp value of scaled control output is given by

$$
\Delta u_s^*(k) = \frac{[\tilde{A}(\Delta \tilde{U}_{-2})\tilde{C}(\Delta \tilde{U}_{-2}) + \tilde{A}(\Delta \tilde{U}_{-1})\tilde{C}(\Delta \tilde{U}_{-1}) + \tilde{A}(\Delta \tilde{U}_{+1})\tilde{C}(\Delta \tilde{U}_{+1}) + \tilde{A}(\Delta \tilde{U}_{+2})\tilde{C}(\Delta \tilde{U}_{+2})]}{\tilde{A}(\Delta \tilde{U}_{-2}) + \tilde{A}(\Delta \tilde{U}_{-1}) + \tilde{A}(\Delta \tilde{U}_{+1}) + \tilde{A}(\Delta \tilde{U}_{+2})}
$$

(5)

where $\tilde{A}(\cdot)$ and $\tilde{C}(\cdot)$ are respectively, the area and centroid corresponding to the inferred output membership function. It is easy to compute the following from Figure 5:

$$
\tilde{A}(\Delta \tilde{U}_{-2}) = \frac{\mu_{-2}}{3} \{ (2B - 3A) - \mu_{-2}(B - 3A) \}$$
\[ \hat{C}(\Delta \bar{U}_{-2}) = \frac{-[(24B^2 - 18AB - 27A^2) - 6\mu_{-2}(B^2 - 9A^2) - 4\mu^2_{-2}(B - 3A)^2]}{18[(2B - 3A) - \mu_{-2}(B - 3A)]} \]
\[ \check{\Delta \bar{U}_{-1}} = \frac{2\mu_{-1}}{3} \{(2B - 3A) - \mu_{-1}(B - 3A)\} \]
\[ \hat{C}(\Delta \bar{U}_{-1}) = \frac{-B}{3} \]
\[ \check{\Delta \bar{U}_{+1}} = \frac{2\mu_{+1}}{3} \{(2B - 3A) - \mu_{+1}(B - 3A)\} \]
\[ \hat{C}(\Delta \bar{U}_{+1}) = \frac{B}{3} \]
\[ \check{\Delta \bar{U}_{+2}} = \frac{\mu_{+2}}{3} \{(2B - 3A) - \mu_{+2}(B - 3A)\} \]
\[ \hat{C}(\Delta \bar{U}_{+2}) = \frac{[24B^2 - 18AB - 27A^2) - 6\mu_{+2}(B^2 - 9A^2) - 4\mu^2_{+2}(B - 3A)^2]}{18[(2B - 3A) - \mu_{+2}(B - 3A)]} \]

3 Mathematical models of the simplest fuzzy PID controller

For simplicity let \( h_1 = h_e, \ h_2 = h_{\Delta e}, \ h_3 = h_{\Delta^2 e}, \ x_1 = e_s(k), \ x_2 = \Delta e_s(k), \) and \( x_3 = \Delta^2 e_s(k). \) Then upon substituting the values of \( \mu_{-2}, \mu_{-1}, \mu_{+1} \) and \( \mu_{+2}, \) defined in Tables 1 and 2, into \( \check{A}(\cdot)s \) and \( \hat{C}(\cdot)s \) of equation (5), and upon simplification we get the models of the following classes of fuzzy controllers:

Class 1 Using maximum OR,
\[ \Delta u_s(k) = \frac{N_1}{D_1} \]

Class 2 Using bounded sum OR,
\[ \Delta u_s(k) = \frac{N_2}{D_2} \]

where

Case 1 In cells (1, 2, 3), (1, 3, 4), (2, 3, 1), (3, 1, 2), (3, 4, 1), (4, 1, 3)
\[ N_1 = N_2 = S_1B \]
\[ D_1 = D_2 = 3 \]
and \( S_1 \) is as defined in Table 1.

Case 2 In cells (2, 2, 2), (4, 4, 4)
\[ N_1 = N_2 = S_1(14B^2 + 6AB - 9A^2) \]
\[ D_1 = D_2 = 18B \]

Case 3 In cells (1, 7, 9), (1, 8, 10), (3, 11, 5), (3, 12, 6)
\[ N_1 = N_2 = S_12B^2h_3x_3 \]
\[ D_1 = D_2 = 3[3A(x_3^2 - h_3^2) - B(x_3^2 - 3h_3^2)] \].
Case 4 In cells (2, 7, 6), (2, 8, 5), (4, 11, 10), (4, 12, 9)
\[ N_1 = N_2 = 6AB\{x_3^4 + S_1(6h_3x_3^2 - 5h_3^2)\} - 9A^2(x_3^4 + S_1h_3^2) - B^2\{x_3^4 + S_1(12h_3x_3^2 - 38h_3^2) - 3h_3^2x_3\} \]
\[ D_1 = D_2 = 9h_3[9A(x_3^2 - h_3^2) - B(3x_3^2 + S_1h_3x_3 - 9h_3^2)]. \]

Case 5 In cells (5, 2, 8), (6, 2, 7), (9, 4, 12), (10, 4, 11)
\[ N_1 = N_2 = 6AB\{x_3^4 + S_1(6h_1x_1^2 - 5h_1^2)\} - 9A^2(x_1^4 + S_1h_1^2) - B^2\{x_1^4 + S_1(12h_1x_1^2 - 38h_1^2) - 3h_1^2x_1\} \]
\[ D_1 = D_2 = 9h_1[9A(x_1^2 - h_1^2) - B(3x_1^2 + S_1h_1x_1 - 9h_1^2)]. \]

Case 6 In cells (5, 3, 11), (6, 3, 12), (9, 1, 7), (10, 1, 8)
\[ N_1 = N_2 = S_12B^2h_1x_1 \]
\[ D_1 = D_2 = 3[3A(x_1^2 - h_1^2) - B(x_1^2 - 3h_1^2)]. \]

Case 7 In cells (7, 6, 2), (8, 5, 2), (11, 10, 4), (12, 9, 4)
\[ N_1 = N_2 = 6AB\{x_3^4 + S_1(6h_2x_2^2 - 5h_2^2)\} - 9A^2(x_3^4 + S_1h_2^2) - B^2\{x_3^4 + S_1(12h_2x_2^2 - 38h_2^2) - 3h_2^2x_2\} \]
\[ D_1 = D_2 = 9h_2[9A(x_2^2 - h_2^2) - B(3x_2^2 + S_1h_2x_2 - 9h_2^2)]. \]

Case 8 In cells (7, 9, 1), (8, 10, 1), (11, 5, 3), (12, 6, 3)
\[ N_1 = N_2 = S_12B^2h_2x_2 \]
\[ D_1 = D_2 = 3[3A(x_2^2 - h_2^2) - B(x_2^2 - 3h_2^2)]. \]

Case 9 In cells (5, 7, 16), (5, 7, 17), (5, 8, 18), (6, 7, 15), (9, 11, 13), (9, 11, 20),
(9, 12, 14), (10, 11, 19)
\[ N_1 = S_1\{(63A^2 + 66AB - 89B^2)h_1^3h_3^3 + (-27A^2 - 18AB + 9B^2)h_1h_3(h_2^2x_1^2 + h_1^2x_3^2) + (27A^2 - 18AB + 3B^2)x_1x_3(h_2^2x_1^2 + h_1^2x_3^2) + S_1x_2^2x_3^2(h_1x_1 + h_1x_3) + (27A^2 - 54AB + 15B^2)h_1h_3x_1^2x_3^2 + (-27A^2 + 18AB + 33B^2)x_1^2x_3^2 + (27A^2 + 54AB - 81B^2)h_1^2h_3x_1 + (9A^2 - 6AB + B^2)(h_1^2x_1^2 + h_1^2x_3^2 + S_1x_1x_3^3) + (27A^2 + 198AB - 225B^2)h_1^2h_3x_3^3 + (-27A^2 - 198AB + 69B^2)h_1^2x_1^2x_3 + (-27A^2 - 54AB + 21B^2)x_1^2x_3^2h_3x_1x_3^2 \]
\[ D_1 = 18h_1h_3[(45A - 35B)h_1^2h_3^2 - S_118(A - B)h_1^2x_1 - (15A - 5B)(h_2^2x_1^2 + h_1^2x_3^2 + S_1x_1x_3^3) + S_16(A - B)h_1^2h_3x_3 - 4Bh_1h_3x_1x_3 - S_1(6A - 2B)h_3x_1^2x_3 + S_1(18A - 6B)h_1x_1x_3^2 \]
\[ N_2 = S_1\{(-63A^2 - 102AB + 149B^2)h_1^3h_3^3 + (-27A^2 - 126AB + 153B^2)h_1^2h_3^2(h_3x_1 + h_1x_3) + (27A^2 - 18AB + 3B^2)[S_1h_2^2x_1^2 + h_1^2x_3^2](h_1h_3 - x_1x_3) + S_1x_1^2x_3^2(h_3x_1 + h_1x_3) + (-9A^2 + 6AB - B^2)h_1^2x_1^2 + S_1(-27A^2 + 162AB - 51B^2)h_1h_3x_1^2x_3 + S_1(-18AB - 81B^2)h_1^2x_1^2 + (27A^2 + 126AB - 45B^2)h_1h_3x_1x_3(h_3x_1 + h_1x_3) \]
\[ D_2 = 18h_1h_3[(-51A + 45B)h_1^2h_3^2 + S_16(A - B)h_1^2h_3(h_3x_1 + h_1x_3) + (9A - 3B)(h_2^2x_1^2 + h_1^2x_3^2) - 4Bh_1h_3x_1x_3 + S_1(-6A + 2B)x_1x_3(h_3x_1 + h_1x_3) + (33A - 11B)x_1^2x_3^2 \].
Case 10 In cells (5, 8, 19), (6, 7, 14), (6, 8, 13), (6, 8, 20), (9, 12, 15), (10, 11, 18), (10, 12, 16), (10, 12, 17)

\[ N_1 = S_1 \{(63A^2 + 66AB - 89B^2)h_3^3 + (-27A^2 - 18AB + 9B^2)h_1 h_3(h_2^3 x_2^2 + h_2^3 x_3^2) + (27A^2 - 18AB + 3B^2)\}x_2 x_3 (h_2^3 x_4^2 + h_2^3 x_5^2) + S_1 x_2^3 x_3 (h_4 x_2 + h_4 x_1) + (27A^2 - 54AB + 15B^2)h_2 h_3 x_2^3 x_3^2 + (-27A^2 + 18AB + 33B^2)h_1 h_2 x_2 x_3 + (27A^2 + 54AB - 81B^2)h_2^3 x_3 + (9A^2 - 6AB + B^2)(h_1^3 x_4^2 + h_1^3 x_5^2 + S_1 x_2^3 x_3^2) + (27A^2 + 198AB - 225B^2)h_1^3 h_3 x_2 + (-27A^2 - 198AB + 69B^2)h_1^2 h_3 x_2 x_3^2 + (-27A^2 - 54AB + 21B^2)h_1^2 h_3 x_2 x_3^2 x_3 \]

\[ D_1 = 18h_1 h_3[(45A - 35B)h_1^3 - S_1 18(A - B)h_1^2 h_3 x_3 - (15A - 5B)(h_1^3 x_4^2 + h_1^3 x_5^2 + x_2^2 x_3^2) + S_1 6(A - B)h_3^2 x_3 x_3 - 4Bh_1 h_3 x_2 x_3 - S_1(6A - 2B)h_3^2 x_2 x_3 + S_1(18A - 6B)h_3 x_2 x_3] \]

\[ N_2 \text{ and } D_2 \text{ are the same as in Case 9.} \]

Case 11 In cells (7, 14, 6), (8, 13, 6), (8, 19, 5), (8, 20, 6), (11, 18, 10), (12, 15, 9), (12, 16, 10), (12, 17, 10)

\[ N_1 = S_1 \{(63A^2 + 66AB - 89B^2)h_3^3 + (-27A^2 - 18AB + 9B^2)h_2 h_3(h_2^3 x_2^2 + h_2^3 x_3^2) + (27A^2 - 18AB + 3B^2)\}x_2 x_3 (h_2^3 x_4^2 + h_2^3 x_5^2) + S_1 x_2^3 x_3 (h_3 x_2 + h_3 x_3) + (27A^2 - 54AB + 15B^2)h_2 h_3 x_2^3 x_3^2 + (-27A^2 + 18AB + 33B^2)h_1 h_2 x_2 x_3 + (27A^2 + 54AB - 81B^2)h_2^3 x_3 + (9A^2 - 6AB + B^2)(h_1^3 x_4^2 + h_1^3 x_5^2 + S_1 x_2^3 x_3^2) + (27A^2 + 198AB - 225B^2)h_1^3 h_3 x_2 + (-27A^2 - 198AB + 69B^2)h_1^2 h_3 x_2 x_3^2 + (-27A^2 - 54AB + 21B^2)h_1^2 h_3 x_2 x_3^2 x_3 \]

\[ D_1 = 18h_1 h_3[(45A - 35B)h_3^3 - S_1 18(A - B)h_3^2 x_3 - 15A - 5B)(h_3^3 x_4^2 + h_3^3 x_5^2 + x_2^2 x_3^2) + S_1 6(A - B)h_3^2 x_3 x_3 - 4Bh_1 h_3 x_2 x_3 - S_1(6A - 2B)h_3^2 x_2 x_3 + S_1(18A - 6B)h_3 x_2 x_3] \]

\[ N_2 = S_1 [(-63A^2 - 102AB + 149B^2)h_3^3 + (-27A^2 - 126AB + 153B^2)h_2^3 (h_3 x_3 + h_2 x_3) + (27A^2 - 18AB + 3B^2)\}S_1(h_3^3 x_4^2 + h_3^3 x_5^2) \]

\[ h_2 (h_2^3 x_3 + x_2 x_3) - x_2^2 x_3^2 (h_3 x_3 + h_2 x_3)] + (-9A^2 + 6AB - B^2)(h_3 x_4^2 + h_3 x_5^2 + S_1 x_2^3 x_3^2) + S_1(-27A^2 - 162AB - 51B^2)h_2 h_3 x_2^3 x_3^2 + S_1(27A^2 - 18AB - 81B^2)h_2^3 x_3 x_3^2 + (27A^2 + 126AB - 45B^2)h_2 h_3 x_2 x_2 x_3 (h_3 x_2 + h_2 x_3) \]

\[ D_2 = 18h_1 h_3[(-51A + 45B)h_3^3 + S_1 6(A - B)h_3^2 (h_3 x_2 + h_2 x_3) + (9A - 3B)(h_3^3 x_4^2 + h_3^3 x_5^2) - 4Bh_2 h_3 x_2 x_3 + S_1(-6A + 2B)x_2 x_3 (h_3 x_2 + h_2 x_3)] + (33A - 11B)x_2 x_3^2 \]

Case 12 In cells (7, 15, 5), (7, 17, 5), (8, 18, 5), (11, 13, 9), (11, 19, 10), (11, 20, 9), (12, 14, 9)

\[ N_1 = S_1 \{(63A^2 + 66AB - 89B^2)h_3^3 + (-27A^2 - 18AB + 9B^2)h_2 h_3(h_2^3 x_2^2 + h_2^3 x_3^2) + (27A^2 - 18AB + 3B^2)\}x_2 x_3 (h_2^3 x_4^2 + h_2^3 x_5^2) + S_1 x_2^3 x_3 (h_3 x_2 + h_3 x_3) + (27A^2 - 54AB + 15B^2)h_2 h_3 x_2^3 x_3^2 + (-27A^2 + 18AB + 33B^2)h_1 h_2 x_2 x_3 + (27A^2 + 54AB - 81B^2)h_2^3 x_3 + (9A^2 - 6AB + B^2)(h_1^3 x_4^2 + h_1^3 x_5^2 + S_1 x_2^3 x_3^2) + (27A^2 + 198AB - 225B^2)h_1^3 h_3 x_2 + (-27A^2 - 198AB + 69B^2)h_1^2 h_3 x_2 x_3^2 + (-27A^2 - 54AB + 21B^2)h_1^2 h_3 x_2 x_3^2 x_3 \]
\[ D_1 = 18h_2h_3[(45A - 35B)h_3^2h_2^3 - S_118(A - B)h_3^2h_2x_3 - (15A - 5B)h_3^2x_3^2 + S_16(A - B)h_3^2x_2x_3 - 4Bh_2h_3x_2x_3 - S_16(A - 2B)h_2x_3^2 + S_1(18A - 6B)h_3x_3^2] \]

\[ N_2 \text{ and } D_2 \text{ are the same as in Case 11.} \]

Case 13  
In cells (13, 6, 8), (14, 6, 7), (15, 9, 12), (16, 10, 12), (17, 10, 12), (18, 10, 11),
(19, 5, 8), (20, 6, 8)

\[ N_1 = S_1\{(63A^2 + 66AB - 89B^2)h_3^3 + (-27A^2 - 18AB + 9B^2)h_1h_2(h_2x_1 + h_1x_2) + (27A^2 - 18AB + 3B^2)[x_1x_2(h^2x_2^3 + h^3x_2^2) + S_1x_2^2x_3(h_1x_2 + h_2x_1)] + (27A^2 - 54AB + 15B^2)h_1h_2x_1^2x_2^2 + (-27A^2 + 18AB + 33B^2)h_1^2h_2x_1x_2^2 + (27A^2 + 54AB - 81B^2)h_1^2h_2x_1 + (9A^2 - 6AB - B^2)[h_2x_1 + S_1x_2x_3^2] + (27A^2 + 198AB - 225B^2)h_1^2x_2^2 + (-27A^2 - 198AB + 69B^2)h_1^2x_2^2 + (-27A^2 - 54AB + 21B^2)h_1^2x_1x_2^2 \]

\[ D_1 = 18h_1h_2[(45A - 35B)h_3^2h_2^3 - S_118(A - B)h_3^2h_2x_3 - (15A - 5B)h_3^2x_3^2 + S_16(A - B)h_3^2x_2x_3 - 4Bh_1h_2x_3x_3 - S_1(6A - 2B)h_1x_1x_2 + S_1(18A - 6B)h_1x_1x_2^2] \]

\[ N_2 = S_1\{-63A^2 - 102AB + 149B^2)h_3^3 + (-27A^2 - 126AB + 153B^2)h_1^2h_2x_1 + h_1x_2) + (27A^2 - 18AB + 3B^2)[S_1x_1x_2(h_1h_2x_1 + x_2x_3^2) + x_1x_2^2x_3(h_2x_1 + h_1x_2)] + (-9A^2 + 6AB - B^2 + h_2x_1 + h_1x_2) + S_1(-27A^2 + 162AB - 51B^2)h_1h_2x_1x_2^2 + S_1(27A^2 - 18AB - 81B^2)h_1^2h_2x_1x_2 + (27A^2 + 126AB - 45B^2)h_1x_1x_2(x_2x_3 + h_1x_2) \]

\[ D_2 = 18h_1h_2[(51A + 45B)h_3^2h_2^3 + S_16(A - B)h_1h_2x_1x_1 + h_1x_2) + (9A - 3B)(h_3^2x_1^2 + h_3x_1^2) - 4Bh_1h_2x_1x_2 + S_1(-6A + 2B)x_1x_2hx_1 + x_1x_2) + (3A - 11B)x_1x_2x_3] \]

Case 14  
In cells (13, 9, 11), (14, 9, 12), (15, 6, 7), (16, 5, 7), (17, 5, 7), (18, 5, 8),
(19, 10, 11), (20, 9, 11)

\[ N_1 = S_1\{(63A^2 + 66AB - 89B^2)h_3^3 + (-27A^2 - 18AB + 9B^2)h_1h_2(h_2x_1 + h_1x_2) + (27A^2 - 18AB + 3B^2)[x_1x_2(h^2x_2^3 + h^3x_2^2) + S_1x_2^2x_3(h_1x_2 + h_2x_1)] + (27A^2 - 54AB + 15B^2)h_1h_2x_1^2x_2^2 + (-27A^2 + 18AB + 33B^2)h_1^2h_2x_1x_2^2 + (27A^2 + 54AB - 81B^2)h_1^2h_2x_1 + (9A^2 - 6AB + B^2)[h_2x_1 + h_1x_2] + S_1x_2x_3^2 + (27A^2 + 198AB - 225B^2)h_1^2x_2^2 + (-27A^2 - 198AB + 69B^2)h_1^2x_2^2 + (-27A^2 - 54AB + 21B^2)h_1^2x_1x_2^2 \]

\[ D_1 = 18h_1h_2[(45A - 35B)h_3^2h_2^3 - S_118(A - B)h_3^2h_2x_3 - (15A - 5B)h_3^2x_3^2 + S_16(A - B)h_3^2x_2x_3 - 4Bh_1h_2x_3x_3 - S_1(6A - 2B)h_1x_1x_2 + S_1(18A - 6B)h_1x_1x_2^2] \]

\[ N_2 \text{ and } D_2 \text{ are the same as in Case 13.} \]

Case 15 In cells  (13, 14, 16), (14, 14, 15), (15, 15, 14), (15, 16, 13), (15, 17, 13),
(15, 17, 20), (16, 18, 13), (16, 18, 20), (17, 18, 20), (18, 19, 18), (19, 13, 16),
(19, 13, 17), (19, 19, 18), (19, 20, 17), (20, 14, 16), (20, 14, 17)

\[ N_1 = (-243A^2 - 270AB + 357B^2)h_1^2h_2^3(h_3x_1 + h_1x_3) + (-243A^2 - 702AB + 693B^2)h_1^2h_2^3x_2 + (-9A^2 + 6AB - B^2)(h_3^2h_3x_1 + \]
Case 16 In cells

Modelling, stability analysis and computational aspects

\[ h_1^3 x_2^3 + h_1^3 h_2^3 x_3^3 + x_2^3 x_3^3 + h_1^2 (288 (AB - B^2) h_2^3 h_3^2 x_2 + (288 AB - 96 B^2) h_1 h_2^3 x_1 x_3) \{ h_1 x_2 - h_3 x_1 \} + (135 A^2 + 198 AB - 81 B^2) h_1 h_2^3 h_3 x_2 (h_1^3 x_2^2 + h_1^2 h_2^2 x_2^2 x_3 + h_2^2 h_3 x_2 x_3^2 + h_1^3 h_2^3 x_2 x_3 + h_1^2 h_2^2 x_2 x_3^2 + h_1^2 h_3^2 x_2 x_3^2) + (189 A^2 - 126 B^2 + 69 B^2) h_1^3 h_2^3 x_2 x_3 + (27 A^2 + 18 AB - 3 B^2) (h_1^3 x_2^2 h_3 x_2 + h_1 x_2) + h_1 x_2 (h_1 x_2 + h_1 x_3) + h_1^2 x_2^2 x_3 (h_2 x_3 + h_3 x_2) + x_1 x_2 x_3 (h_1^2 x_2^2 x_3 + h_2 x_2^2 x_3 + h_2^2 x_2^2 x_3 + h_3^2 x_2^2 x_3 + h_1 x_2 x_3 + h_1 h_2 x_1 x_2 x_3^2) + (27 A^2 + 162 AB - 51 B^2) h_1 h_2 h_3 x_1 x_2 x_3 (h_3 x_1 + h_1 x_1) + (81 A^2 + 54 AB - 9 B^2) x_1 x_2 x_3 (h_2^3 x_2^2 + h_1^3 x_2^2 x_3 + h_2 h_3 x_2 x_3 + h_1 h_2 x_1 x_2 x_3^2) + (27 A^2 + 306 AB - 99 B^2) h_1 h_2 h_3 x_1 x_2 x_3^2 \]

\[ \begin{align*}
D_1 &= 18 h_1 h_2 h_3 (-126 A + 90 B) h_1^3 h_2^3 h_3^2 + 1 \{ 72 A - 56 B \} h_1^3 h_2^2 h_3 (h_1 x_1 - h_3 x_1) + (184 - 6 B) (h_1^3 h_2^3 x_1^2 + h_1^2 h_2^2 x_2 + h_1^2 h_2^2 x_2) + \frac{h_1^2 x_2^2 x_3 + x_1^2 x_2 x_3^2 + 24 (A - B) (h_1 h_2 h_3 x_1 x_3 - h_1 h_2 h_3 x_1 x_2 - h_2 h_3 x_2 x_3) + (24 A - 8 B) \{ h_2 h_3 x_1 x_3 + h_1 h_2 x_1 x_3 - h_3 x_1 x_2 x_3 - S_1 \} (h_2 h_3 x_1 x_3 + h_1 h_2 x_1 x_3 + h_3 x_1 x_2 x_3 - h_1 h_2 x_1 x_2 x_3 - h_1 h_2 x_1 x_2 x_3 - h_1 h_2 x_1 x_2 x_3) \\
N_2 &= \frac{(-243 A^2 - 414 AB + 597 B^2) h_1^3 h_2^3 h_3^2 (h_2 h_3 x_1 + h_1 h_3 x_2 + h_1 h_3 x_3) + (9 A^2 + 6 AB - 2 B^2) (h_1^3 h_2^3 x_1^2 + h_1^2 h_2^2 x_2 + h_1^2 h_2^2 x_2) + x_1 x_2 x_3 + (135 A^2 - 90 AB + 15 B^2) h_1 h_2 h_3 [h_2 x_1 x_2 x_3 (h_2 x_1 x_3) + h_2 x_1 x_3 (h_1 x_3) + h_1^2 x_2 x_3 (h_2 x_3 + h_2 x_3) + (27 A^2 + 18 AB - 3 B^2) (h_1^3 x_2^2 x_3 + h_1^2 x_3 x_3 + h_3^2 x_2^2 x_3 + h_1^2 x_2^2 x_3 + h_1 x_2 x_3 + h_1 h_2 x_2 x_3 x_3) + (-24 A + 8 B) x_1 x_2 x_3 (h_2 h_3 x_2 + h_3 x_1 x_2 x_3 + h_1 h_2 x_1 x_2 x_3) + (27 A^2 + 504 AB - 196 B^2) h_1 h_2 x_1 x_2 x_3 (h_2 x_1 x_3 + h_1 h_2 x_2 x_3 + h_1 h_2 x_3) \\
D_2 &= 18 h_1 h_2 x_3 (\{ -222 A + 186 B \} h_1^3 h_2^3 h_3^2 + (18 A - 6 B) (h_1^2 h_2^2 x_1 x_2 + h_1^2 h_2^2 x_2 + h_1^2 h_2^2 x_2 + h_1^2 h_2^2 x_2 + h_1^2 h_2^2 x_2 + h_1^2 h_2^2 x_2 + h_1^2 h_2^2 x_2 + h_1^2 h_2^2 x_2 + h_1^2 h_2^2 x_2 + h_1^2 x_1 x_2 x_3) + (24 A - B) h_1 h_2 h_3 (h_1 x_1 + h_1 h_2 x_1 x_2 x_3 + h_1 x_2 x_3) (114 A - 38 B) x_1 x_2 x_3) \end{align*} \]
The control surface generated by $\Delta u_\alpha(k)$ is continuous at any point in the 3D input space.

2. The magnitude of incremental control effort increases monotonically as the distance of the input point increases from the origin of 3D input space.

3. The expression for class 1 fuzzy PID controller in Case 15 of Section 3 can be rewritten as

$$\Delta u_\alpha(k) = \frac{\gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3}{D_1}$$

(6)
where

\[ \gamma_1 = (-243A^2 - 270AB + 357B^2)h_1^3h_3^3h_5^3 + (-9A^2 + 6AB - B^2)(h_3^3h_x^3_1 + \frac{1}{4}x_1^2x_2^2x_3^2) - 288(AB - B^2)h_1^3h_2^3h_x^3_2 - (288AB - 96B^2)h_1^3h_3^3h_x^3_1x_2x_3 + (135A^2 + 198B^2)h_1^2h_3^3h_x^3_1x_2 + (135A^2 + 54AB - 33B^2)h_1^3h_2^3h_x^3_1 + (63A^2 - 42AB + 23B^2)h_1^2h_2^3h_x^3_2x_3 + (-27A^2 + 18AB - 3B^2)(h_2^2h_1^2x_1^2x_2^2 + h_3^3h_x^3_1x_2^2 + h_1^2x_1^2x_3^2) + (-27A^2 + 162AB - 51B^2)h_1h_2h_3^3x_2^3x_3 + (-81A^2 + 54AB - 9B^2)h_2h_3x_2^3x_3(h_2h_3 + x_2x_3), \]

\[ \gamma_2 = (-243A^2 - 270AB + 357B^2)h_1^3h_2^3h_x^3_3 + (-9A^2 + 6AB - B^2)(h_3^3h_x^3_2 + \frac{1}{4}x_1^2x_2^2x_3^2) + (135A^2 + 54AB - 33B^2)h_1^2h_2^2h_x^3_3(h_3x_1 + h_1x_3) + (63A^2 - 42AB + 23B^2)h_1^2h_2^2h_x^3_1x_3 + (-27A^2 + 18AB - 3B^2)(h_1^2h_x^3_2x_3 + h_1^2h_x^3_1x_3^2 + h_1^2x_2^2x_3^2) + (-81A^2 + 54AB - 9B^2)h_1x_3x_2^2x_3(h_1h_3 + x_2x_3), \]

\[ \gamma_3 = (-243A^2 - 270AB + 357B^2)h_1^3h_2^3h_x^3_3 + (-9A^2 + 6AB - B^2)(h_3^3h_x^3_2 + \frac{1}{4}x_1^2x_2^2x_3^2) + 288(AB - B^2)h_1^2h_2^2h_x^3_2x_3 + (288AB - 96B^2)h_1^2h_3^3h_x^3_2x_3 + (135A^2 + 198B^2)h_1^2h_2^3h_x^3_2x_3 + (135A^2 + 54AB - 33B^2)h_1^2h_2^3h_x^3_2 + (63A^2 - 42AB + 23B^2)h_1^2h_2^2h_x^3_1x_3 + (-27A^2 + 18AB - 3B^2)(h_1^2h_x^3_2x_3 + h_1^2h_x^3_1x_3^2 + h_1^2x_2^2x_3^2) + (-27A^2 + 162AB - 51B^2)h_1^2h_2x_1^2x_3^2 + (-81A^2 + 54AB - 9B^2)h_1^2x_1^2x_3^2(h_2h_3 + x_2x_3) + D_1 \text{ defined in Case 15 of Section 3.} \]

Similarly, the expression for class 2 fuzzy PID controller in Case 15 of Section 3 can be rewritten as

\[ \Delta u_a(k) = \frac{\gamma_4x_1 + \gamma_5x_2 + \gamma_6x_3}{D_2} \tag{7} \]

where

\[ \gamma_4 = (-243A^2 - 414AB + 597B^2)h_1^3h_3^3h_5^3 + (-9A^2 + 6AB - B^2)(h_3^3h_x^3_1 + \frac{1}{4}x_1^2x_2^2x_3^2) + (135A^2 - 90AB + 15B^2)h_1^2h_2^3h_x^3_1(h_3x_2 + h_2x_3) + (-27A^2 + 18AB - 3B^2)(h_1^2h_x^3_2x_3 + h_1^2h_x^3_1x_3^2 + h_1^2x_2^2x_3^2) + (63A^2 - 42AB - 105B^2)h_1^2h_2^3h_x^3_2x_3 + (-27A^2 + 54AB - 9B^2)h_2h_3x_2^3x_3(h_2h_3 + x_3), \]

\[ \gamma_5 = (-243A^2 - 414AB + 597B^2)h_1^3h_2^3h_x^3_3 + (-9A^2 + 6AB - B^2)(h_3^3h_x^3_2 + \frac{1}{4}x_1^2x_2^2x_3^2) + (135A^2 - 90AB + 15B^2)h_1^2h_2^3h_x^3_2(h_3x_1 + h_1x_3) + (-27A^2 + 18AB - 3B^2)(h_1^2h_x^3_2x_3 + h_1^2h_x^3_1x_3^2 + h_1^2x_2^2x_3^2) + (63A^2 - 42AB - 105B^2)h_1^2h_2^3h_x^3_1x_3 + (-27A^2 + 54AB - 9B^2)h_1h_3x_2^3x_3(h_1h_3 + x_1x_3), \]

\[ \gamma_6 = (-243A^2 - 414AB + 597B^2)h_1^3h_3^3h_5^3 + (-9A^2 + 6AB - B^2)(h_3^3h_x^3_1 + \frac{1}{4}x_1^2x_2^2x_3^2) + (135A^2 - 90AB + 15B^2)h_1^2h_2^3h_x^3_1(h_3x_1 + h_1x_3) + (-27A^2 + 18AB - 3B^2)(h_1^2h_x^3_2x_3 + h_1^2h_x^3_1x_3^2 + h_1^2x_2^2x_3^2) + (63A^2 - 42AB - 105B^2)h_1^2h_2^3h_x^3_1x_3 + (-27A^2 + 54AB - 9B^2)h_1h_2x_1^2x_3^2(h_1h_2 + x_1x_2) + \]

In view of equations (6) and (7) the simplest fuzzy PID controllers are different nonlinear PID controllers. They are variable gain controllers as \( \gamma_1, \gamma_2, ..., \gamma_6 \) are functions of all the three inputs.
Fuzzy PID controllers are variable structure controllers as their structures change in the input cells.

The minimum incremental control effort, given by \(\frac{-(14B^2+6AB-9A^2)}{18B}\), occurs at \((e_s(k), \Delta e_s(k), \Delta^2 e_s(k)) = (-h_e, -h_{\Delta e}, -h_{\Delta^2 e})\).

The maximum incremental control effort is zero at the origin of 3D input space.

The maximum incremental control effort, given by \(\frac{(14B^2+6AB-9A^2)}{18B}\), occurs at \((e_s(k), \Delta e_s(k), \Delta^2 e_s(k)) = (h_e, h_{\Delta e}, h_{\Delta^2 e})\).

5 Computational aspects of the simplest fuzzy PID controller

Conventional PID controllers are still widely used in industries because they provide control output quickly. The computational delay introduced by them in the loop is almost insignificant. In fact, it is the smallest when compared with any other control scheme. It can be seen from the mathematical model of PID controllers [equation (2)] that they require only two mathematical operations (additions). But conventional PID controllers do not work satisfactorily for nonlinear, higher order and time-delay systems. They require only two mathematical operations (additions). The total number of mathematical operations and memory locations required during the implementation of class 1 controller are 449 and 32, respectively. Thus, as far as computational aspects are concerned, class 2 controller is better than class 1 controller.

From the mathematical models of class 1 fuzzy controller in different cells presented in Section 3, it can be observed that the computational and memory burdens on digital
computer are the same when all the three inputs lie in the inner cuboid (Case d in Table 1) and are the highest when compared to that in Cases a, b, and c. This is true for class 2 fuzzy controller also. As we are comparing the computational and memory burdens on digital computer during the implementation of different classes of fuzzy controllers it is good enough to consider the cell (13, 14, 16) where the computational effort is maximum. The number of mathematical operations required to compute \( e_s(k), \Delta e_s(k), \) and \( \Delta^2 e_s(k) \) using \( e(k), e(k-1), e(k-2), S_e, S_{\Delta e} \) and \( S_{\Delta^2 e} \) is the same for both the classes of fuzzy PID controllers and hence not counted here in the relative assessment.

It has been observed that the time taken by the digital computer with 3.2 GHz processor to compute \( \Delta u_s(k) \) from \( e_s(k), \Delta e_s(k), \) and \( \Delta^2 e_s(k) \) for class 1 and class 2 controllers are \( 44e - 4 \) sec and \( 36e - 4 \) sec respectively. The computational time-delay should be less than the sampling period \( T \). We believe that this quantitative analysis helps the control practitioners in implementing fuzzy controllers.

### 6 BIBO stability analysis of feedback systems containing the simplest fuzzy PID controller

Using the Small Gain theorem (Desoer and Vidyasagar, 1975) we establish the sufficient condition for BIBO stability of feedback systems containing the simplest fuzzy PID controller.

**Figure 7** A feedback system

Consider the feedback system shown in Figure 7. According to the small gain theorem, if \( \gamma_1(G_1) \), the gain of \( G_1 \), and \( \gamma_2(G_2) \), the gain of \( G_2 \), have a product smaller than unity, then any bounded-input pair \((u_1, u_2)\) produces a bounded-output pair \((y_1, y_2)\). We consider the general case wherein the process \( G_2 \) under control is nonlinear, denoted by \( N \). If the plant is continuous-time plant, then we discretise it and use in digital computer simulation. Hence, by defining \( r(k) = u_1(k), u(k - 1) = u_2(k), \Delta u(k) = y_1(k) = G_1e_1(k), y(k) = y_2(k) = Ne_2(k), e(k) = e_1(k) \) and \( u(k) = e_2(k) \) in Figure 7, we obtain an equivalent closed-loop system as shown in Figure 1. Let

\[
M_e = \sup_{k \geq 0} |e(k)| = \sup_{k \geq 0} |e_1(k)| ; S_e \cdot M_e = h_e = h_1
\]

\[
M_{\Delta e} = \sup_{k \geq 0} |\Delta e(k)| = \sup_{k \geq 0} |e(k) - e(k - 1)| \leq 2M_e ;
\]

\[
S_{\Delta e} \cdot M_{\Delta e} = h_{\Delta e} = h_2
\]

\[
M_{\Delta^2 e} = \sup_{k \geq 0} |\Delta^2 e(k)| = \sup_{k \geq 0} |\Delta e(k) - \Delta e(k - 1)| \leq 2M_{\Delta e} or 4M_e ;
\]

\[
S_{\Delta^2 e} \cdot M_{\Delta^2 e} = h_{\Delta^2 e} = h_3
\]
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From the mathematical model of class 2 fuzzy controller in the cell (13,14,16) we get

\[ \| \Delta u(k) \| = \| y_1(k) \| = \| G_1 e_1(k) \| = \| Q \| \]

\[ \leq \frac{14B^2 + 6AB - 9A^2}{54BS_\Delta u h_1 h_2 h_3} \left[ (h_2 h_3 S_e + h_1 h_3 S_\Delta e + h_1 h_2 S_\Delta^2 e) |e_1(k)| \right. \\
\left. + (h_1 h_3 S_\Delta e + 3h_1 h_2 S_\Delta^2 e) M_e \right] \]

where \( Q = N_2 \) and \( R = D_2 \) presented in Case 15 of Section 3,

so that

\[ \gamma_1 = \frac{(14B^2 + 6AB - 9A^2)(h_2 h_3 S_e + h_1 h_3 S_\Delta e + h_1 h_2 S_\Delta^2 e)}{54BS_\Delta u h_1 h_2 h_3} \tag{8} \]

Considering the other class of controller as a subsystem instead of class 2 controller, the expression obtained for \( \gamma_1 \) is the same as in equation (8). Next, we have

\[ \| y_2(k) \| = \| G_2 e_2(k) \| = \| Ne_2(k) \| \leq \| N \| |e_2(k)| \]

so that

\[ \gamma_2 = \| N \| \tag{9} \]

The gain \( \gamma_2 \) of a nonlinear system can be computed using Theorem 5.1 in Khalil (2002). At origin, \( \| \Delta u(k) \| = 0 \). This implies that \( \gamma_1 = 0 \). Since \( \gamma_1 = 0 \) and \( \gamma_2 = \| N \| \), to ensure \( \gamma_1 \gamma_2 < 1 \), \( \| N \| < \infty \) follows. So, the sufficient condition for the nonlinear fuzzy PID control system to be BIBO stable can be stated as follows:

**Theorem 1:** A sufficient condition for the closed-loop system shown in Figure 1 with the fuzzy PID controller (class 1, class 2) to be BIBO stable is:

1. the given nonlinear process \( N \) has a bounded norm, i.e., \( \| N \| \leq \infty \)
2. the parameters \( S_e, S_\Delta e, S_\Delta^2 e, h_1, h_2, h_3, A, B \) and \( S_\Delta u \) of fuzzy PID controller should satisfy the inequality

\[ \frac{(14B^2 + 6AB - 9A^2)(h_2 h_3 S_e + h_1 h_3 S_\Delta e + h_1 h_2 S_\Delta^2 e)}{54BS_\Delta u h_1 h_2 h_3} \| N \| < 1 \]

7 Conclusions

In this paper, two new mathematical models for a fuzzy PID controller have been derived using L-type, \( \Gamma \)-type and \( \Pi \)-type fuzzy membership functions, algebraic product triangular norm, maximum/bounded sum triangular co-norm, Mamdani minimum inference method, and CoS defuzzification method. The models obtained are shown to be nonlinear. Sufficient conditions for a closed loop system with fuzzy PID controller in the loop to be BIBO stable are established. We believe that this work helps in replacing conventional PID controllers by fuzzy controllers for better performance.
References


