A robust second order numerical method for a weakly coupled system of singularly perturbed reaction-diffusion problem with discontinuous source term

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Abstract: In this paper, a fitted mesh numerical method on Shishkin mesh is proposed to solve a weakly coupled system of two singularly perturbed reaction-diffusion equations containing equal diffusion parameters with discontinuous source terms. This method uses the standard centred finite difference scheme constructed on piecewise-uniform Shishkin mesh with the average of the source terms on either side of the point of discontinuity and then the problem is solved by an iterative procedure. An error analysis is carried out and the method ensures that the parameter-uniform convergence of almost the second order. Numerical results are provided to confirm the theoretical results and compares well with the existing results.

Keywords: singular perturbation problem; SPP; weakly coupled reaction-diffusion system; fitted mesh method; Shishkin mesh; discontinuous source term; parameter-uniform.

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1 Introduction

Singular perturbation problems (SPPs) arise in many fields of applied mathematics and engineering and biological sciences such as fluid dynamics, electrical and electronic circuits and systems, electrical power systems, aerospace systems, nuclear reactors, biology and ecology (Naidu, 1988). Linearised Navier-stokes equations at high Reynolds number, heat transport problems with large péclet numbers, magneto-hydrodynamics duct problems at high Hartman number, drift diffusion equation of semiconductor device modelling, Wentzel, Kramers and Brillouin (WKB) problems are some of the applications of SPPs. There is a vast literature describing various numerical methods to solve the SPPs but they mostly dealt with single singularly perturbed differential equations. A very few works have been reported in the case of coupled system of singularly perturbed second order differential equations for both convection-diffusion and reaction-diffusion problems particularly on non-smooth data (Basha and Shanthi, 2015a, 2016, 2017; Paramasivam et al., 2014; Rao and Chawla, 2013; Tamilselvan et al., 2007). The coupled system of singularly perturbed reaction-diffusion problems have applications in the modelling of different phenomena such as the turbulent interaction of waves and currents (Thomas, 1998; Madden and Stynes, 2003), predator prey population dynamics (Kan-On and Mimura, 1998), electro analytic chemistry and mass transfer processes in multi-component systems (Shishkin, 1995). A linearisation of the Navier-Stokes equations written in rotation form will yield a singularly perturbed reaction-diffusion system for large Reynolds number (Linss and Stynes, 2009).

A standard finite difference method is proved uniformly convergent on a fitted piecewise uniform Shishkin mesh for a single equation reaction-diffusion problem (Miller et al., 2012). The same approach for coupled system of two singularly perturbed reaction-diffusion problems, with diffusion coefficients $\varepsilon_1, \varepsilon_2$, was originally proposed by Shishkin (1995) and identified three different cases:

1 $0 < \varepsilon_1 = \varepsilon_2 \ll 1$

2 $0 < \varepsilon_1 \ll \varepsilon_2 = 1$

3 $0 < \varepsilon_1 \leq \varepsilon_2 \ll 1.$

Shishkin presented a fitted mesh for each case and showed that the orders of convergence are almost first order, $i.e., O(N^{-1} \ln N)$ for case 1, $O(N^{-2/5})$ for case 2 and $O(N^{-1/4})$ for case 3. Matthews et al. (2000) proved that the method is almost first order for case 1. This has been improved to almost second order by Linss and Madden (2002). Matthews et al. (2002) showed that the method for case 2 is almost second order convergence. For case 3, Linss and Madden (2004) established that the method is almost second order convergence. In Matthews et al. (2000, 2002) and Matthews (2000) proposed an iterative scheme along with the fitted mesh method for smooth case. Das and Natesan (2014) derived optimal error estimates using mesh equidistribution technique for singularly perturbed system of reaction-diffusion boundary value problems for smooth case which lead to the second order parameter uniform convergence. Tamilselvan et al. (2007) developed a numerical method using fitted piecewise uniform Shishkin mesh for the coupled system of singularly perturbed reaction-diffusion equations for case 1 with discontinuous source term and obtained almost first order uniform convergence of $O(N^{-1} \ln N)$. A singularly perturbed linear second order ordinary differential equations of reaction-diffusion type with discontinuous source term for differing parameters was studied by Paramasivam et al. (2014) using fitted mesh method and obtained the first order parameter-uniform convergence of $O(N^{-1}\ln N)$. Ramesh Babu and Ramanujam (2009) proposed an almost second order finite element method on Shishkin and Bakhvalov-Shishkin meshes for a weakly coupled system of two singularly perturbed reaction-diffusion equations containing equal diffusion parameters with discontinuous source term. Basha and Shanthi (2015b) developed an almost second order parameter-uniform numerical methods by considering two hybrid difference schemes using fitted mesh method on Shishkin mesh for the same problem. In that paper, the authors used the central difference scheme in the inner and outer regions for difference scheme-I (DS-I) whereas a combination of central difference scheme and cubic spline difference scheme is employed in outer and inner regions respectively for difference scheme-II (DS-II). At the point of discontinuity, second order one sided difference approximations are used for both the methods.

Motivated by the above works, we have developed a robust and easy to implement second order parameter-uniform numerical method using centered finite difference scheme on fitted piecewise uniform Shishkin mesh by considering the average of the source terms on either side of the point of discontinuity with an iterative procedure for the weakly coupled system of singularly perturbed reaction-diffusion equations with discontinuous source term.

Let $\Omega = (0,1), \Omega^- = (0,d), \Omega^+ = (d,1), d \in \Omega$ and $y_1, y_2 \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^1(\Omega) \cap \mathcal{C}^2(\Omega^- \cup \Omega^+), \overline{y} = (y_1, y_2)^T$. Also, let the jump at $d \in \Omega$ in any function is given by $[\omega](d) = \omega(d^+) - \omega(d^-)$. Consider the following weakly coupled system of singularly perturbed reaction-diffusion equations with discontinuous source term:

$$P_1\overline{y}(x) \equiv -\varepsilon y_1''(x) + a_{11}(x)y_1(x) + a_{12}(x)y_2(x) = f_1(x), \forall\, x \in \Omega^- \cup \Omega^+ \quad (1)$$

$$P_2\overline{y}(x) \equiv -\varepsilon y_2''(x) + a_{21}(x)y_1(x) + a_{22}(x)y_2(x) = f_2(x), \forall\, x \in \Omega^- \cup \Omega^+ \quad (2)$$

$$y_1(0) = y_{1,0},\ y_1(1) = y_{1,1},\ y_2(0) = y_{2,0},\ y_2(1) = y_{2,1}, \quad (3)$$

$$|[f_1](d)| \leq C,\ \ |[f_2](d)| \leq C. \quad (4)$$

where ε is a small parameter such that $0 < \varepsilon \ll 1$. The above system of equations can also be expressed in matrix-vector form as follows:

$$\mathbf{P}_\varepsilon \overline{y} \equiv \begin{pmatrix} P_1\overline{y} \\ P_2\overline{y} \end{pmatrix} \equiv \begin{pmatrix} -\varepsilon \dfrac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon \dfrac{d^2}{dx^2} \end{pmatrix} \overline{y} + A(x)\overline{y} = \overline{f}(x), \quad x \in \Omega^- \cup \Omega^+ \quad (5)$$

with the boundary conditions

$$\overline{y}(0) = \overline{y}_0, \quad \overline{y}(1) = \overline{y}_1, \quad (6)$$

where $\overline{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $\quad A(x) = \begin{pmatrix} a_{11}(x)\; a_{12}(x) \\ a_{21}(x)\; a_{22}(x) \end{pmatrix}$, $\quad \overline{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$ and $\overline{y}_{(0)} = \begin{pmatrix} y_{1,0} \\ y_{2,0} \end{pmatrix}$, $\quad \overline{y}_{(1)} = \begin{pmatrix} y_{1,1} \\ y_{2,1} \end{pmatrix}$.

Assume that

$$a_{11}(x) > |a_{12}(x)|, a_{22}(x) > |a_{21}(x)|,\ a_{12}(x) \leq 0,\ a_{21}(x) \leq 0, \forall\ x \in \Omega. \tag{7}$$

and $\alpha > 0$ is a constant such that

$$\alpha = \min_{\overline{\Omega}}\{\alpha_1, \alpha_2\}, \alpha_1 = \min_{\overline{\Omega}}\{a_{11}(x) + a_{12}(x)\}, \alpha_2 = \min_{\overline{\Omega}}\{a_{21}(x) + a_{22}(x)\}. \tag{8}$$

It is also assumed that the source terms f_1, f_2 are sufficiently smooth on $\overline{\Omega} \setminus \{d\}$; a single discontinuity in $f_i(x), i = 1, 2$ occur at the point $d \in \Omega$. This discontinuity gives rise to interior layers in the solution of the problem in addition to the boundary layers at the both end points. Since $f_i, i = 1, 2$ are discontinuous at d, the solution $\overline{y}$ of (1)–(4) does not necessary to have a continuous second order derivative at the point d. But the first derivative of the solution exists and is continuous.

The rest of the paper is organised as follows. Section 2 presents some analytical results of the solution of the problem (1)–(4). Discretisation of the continuous problem is described in Section 3. Section 4 provides error analysis and iterative procedure is given in Section 5. Two numerical examples are provided in Section 6 to validate the theoretical results. The paper ends with conclusions in Section 7.

Throughout this paper, C denotes a generic positive constant that is independent of the singular perturbation parameter ε and the discretisation parameter N of the discrete problem. The norm which is suitable for studying the convergence of numerical solution to the exact solution of a singular perturbation problem is the maximum norm defined by $\| y \| = \sup_{0 \leq x \leq 1} |y(x)|$ for any scalar-valued function y and $\| \overline{y} \| = \max_{1 \leq k \leq 2} |y_k|$ for any vector-valued function $\overline{y}$.

2 Theoretical results

In this section, the maximum principle, stability result and bounds on the solution and its derivatives are established for the boundary value problem (BVP)(1)-(4).

Lemma 1: The problem (1)–(4) has a solution $\overline{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$ with $y_1, y_2 \in \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega^- \cup \Omega^+)$.

Proof: The proof of this Lemma can be found in Tamilselvan et al. (2007).

Lemma 2 (Maximum principle): Suppose $y_1, y_2 \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^2(\Omega^- \cup \Omega^+)$ and $A(x)$ satisfy (7) and (8). Further suppose that $\overline{y} = (y_1, y_2)^T$ satisfies $\overline{y}(0) \geq \overline{0}, \overline{y}(1) \geq \overline{0}, P_1\overline{y}(x) \geq \overline{0}, P_2\overline{y}(x) \geq \overline{0}$ in $\Omega^- \cup \Omega^+$ and $[\overline{y}](d) = \overline{0}$, $[\overline{y}'](d) \leq \overline{0}$. Then $\overline{y}(x) \geq \overline{0}, \forall\ x \in \overline{\Omega}$.

Proof: Let $y_1(m) = \min_{x \in \overline{\Omega}}\{y_1(x)\}$ and $y_2(n) = \min_{x \in \overline{\Omega}}\{y_2(x)\}$. Without loss of generality, assume that $y_1(m) \leq y_2(n)$. If $y_1(m) \geq 0$, then there is nothing to prove. So, let $y_1(m) < 0$, then it will be shown that this leads to a contradiction. Note that $m \neq \{0, 1\}$, and $y_1'(m) = 0, y_1''(m) \geq 0$.
Therefore, either $m \in \Omega^- \cup \Omega^+$ or $m = d$.

1 Case 1: let $m \in \Omega^- \cup \Omega^+$. Then, we have

$$\begin{aligned} P_1\overline{y}(m) &= -\varepsilon y_1''(m) + a_{11}(m)y_1(m) + a_{12}(m)y_2(m) \\ &= -\varepsilon y_1''(m) + (a_{11}(m) + a_{12}(m))y_1(m) \\ &\quad +(y_2(m) - y_1(m))a_{12}(m) \\ &< 0, \end{aligned}$$

which contradicts the hypothesis of the Lemma.

2 Case 2: let $m = d$.
In this case, the argument depends on whether or not y_1 is differentiable at d. If $y_1'(d)$ does not exist, then $[y_1'](d) \neq 0$. Since $y_1'(d-) \leq 0, \quad y_1'(d+) \geq 0$, it is clear that $[y_1'](d) > 0$, which is a contradiction. On the other hand, if y_1 is differentiable at d *i.e.*, $y_1'(d)$ exists, then $y_1'(d) = 0$ and $y_1' \in \mathcal{C}^1(\Omega)$. Since $y_1(d) < 0$, there exists a neighbourhood $N_h = (d-h, d)$ such that $y_1(x) < 0$ and $y_1(x) < y_2(x), \forall\, x \in N_h$.

Let $x_1 \neq d, x_1 \in N_h$ be a point such that $y_1(x_1) > y_1(d)$. It follows from the mean value theorem that, for some $x_2 \in N_h, y_1'(x_2) = \frac{y_1(d)-y_1(x_1)}{d-x_1} < 0$, and for some $x_3 \in N_h, y_1''(x_3) = \frac{y_1'(d)-y_1'(x_2)}{d-x_2} = \frac{-y_1'(x_2)}{d-x_2} > 0$.

Also note that $y_1(x_3) < 0, y_1'(x_3) = 0$, since $x_3 \in N_h$. Thus,

$$\begin{aligned} P_1\overline{y}(x_3) &= -\varepsilon y_1''(x_3) + a_{11}(x_3)y_1(x_3) + a_{12}(x_3)y_2(x_3) \\ &= -\varepsilon y_1''(x_3) + (a_{11}(x_3) + a_{12}(x_3))y_1(x_3) \\ &\quad +(y_2(x_3) - y_1(x_3))a_{12}(x_3) \\ &< 0, \end{aligned}$$

which is a contradiction. Similarly, $P_2\overline{y}(x)$ can be dealt. Hence, $\overline{y}(x) \geq \overline{0}, \forall\, x \in \overline{\Omega}$.

Lemma 3 (Stability result): Let the coupling matrix $A(x)$ satisfy (7) and (8) and if $\overline{y} = (y_1, y_2)^T$ be the solution of (1)-(2), then for $j = 1, 2$ and $x \in \overline{\Omega}$,
$|y_j(x)| \leq \max\{\|\overline{y}(0)\|, \|\overline{y}(1)\|, \frac{1}{\alpha}\|\overline{f}\|_{\Omega^-\cup\Omega^+}\}$.

Proof: Define two barrier functions $\overline{\omega}^{\pm}(x) = (\omega_1^{\pm}(x), \omega_2^{\pm}(x))^T$ as

$$\overline{\omega}^{\pm}(x) = \left(\max\{\|\overline{y}(0)\|, \|\overline{y}(1)\|, \frac{1}{\alpha}\|\overline{f}\|_{\Omega^-\cup\Omega^+}\}\right)\overline{e} \pm \overline{y}(x),$$

where $\overline{e} = (1,1)^T$ is the unit column vector. Clearly, $\overline{\omega}^{\pm}(0) \geq \overline{0}, \quad \overline{\omega}^{\pm}(1) \geq \overline{0}$ and $\mathbf{P}_\varepsilon\overline{\omega}^{\pm}(x) \geq \overline{0}, \forall\, x \in \Omega^- \cup \Omega^+$. Further, $[(\overline{\omega}^{\pm})](d) = \pm[\overline{y}](d) = \overline{0}$ and $[(\overline{\omega}^{\pm})'](d) = \pm[\overline{y}'](d) = \overline{0}$. Therefore, by maximum principle, it follows that $\overline{\omega}^{\pm}(x) \geq \overline{0}$ on $\overline{\Omega}$.

2.1 Derivative estimates

Lemma 4: Let $\overline{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$ be the solution of (1)-(4). Then for $k = 1, 2$ and $\forall x \in \overline{\Omega} \setminus \{d\}$

$$| y_j^{(k)} | \leq C(1 + \varepsilon^{-\frac{k}{2}}),\ j = 1, 2,$$

and

$$| y_j^{(3)} | \leq C\varepsilon^{-\frac{3}{2}},\ j = 1, 2.$$

Proof: Following the procedure adopted in Tamilselvan et al. (2007), this Lemma can be proved. To derive the sharper bounds on the derivatives of the solution, $\overline{y}$ is decomposed into smooth $\overline{v}$ and singular $\overline{w}$ components as $\overline{y} = \overline{v} + \overline{w}$.

The components $\overline{v}$ and $\overline{w}$ are defined as the solutions of

$$\mathbf{P}_\varepsilon \overline{v}(x) = \overline{f}(x), x \in \Omega^- \cup \Omega^+,$$

$$\overline{v}(x) = A^{-1}(x)\overline{f}(x), x \in \{0, d^-, d^+, 1\}.$$

$$\mathbf{P}_\varepsilon \overline{w}(x) = 0, x \in \Omega^- \cup \Omega^+,$$

$$\overline{w}(x) = \overline{u}(x) - \overline{v}(x), x \in \{0, 1\},$$

$$[\overline{w}](d) = -[\overline{v}](d), [\overline{w}'](d) = -[\overline{v}'](d).$$

Lemma 5: The smooth and singular components $\overline{v}$ and $\overline{w}$ of $\overline{y}$ satisfy the bounds

$$| v_j^{(k)}(x) | \leq \begin{cases} C(1 + \varepsilon^{(1-k/2)}\mathbb{B}l(x, \alpha)),\ x \in \Omega^- \\ C(1 + \varepsilon^{(1-k/2)}\mathbb{B}r(x, \alpha)),\ x \in \Omega^+,\ j = 1, 2, \end{cases} \quad \text{and}$$

$$| w_j^{(k)}(x) | \leq \begin{cases} C\varepsilon^{(-k/2)}\mathbb{B}l(x, \alpha),\ x \in \Omega^- \\ C\varepsilon^{(-k/2)}\mathbb{B}r(x, \alpha),\ x \in \Omega^+,\ j = 1, 2, \end{cases}$$

for each $k, 0 \leq k \leq 3$, where the boundary layer functions are defined by

$$\mathbb{B}l(x, \alpha) = e^{-x\sqrt{\alpha/\varepsilon}} + e^{-(d-x)\sqrt{\alpha/\varepsilon}},$$
$$\mathbb{B}r(x, \alpha) = e^{-(x-d)\sqrt{\alpha/\varepsilon}} + e^{-(1-x)\sqrt{\alpha/\varepsilon}}.$$

and α is defined in (8).

Proof: The bounds on the smooth and singular components and their derivatives can be obtained by adopting the technique used in Tamilselvan et al. (2007).

3 Discrete problem

In this section, we describe the fitted mesh method for the problem (1)–(4). On $\Omega^- \cup \Omega^+$ a piecewise uniform Shishkin mesh of N mesh intervals is constructed as follows. The interval Ω^- is subdivided into three subintervals as $[0, \tau_1] \cup [\tau_1, d - \tau_1] \cup [d - \tau_1, d]$,for some τ_1 that satisfies $0 < \tau_1 \leq d/4$. On $[0, \tau_1]$ and $[d - \tau_1, d]$ a uniform mesh with $\dfrac{N}{8}$ mesh-intervals is placed while $[\tau_1, d - \tau_1]$ has a uniform mesh with $\dfrac{N}{4}$ mesh intervals. The sub-intervals $[d, d + \tau_2], [d + \tau_2, 1 - \tau_2], [1 - \tau_2, 1]$ of $\overline{\Omega}^+$ are treated analogously for some τ_2 satisfying $0 < \tau_2 \leq (1 - d)/4$. The interior points of the mesh are denoted by $\Omega_\varepsilon^N = \{x_i : 1 \leq i \leq \dfrac{N}{2} - 1\} \cup \{x_i : \dfrac{N}{2} + 1 \leq i \leq N - 1\}$. Clearly $x_{N/2} = d$ and $\overline{\Omega}_\varepsilon^N = \{x_i\}_0^N$. Note that this mesh is a uniform mesh when $\tau_1 = \dfrac{d}{4}$ and $\tau_2 = \dfrac{1-d}{4}$. Let τ_1 and τ_2 be the functions of N and ε can be chosen as $\tau_1 = \min\left\{\dfrac{d}{4}, 2\sqrt{\varepsilon/\alpha}\ln N\right\}$ and $\tau_2 = \min\left\{\dfrac{1-d}{4}, 2\sqrt{\varepsilon/\alpha}\ln N\right\}$. The six mesh widths are given by $h_1 = 8\tau_1/N, \quad h_2 = 4(d - 2\tau_1)/N, \quad h_3 = 8\tau_1/N, h_4 = 8\tau_2/N, \quad h_5 = 4(1 - d - 2\tau_2)/N, \quad h_6 = 8\tau_2/N$. On the piecewise-uniform mesh $\overline{\Omega}_\varepsilon^N$ a standard centred finite difference operator is used.

The fitted mesh method for (1)-(3) $\forall\, x_i \in \Omega_\varepsilon^N$ is

$$P_1^N \overline{Y}(x_i) \equiv -\varepsilon\delta^2 Y_1(x_i) + a_{11}(x_i)Y_1(x_i) + a_{12}(x_i)Y_2(x_i) = f_1(x_i), \tag{9}$$
$$P_2^N \overline{Y}(x_i) \equiv -\varepsilon\delta^2 Y_2(x_i) + a_{21}(x_i)Y_1(x_i) + a_{22}(x_i)Y_2(x_i) = f_2(x_i). \tag{10}$$
$$Y_1(0) = Y_{1,0},\ Y_1(N) = Y_{1,N},\ Y_2(0) = Y_{2,0},\ Y_2(N) = Y_{2,N}, \tag{11}$$

and at the point of discontinuity $x_i = x_{\frac{N}{2}} = d$,

$$P_1^N \overline{Y}(d) \equiv -\varepsilon\delta^2 Y_1(d) + a_{11}(d)Y_1(d) + a_{12}(d)Y_2(d) = \tilde{f}_1(d), \tag{12}$$
$$P_2^N \overline{Y}(d) \equiv -\varepsilon\delta^2 Y_2(d) + a_{21}(d)Y_1(d) + a_{22}(d)Y_2(d) = \tilde{f}_2(d), \tag{13}$$

where

$$\delta^2 \overline{Y}(x_i) = \tfrac{(D^+ - D^-)\overline{Y}(x_i)}{\overline{h_i}}, \quad D^+ \overline{Y}(x_i) = \tfrac{\overline{Y}(x_{i+1}) - \overline{Y}(x_i)}{h_{i+1}},$$
$$D^- \overline{Y}(x_i) = \tfrac{\overline{Y}(x_i) - \overline{Y}(x_{i-1})}{h_i}, \quad \tilde{f}_j(d) = \frac{f_j\left(x_{\frac{N}{2}-1}\right) + f_j\left(x_{\frac{N}{2}+1}\right)}{2},\ j = 1, 2,$$
$$h_i = x_i - x_{i-1}, \quad h_{i+1} = x_{i+1} - x_i, \quad \overline{h_i} = \tfrac{h_i + h_{i+1}}{2}.$$

Analogous to the continuous problem, the following discrete maximum principle and discrete stability result can be proved for the discrete problem.

Lemma 6 (Discrete maximum principle): For any mesh function $\overline{\Psi}(x_i)$, assume that $\overline{\Psi}_0 \geq \overline{0}, \overline{\Psi}_N \geq \overline{0}, \mathbf{P}_\varepsilon^N \overline{\Psi} \geq \overline{0}\ \forall\, x_i \in \Omega^N$ and $D^+ \overline{\Psi}_{\frac{N}{2}} - D^- \overline{\Psi}_{\frac{N}{2}} \leq \overline{0}$. Then $\overline{\Psi}(x_i) \geq$

$\overline{0}\ \forall\ x_i \in \overline{\Omega}^N$. *Lemma 7 (Discrete stability):* If $\overline{Z}(x_i) = (Z_1(x_i), Z_2(x_i))^T$ is any mesh function such that $\overline{Z}_0 \geq \overline{0}, \overline{Z}_N \geq \overline{0}$, then $\forall\ x_i \in \overline{\Omega}^N$,

$$|Z_j(x_i)| \leq \max\left(\|\overline{Z}(x_0)\|, \|\overline{Z}(x_N)\|, \frac{1}{\alpha}\|\mathbf{P}_\varepsilon^N \overline{Z}(x_i)\| \right), \quad j = 1, 2.$$

4 Error analysis

In this section, we derive an error estimate for the numerical solution obtained by the scheme (9)–(13) for the problem (1)–(4).

By classical estimates and bounds on the derivatives of regular and singular components for all $i \neq \frac{N}{2}, k = 1, 2$, we have

$$\left|\varepsilon\left(\frac{d^2}{dx^2} - \delta^2\right) v_k(x_i)\right| \leq \begin{cases} C\varepsilon(x_{i+1} - x_{i-1})|v_k|_3, \\ C\varepsilon h^2 |v_k|_4 \leq CN^{-2},\ x_{i+1} - x_i = x_i - x_{i-1} = h \end{cases} \tag{14}$$

and $\left|\varepsilon\left(\frac{d^2}{dx^2} - \delta^2\right) w_k(x_i)\right| \leq C\varepsilon(x_{i+1} - x_{i-1})|w_k|_3,$
$C\varepsilon h^2 |w_k|_4 \leq CN^{-2},$
$x_{i+1} - x_i = x_i - x_{i-1} = h,$
$C\varepsilon \max\limits_{x\in[x_{i-1}, x_{i+1}]} |w_k''(x_i)|$. Using the inequality (4) in the outer-layer regions $[\tau_1, d - \tau_1] \cup [d + \tau_2, 1 - \tau_2]$, we have

$$\left|\varepsilon\left(\frac{d^2}{dx^2} - \delta^2\right) w_k(x_i)\right| \leq CN^{-2},\ k = 1, 2.$$

Using the inequality (4) within the layers $(0, \tau_1) \cup (d - \tau_1, d)$, we have

$$\left|\varepsilon\left(\frac{d^2}{dx^2} - \delta^2\right) w_k(x_i)\right| \leq C\tau_1^2\varepsilon^{-1}N^{-2} \leq CN^{-2}(\ln N)^2,\ k = 1, 2$$

and for $(d, d + \tau_2) \cup (1 - \tau_2, 1)$, we have

$$\left|\varepsilon\left(\frac{d^2}{dx^2} - \delta^2\right) w_k(x_i)\right| \leq C\tau_2^2\varepsilon^{-1}N^{-2} \leq CN^{-2}(\ln N)^2,\ k = 1, 2.$$

Thus, we get

$$\left|\varepsilon\left(\frac{d^2}{dx^2} - \delta^2\right) w_k(x_i)\right| \leq CN^{-2}(\ln N)^2,\ k = 1, 2,\ x_i \neq d.$$

Using the decomposition $\overline{y} = \overline{v} + \overline{w}$ and the bounds on the derivatives of these components for $x_i \neq d$, we get

$$|\mathbf{P}_\varepsilon^N(\overline{Y} - \overline{y})(x_i)| \leq CN^{-2}(\ln N)^2,\ x_i \neq d. \tag{15}$$

Lemma 8: At the point of discontinuity $x_i = d$, the error satisfies the following estimate:

$$|\mathbf{P}_\varepsilon^N(\overline{Y} - \overline{y})(d)| \leq CN^{-1}\ln N.$$

Proof: Similar to the procedure adopted in Rao and Chawla (2013) and De Falco and O'Riordan (2010), the proof is given as follows:

At the point $x_i = x_{\frac{N}{2}} = d$, we have $h_{\frac{N}{2}} = h_{(\frac{N}{2})+1} = h$. Therefore,

$$P_{1,\varepsilon}^{N}(\overline{Y} - \overline{y})(d) = P_{1,\varepsilon}^{N}\overline{Y}(d) - \overline{y}(d) = \tilde{f}_1(d) - P_{1,\varepsilon}^{N}\overline{y}(d)$$

$$= \tilde{f}_1(d) + \frac{\varepsilon}{h^2}\int_{t=d}^{d+h}\int_{s=d}^{t} y_1''(s)ds\,dt - \frac{\varepsilon}{h^2}\int_{t=d-h}^{d}\int_{s=d}^{t} y_1''(s)ds\,dt$$

$$- a_{11}(d)y_1(d) - a_{12}(d)y_2(d)$$

$$= \frac{1}{h^2}\int_{t=d}^{d+h}\int_{s=d}^{t}\int_{r=s}^{d+h} + \frac{1}{h^2}\int_{t=d-h}^{d}\int_{s=d}^{t}\int_{r=d-h}^{s} (f_1 - a_{11}y_1 - a_{12}y_2)'(r)dr\,ds\,dt$$

$$- a_{11}(d)y_1(d) - a_{12}(d)y_2(d) + \frac{1}{2}(a_{11}(d-h)y_1(d-h) - a_{12}(d-h)y_2(d-h))$$

$$+ \frac{1}{2}(a_{11}(d+h)y_1(d+h) - a_{12}(d+h)y_2(d+h))$$

$$= \frac{1}{h^2}\int_{t=d}^{d+h}\int_{s=d}^{t}\int_{r=s}^{d+h} + \frac{1}{h^2}\int_{t=d-h}^{d}\int_{s=d}^{t}\int_{r=d-h}^{s} (f_1 - a_{11}y_1 - a_{12}y_2)'(r)dr\,ds\,dt$$

$$+ \frac{1}{2}\int_{t=d}^{d-h}(a_{11}(t)y_1(t) - a_{12}(t)y_2(t))'dt + \frac{1}{2}\int_{t=d}^{d+h}(a_{11}(t)y_1(t) - a_{12}(t)y_2(t))'dt,$$

which implies that

$$\left|P_{1,\varepsilon}^{N}(\overline{Y} - \overline{y})(d)\right| \le CN^{-1}\ln N. \tag{16}$$

Similarly, we obtain

$$\left|P_{2,\varepsilon}^{N}(\overline{Y} - \overline{y})(d)\right| \le CN^{-1}\ln N. \tag{17}$$

Combining the results (16) and (17), we get the desired result.

Theorem 9: Let $\overline{y}$ and $\overline{Y}$ be the exact and the numerical solutions of the problem (1)–(4) respectively. Then, for sufficiently large N, we have

$$\|(\overline{Y} - \overline{y})(x_i)\| \le CN^{-2}(\ln N)^2,\ \forall\, x_i \in \overline{\Omega}_{\varepsilon}^{N}.$$

Proof: By following the similar procedure adopted in De Falco and O'Riordan (2010), the proof is given as follows:

Consider $\tau_1 < \frac{1}{4}$ and $\tau_2 < \frac{1}{4}$.
Define the mesh functions η_1 and η_2 as

$$\eta_1(x_j) = \prod_{i=1}^{j}\left(1 + \frac{\sqrt{\alpha}h_i}{\sqrt{2\varepsilon}}\right)$$

and

$$\eta_2(x_j) = \prod_{i=j}^{N} \left(1 + \frac{\sqrt{\alpha} h_i}{\sqrt{2\varepsilon}}\right)^{-1}.$$

It is easy to see that

$$D^-\eta_1(x_i) = \left(\frac{\frac{\sqrt{\alpha}}{\sqrt{2\varepsilon}}}{1 + \frac{\sqrt{\alpha} h_i}{\sqrt{2\varepsilon}}}\right)\eta_1(x_i), \quad D^+\eta_1(x_i) = \left(\frac{\sqrt{\alpha}}{\sqrt{2\varepsilon}}\right)\eta_1(x_i),$$

$$D^+\eta_2(x_i) = -\left(\frac{\frac{\sqrt{\alpha}}{\sqrt{2\varepsilon}}}{1 + \frac{\sqrt{\alpha} h_i}{\sqrt{2\varepsilon}}}\right)\eta_2(x_i), \quad D^-\eta_2(x_i) = -\left(\frac{\sqrt{\alpha}}{\sqrt{2\varepsilon}}\right)\eta_2(x_i).$$

Define the three barrier functions ψ_1, ψ_2 and ψ_3 as follows:

$$\psi_1(x_i) = \begin{cases} \frac{x_i}{\tau_1}, & 0 \le x_i \le \tau_1, \\ 1, & \tau_1 \le x_i \le 1 - \tau_2, \\ \frac{1 - x_i}{1 - \tau_2}, & 1 - \tau_2 \le x_i \le 1, \end{cases}$$

$$\psi_2(x_i) = \begin{cases} \frac{\eta_1(x_i)}{\eta_1(d - \tau_1)}, & 0 \le x_i \le d - \tau_1, \\ 1, & d - \tau_1 \le x_i \le d + \tau_2, \\ \frac{\eta_2(x_i)}{\eta_2(d + \tau_2)}, & d + \tau_2 \le x_i \le 1, \end{cases}$$

$$\psi_3(x_i) = \begin{cases} \frac{\eta_1(x_i)}{\eta_1(d)}, & 0 \le x_i \le d, \\ \frac{\eta_2(x_i)}{\eta_2(d)}, & d \le x_i \le 1. \end{cases}$$

Note that $\eta_1(x_i) = x_i$, if $\tau_1 = \frac{1}{4}$ and $\eta_2(x_i) = 1 - x_i$, if $\tau_2 = \frac{1}{4}$ can be used.
From Lemma 4, we have

$$\left|\mathbf{P}_{\varepsilon}^{N}(\overline{Y} - \overline{y})(d)\right| \le CN^{-1}\ln N, \text{ for } x_i = d. \tag{17}$$

Now, choose the mesh functions $\overline{\zeta}^{\pm}$ given by

$$\zeta_k{}^{\pm}(x_i) = CN^{-2}(\ln N)^2 \left[1 + \sum_{j=1}^{3}\psi_j(x_i)\right] \pm (e_k(x_i)),\ k = 1, 2,$$

where $e_k(x_i) = Y_k(x_i) - y_k(x_i),\ k = 1, 2.$

It is not hard to see that $\overline{\zeta}^{\pm}(0) \ge 0, \overline{\zeta}^{\pm}(1) \ge 0$ and $\mathbf{P}^N\overline{\zeta}^{\pm}(x_i) \ge 0$. Therefore, applying the discrete maximum principle to $\overline{\zeta}^{\pm}(x_i)$, we conclude that

$$\|(\overline{Y} - \overline{y})(x_i)\| \le CN^{-2}(\ln N)^2,\ \forall\, x_i \in \overline{\Omega}_{\varepsilon}^{N}.$$

Hence the proof.

5 Iterative procedure

Equations (9)–(10) lead to the system of linear equations of order $2(N-1)$ x $2(N-1)$ of the form

$$\begin{pmatrix} T_1 & D_1 \\ D_2 & T_2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}. \tag{18}$$

i.e.,

$$T_1 Y_1 = F_1 - D_1 Y_2, \tag{19}$$

$$T_2 Y_2 = F_2 - D_2 Y_1, \tag{20}$$

where T_1, T_2 are the tridiagonal matrices, F_1, F_2 are column vectors and D_1, D_2 are the diagonal matrices.

To solve the test example, the following iterative procedure is used:

$$T_1 Y_1^{(k)} = F_1 - D_1 Y_2^{(k)}, \tag{21}$$

$$T_2 Y_2^{(k+1)} = F_2 - D_2 Y_1^{(k)}. \tag{22}$$

In the computations, Y_2 was initialised to $(0.1, 0.1, ..., 0.1)^T$ and the stopping criteria were set to be $\|Y_2^{(k+1)} - Y_2^{(k)}\| < 10^{-12}$ and $\|Y_1^{(k+1)} - Y_1^{(k)}\| < 10^{-12}$.

The iterative scheme (21)–(22) converges to the solution of the matrix system (18) (Matthews et al., 2000).

Table 1 Maximum point-wise errors $E_{1,\varepsilon}^N$, ε-uniform errors E_1^N and the uniform rates of convergence p_1^N, p_1^{*N} for the solution y_1 of the Example 1

ε	*Number of mesh points N*					
	64	*128*	*256*	*512*	*1,024*	*2,048*
10^{-1}	6.4847E-05	1.6214E-05	4.0507E-06	1.0098E-06	2.4954E-07	5.9472E-08
10^{-2}	4.2893E-04	1.0740E-04	2.6842E-05	6.6916E-06	1.6533E-06	3.9364E-07
10^{-3}	4.0754E-03	1.0463E-03	2.6200E-04	6.5374E-05	1.6157E-05	3.8470E-06
10^{-4}	1.7181E-02	6.1908E-03	2.0347E-03	6.4394E-04	1.6150E-04	3.8466E-05
10^{-5}	1.7178E-02	6.1904E-03	2.0289E-03	6.3239E-04	1.9508E-04	5.4228E-05
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
10^{-15}	1.7178E-02	6.1904E-03	2.0289E-03	6.3239E-04	1.9508E-04	5.4228E-05
E_1^N	1.7181E-02	6.1908E-03	2.0347E-03	6.4394E-04	1.9508E-04	5.4228E-05
p_1^N	1.47	1.61	1.66	1.72	1.85	-
p_1^{*N}	1.89	1.99	2.00	2.03	2.14	-

6 Numerical examples

In this section, two examples are provided to illustrate the computational methods discussed in this article.

For a finite set of values $\varepsilon = \{10^{-1}, 10^{-2}, ..., 10^{-15}\}$, the maximum point-wise errors $E^N_{j,\varepsilon}$ are computed as $E^N_{j,\varepsilon} = \max_{x_i \in \overline{\Omega}^N_\varepsilon} \left| Y^N_j - \tilde{Y}^{8192}_j \right|$ for $i = 0 \leq i \leq N,\ j = 1, 2$, where $\tilde{Y}^{8192}_j$ is the piecewise linear interpolant of the mesh function Y^{8192}_j onto [0, 1]. From these values, the ε-uniform maximum point-wise difference is calculated by $E^N_j = \max_{\varepsilon} E^N_{j,\varepsilon}, j = 1, 2$. The order of convergence is computed by $p^N_j = \log_2 \left(\frac{E^N_j}{E^{2N}_j} \right), j = 1, 2$. Also, the order of convergence is calculated using the formula (O'Riordan et al., 2008) given by

$$p_j{}^{*N} = \frac{\ln E^N_j - \ln E^{2N}_j}{\ln(2 \ln N) - \ln(\ln(2N))}, j = 1, 2.$$

Example 1: (Constant coefficient problem)

$$\begin{aligned} -\varepsilon y_1''(x) + 2y_1(x) - y_2(x) &= f_1(x), \quad x \in \Omega^- \cup \Omega^+, \\ -\varepsilon y_2''(x) - y_1(x) + 2y_2(x) &= f_2(x), \quad x \in \Omega^- \cup \Omega^+, \\ \overline{y}(0) = \overline{0}, \quad \overline{y}(1) &= \overline{0}, \end{aligned}$$

where

$$f_1(x) = \begin{cases} 1, & \text{for} \quad 0 \leq x \leq 0.5, \\ 0.8, & \text{for} \quad 0.5 < x \leq 1. \end{cases}$$

and

$$f_2(x) = \begin{cases} 2, & \text{for} \quad 0 \leq x \leq 0.5, \\ 1.8, & \text{for} \quad 0.5 < x \leq 1. \end{cases}$$

Example 2: (Variable coefficient problem)

$$\begin{aligned} -\varepsilon y_1''(x) + 2(x+1)^2 y_1(x) - (1+x^3) y_2(x) &= f_1(x), \quad x \in \Omega^- \cup \Omega^+, \\ -\varepsilon y_2''(x) - 2\cos\left(\frac{\pi}{4}x\right) y_1(x) + (1+\sqrt{2}) e^{1-x} y_2(x) &= f_2(x), \quad x \in \Omega^- \cup \Omega^+, \\ \overline{y}(0) = \overline{0}, \quad \overline{y}(1) &= \overline{0}, \end{aligned}$$

where

$$f_1(x) = \begin{cases} 2e^x, & \text{for} \quad 0 \leq x \leq 0.5, \\ 1, & \text{for} \quad 0.5 < x \leq 1. \end{cases}$$

and

$$f_2(x) = \begin{cases} 10x + 1, & \text{for} \quad 0 \leq x \leq 0.5, \\ 2, & \text{for} \quad 0.5 < x \leq 1. \end{cases}$$

The values of α are taken as 1 and 0.787 for Example 1 and Example 2 respectively for numerical computations. From Tables 1–4, it is noted that the obtained numerical results are almost the second order uniformly convergent with respect to the singular perturbation parameter ε. These results are compared well with the existing results and can be seen from Table 5 and Table 6 for Example 1 and Example 2 respectively. As per the prediction, left and right boundary layers and interior layers can be seen from the solution and error plots given in Figure 1 and Figure 2 of Example 1 and Example 2 respectively when $\varepsilon = 10^{-4}$ and $N = 256$.

Figure 1 Numerical solution and error plots for the components y_1 and y_2 of Example 1 when $\varepsilon = 10^{-4}, N = 256$ (see online version for colours)

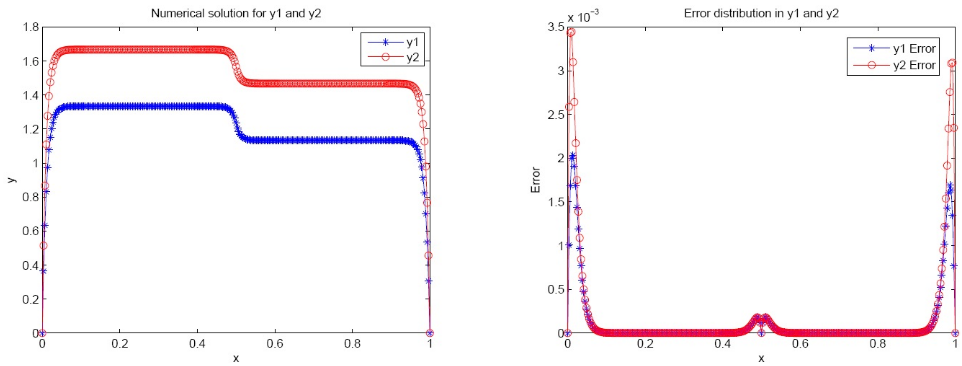

Table 2 Maximum point-wise errors $E^N_{2,\varepsilon}$, ε-uniform errors E^N_2 and the uniform rates of convergence p^N_2, p^{*N}_2 for the solution y_2 of the Example 1

ε	*Number of mesh points N*					
	64	*128*	*256*	*512*	*1,024*	*2,048*
10^{-1}	1.0116E-04	2.5291E-05	6.3185E-06	1.5751E-06	3.8921E-07	9.2737E-08
10^{-2}	7.2743E-04	1.8285E-04	4.5706E-05	1.1395E-05	2.8152E-06	6.7030E-07
10^{-3}	6.9814E-03	1.7843E-03	4.5246E-04	1.1295E-04	2.7916E-05	6.6478E-06
10^{-4}	2.8199E-02	9.8866E-03	3.4441E-03	1.1097E-03	2.7875E-04	6.6437E-05
10^{-5}	2.8194E-02	9.8601E-03	3.4392E-03	1.0845E-03	3.2786E-04	9.5871E-05
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
10^{-15}	2.8194E-02	9.8601E-03	3.4392E-03	1.0845E-03	3.2786E-04	9.5871E-05
E^N_2	2.8199E-02	9.8866E-03	3.4441E-03	1.1097E-03	3.2786E-04	9.5871E-05
p^N_2	1.51	1.52	1.63	1.76	1.77	-
p^{*N}_2	1.94	1.88	1.97	2.07	2.06	-

Table 3 Maximum point-wise errors $E_{1,\varepsilon}^N$, ε-uniform errors E_1^N and the uniform rates of convergence p_1^N, p_1^{*N} for the solution y_1 of the Example 2

ε	*Number of mesh points N*					
	64	*128*	*256*	*512*	*1,024*	*2,048*
10^{-1}	3.9967E-04	1.0023E-04	2.5078E-05	6.2560E-06	1.5462E-06	3.6824E-07
10^{-2}	1.1478E-03	2.8855E-04	7.2233E-05	1.8020E-05	4.4536E-06	1.0606E-06
10^{-3}	6.6827E-03	1.7412E-03	4.3802E-04	1.0963E-04	2.7095E-05	6.4519E-06
10^{-4}	3.2849E-02	1.3079E-02	4.5048E-03	1.1405E-03	2.8287E-04	6.7418E-05
10^{-5}	3.3261E-02	1.3206E-02	4.5265E-03	1.4225E-03	4.3289E-04	1.2545E-04
10^{-6}	3.3398E-02	1.3263E-02	4.5468E-03	1.4291E-03	4.3492E-04	1.2602E-04
10^{-7}	3.3442E-02	1.3281E-02	4.5533E-03	1.4312E-03	4.3556E-04	1.2620E-04
10^{-8}	3.3456E-02	1.3286E-02	4.5553E-03	1.4319E-03	4.3577E-04	1.2626E-04
10^{-9}	3.3461E-02	1.3288E-02	4.5559E-03	1.4321E-03	4.3583E-04	1.2628E-04
10^{-10}	3.3462E-02	1.3289E-02	4.5562E-03	1.4322E-03	4.3585E-04	1.2628E-04
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
10^{-15}	3.3463E-02	1.3289E-02	4.5562E-03	1.4322E-03	4.3586E-04	1.2628E-04
E_1^N	3.3463E-02	1.3289E-02	4.5562E-03	1.4322E-03	4.3586E-04	1.2628E-04
p_1^N	1.33	1.54	1.67	1.72	1.79	-
p_1^{*N}	1.71	1.91	2.01	2.02	2.07	-

Table 4 Maximum point-wise errors $E_{2,\varepsilon}^N$, ε-uniform errors E_2^N and the uniform rates of convergence p_2^N, p_2^{*N} for the solution y_2 of the Example 2

ε	Number of mesh points N					
	64	*128*	*256*	*512*	*1,024*	*2,048*
10^{-1}	8.5571E-04	2.1409E-04	5.3501E-05	1.3338E-05	3.2954E-06	7.8468E-07
10^{-2}	2.5106E-03	6.3186E-04	1.5814E-04	3.9438E-05	9.7449E-06	2.3203E-06
10^{-3}	1.0341E-02	2.6991E-03	6.9681E-04	1.7425E-04	4.3084E-05	1.0260E-05
10^{-4}	3.1178E-02	1.4194E-02	4.9644E-03	1.3025E-03	3.2586E-04	7.7776E-05
10^{-5}	3.1342E-02	1.3325E-02	4.5502E-03	1.4674E-03	4.3707E-04	1.2512E-04
10^{-6}	3.1398E-02	1.3348E-02	4.5587E-03	1.4341E-03	4.3792E-04	1.2536E-04
10^{-7}	3.1416E-02	1.3356E-02	4.5614E-03	1.4350E-03	4.3820E-04	1.2544E-04
10^{-8}	3.1422E-02	1.3358E-02	4.5623E-03	1.4353E-03	4.3828E-04	1.2546E-04
10^{-9}	3.1424E-02	1.3359E-02	4.5625E-03	1.4354E-03	4.3831E-04	1.2547E-04
10^{-10}	3.1424E-02	1.3359E-02	4.5626E-03	1.4354E-03	4.3832E-04	1.2547E-04
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
10^{-15}	3.1425E-02	1.3359E-02	4.5627E-03	1.4354E-03	4.3833E-04	1.2548E-04
E_2^N	3.1425E-02	1.4194E-02	4.9644E-03	1.4674E-03	4.3833E-04	1.2548E-04
p_2^N	1.15	1.52	1.76	1.74	1.80	-
p_2^{*N}	1.47	1.88	2.12	2.06	2.09	-

Table 5 Comparison of maximum errors and rates of convergence obtained by the present method with the existing methods for the solution components y_1 and y_2 of Example 1

N	*64*	*128*	*256*	*512*	*1,024*
			Method by Tamilselvan et al. (2007)		
E_1^N	4.1490E-02	3.4330E-02	2.9820E-02	2.6510E-02	2.3940E-02
p_1^N	0.27	0.20	0.17	0.15	-
E_2^N	2.8910E-01	1.8030E-01	1.1170E-01	6.4340E-02	4.0890E-02
p_2^N	0.57	0.69	0.80	0.65	-
			Method by Basha and Shanthi (2015b) using DS-I		
E_1^N	1.1723E-02	4.3482E-03	1.9511E-03	8.3108E-04	3.1086E-04
p_1^N	1.43	1.16	1.23	1.42	-
p_1^{*N}	1.84	1.43	1.48	1.67	-
E_2^N	1.9295E-02	5.3117E-03	2.7851E-03	1.1848E-03	5.0802E-04
p_2^N	1.86	0.93	1.23	1.22	-
p_1^{*N}	2.39	1.15	1.49	1.44	-
			Method by Basha and Shanthi (2015b) using DS-II		
E_1^N	3.0889E-02	1.0827E-02	3.9968E-03	1.3227E-03	4.1642E-04
p_1^N	1.51	1.44	1.60	1.67	-
p_1^{*N}	1.95	1.78	1.92	1.97	-
E_2^N	5.1236E-02	1.9647E-02	6.7255E-03	2.4320E-03	7.7620E-04
p_2^N	1.38	1.55	1.47	1.65	-
p_1^{*N}	1.78	1.92	1.77	1.94	-
			Present method		
E_1^N	1.7181E-02	6.1908E-03	2.0347E-03	6.4394E-04	1.9508E-04
p_1^N	1.47	1.61	1.66	1.72	-
p_1^{*N}	1.89	1.99	2.00	2.03	-
E_2^N	2.8199E-02	9.8866E-03	3.4441E-03	1.1097E-03	3.2786E-04
p_2^N	1.51	1.52	1.63	1.76	-
p_1^{*N}	1.94	1.88	1.97	2.07	-

Figure 2 Numerical solution and error plots for the components y_1 and y_2 of the Example 2 when $\varepsilon = 10^{-4}, N = 256$ (see online version for colours)

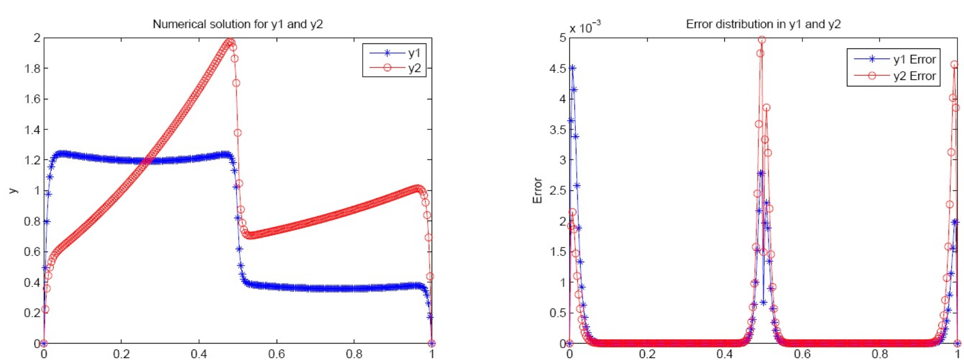

Table 6 Comparison of maximum errors and rates of convergence obtained by the present method with the existing methods for the solution components y_1 and y_2 of Example 2

N	*64*	*128*	*256*	*512*	*1,024*
			Method by Tamilselvan et al. (2007)		
E_1^N	4.0072E-02	3.3085E-02	2.8139E-02	2.4799E-02	2.2319E-02
p_1^N	0.28	0.23	0.18	0.15	-
E_2^N	4.4390E-02	2.7249E-02	2.3387E-02	2.0654E-02	1.8600E-02
p_2^N	0.70	0.22	0.18	0.15	-
			Method by Basha and Shanthi (2015b) using DS-I		
E_1^N	1.8817E-02	5.2355E-03	2.6245E-03	1.1146E-03	4.9558E-04
p_1^N	1.85	1.00	1.24	1.17	-
p_1^{*N}	2.37	1.23	1.49	1.38	-
E_2^N	1.8300E-02	6.2637E-03	2.3580E-03	1.0487E-03	4.7023E-04
p_2^N	1.55	1.41	1.17	1.16	-
p_1^{*N}	1.99	1.75	1.41	1.36	-
			Method by Basha and Shanthi (2015b) using DS-II		
E_1^N	3.0889E-02	1.0827E-02	3.9968E-03	1.3227E-03	4.1642E-04
p_1^N	1.51	1.44	1.60	1.67	-
p_1^{*N}	1.95	1.78	1.92	1.97	-
E_2^N	4.7420E-02	1.9077E-02	6.4871E-03	2.3736E-03	7.8190E-04
p_2^N	1.31	1.56	1.45	1.60	-
p_1^{*N}	1.69	1.93	1.75	1.89	-
			Present method		
E_1^N	3.3462E-02	1.3289E-02	4.5562E-03	1.4322E-03	4.3586E-04
p_1^N	1.33	1.54	1.67	1.72	-
p_1^{*N}	1.71	1.91	2.01	2.02	-
E_2^N	3.1425E-02	1.4194E-02	4.9644E-03	1.4674E-03	4.3832E-04
p_2^N	1.15	1.52	1.76	1.74	-
p_1^{*N}	1.47	1.88	2.12	2.06	-

7 Conclusions

In this article, it is considered that a fitted mesh finite difference method using the classical central difference scheme constructed on Shishkin mesh with the help of an iterative procedure described in Matthews et al. (2000, 2002) and Matthews (2000) to solve the weakly coupled system of singularly perturbed reaction-diffusion problems with discontinuous source term. At the point of discontinuity, the average of the source terms is taken on either side of that point. Error analysis is carried out and numerical results are provided which are in agreement with the theoretical results. The proposed method is proved to be an efficient and robust numerical method that leads to the results of almost second order parameter-uniform convergence better than the existing results.

References

Basha, P.M. and Shanthi, V. (2015a) 'A numerical method for singularly perturbed second order coupled system of convection-diffusion Robin type boundary value problems with discontinuous source term', *International Journal of Applied and Computational Mathematics*, Vol. 1, No. 3, pp.381–397.

Basha, P.M. and Shanthi, V. (2015b) 'Robust computational methods for a coupled system of singularly perturbed reaction-diffusion equations with discontinuous source term', *Journal of Modern Methods in Numerical Mathematics*, Vol. 6, No. 2, pp.64-85.

Basha, P.M. and Shanthi, V. (2016) 'Fitted mesh method for a weakly coupled system of singularly perturbed reaction-convection-diffusion problems with discontinuous source term', *Ain Shams Engineering Journal* (article in press).

Basha, P.M. and Shanthi, V. (2017) 'A parameter-uniform second order numerical method for a weakly coupled system of singularly perturbed convection–diffusion equations with discontinuous convection coefficients and source terms', *Calcolo*, Vol. 54, No. 3, pp.1027–1053.

Das, P. and Natesan, S. (2014) 'Optimal error estimate using mesh equidistribution technique for singularly perturbed system of reaction-diffusion boundary value problems', *Applied Mathematics and Computation*, Vol. 249, pp.265–277.

De Falco, C. and O'Riordan, E. (2010) 'Interior layers in a reaction-diffusion equation with a discontinuous diffusion coefficient', *International Journal of Numerical Analysis and Modeling*, Vol. 7, No. 3, pp.444–461.

Kan-On, Y. and Mimura, M. (1998) 'Singular perturbation approach to a 3-component reaction-diffusion system arising in population dynamics', *SIAM Journal on Mathematical Analysis*, Vol. 29, No. 6, pp.1519–1536.

Linss, T. and Madden, N. (2002) 'A robust difference scheme for a system of coupled singularly perturbed reaction-diffusion equations', *Preprint MATH-NM-12-02, Institut fur Numerische Mathematik*, TU Dresden.

Linss, T. and Madden, N. (2004) 'Accurate solution of a system of coupled singularly perturbed reaction-diffusion equations', *Computing*, Vol. 73, pp.121–133.

Linss, T. and Stynes, M. (2009) 'Numerical solution of systems of singularly perturbed differential equations', *Computational Methods in Applied Mathematics*, Vol. 9, No. 2, pp.165–191.

Madden, N. and Stynes, M. (2003) 'A uniformly convergent numerical method for a coupled system of two singularly perturbed linear reaction-diffusion problems', *IMA Journal of Numerical Analysis*, Vol. 23, No. 4, pp.627–644.

Matthews, S., Miller, J.J.H. and O'Riordan, E. (2000) 'A parameter robust numerical method for a system of singularly perturbed ordinary differential equations', in *Analytical and Numerical Methods for Convection-Dominated and Singularly Perturbed Problems*, Nova Science Publishers, New York, pp.219–224.

Matthews, S., O'Riordan, E. and Shishkin, G.I. (2002) 'A numerical method for a system of singularly perturbed reaction-diffusion equations', *Journal of Computational and Applied Mathematics*, Vol. 145, No. 1, pp.151–166.

Matthews, S. (2000) *Parameter Robust Numerical Methods for a System of Two Coupled Singularly Perturbed Reaction-diffusion Equations*, Master thesis, School of Mathematical Sciences, Dublin City University, Dublin, Ireland.

Miller, J.J.H., O'Riordan, E. and Shishkin, G.I. (2012) *Fitted Numerical Methods for Singular Perturbation Problems*, revised ed., World-Scientific, Singapore.

Naidu, D.S. (1988) *Singular Perturbation Methodology in Control Systems*, Peter Peregrinus Ltd. on behalf of Institution of Electrical Engineers, London, UK.

O'Riordan, E., Stynes, J. and Stynes, M. (2008) 'A parameter-uniform finite difference method for a coupled system of convection-diffusion two-point boundary value problems', *Numer. Math. Theor. Meth. Appl.*, Vol. 1, No. 2, pp.176-197.

Paramasivam, M., Miller, J.J.H. and Valarmathi, S. (2014) 'Parameter-uniform convergence for a finite difference method for a singularly perturbed linear reaction-diffusion system with discontinuous source terms', *International Journal of Numerical Analysis and Modeling*, Vol. 11, No. 2, pp.385–399.

Ramesh Babu, A. and Ramanujam, N. (2009) 'An almost second order FEM for a weakly coupled system of two singularly perturbed differential equations of reaction-diffusion type with discontinuous source term', *Neural, Parallel, and Scientific Computations*, Vol. 17, No. 2, pp.147–168.

Rao, S.C.S. and Chawla, S. (2013) 'Interior layers in coupled system of two singularly perturbed reaction-diffusion equations with discontinuous source term', in *Proceedings of 5th International Conference, Numerical Analysis and its Applications 2012*, LNCS 8236, Lozenetz, Bulgaria, June 2012, pp.445–453.

Shishkin, G.I. (1995) 'Mesh approximations of singularly perturbed boundary value problems for systems of elliptic and parabolic equations', *Computational Mathematics and Mathematical Physics*, Vol. 4, No. 35, pp.429–446.

Tamilselvan, A., Ramanujam N. and Shanthi, V. (2007) 'A numerical method for singularly perturbed weakly coupled system of two second order ordinary differential equations with discontinuous source term', *Journal of Computational and Applied Mathematics*, Vol. 202, No. 2, pp.203–216.

Thomas, G.P. (1998) 'Towards an improved turbulence model for wave-current interactions', in *2nd Annual Report to EU MAST-III Project the Kinematics and Dynamics of Wave-Current Interactions*.