A nonlinear quarter-car active suspension design based on feedback linearisation and $H_\infty$ control

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Abstract: The nonlinear behaviour of suspension elements is substantial when the vehicles experience large road disturbances. These nonlinearities lead to performance deterioration of active suspension systems, which in turn degrades ride comfort, road holding and road handling. Standard control trends on the active suspension count on linear models to benefit from the well established linear control theory, while neglecting the nonlinear dynamics of the suspension systems. In this study, the quarter-car model has a nonlinear suspension spring with a hysteretic nature. The presented design is based on the combination of feedback linearisation (FBL) and $H_\infty$ controller. This approach is selected to take in consideration the nonlinear behaviour of the suspension system, while maintaining the opportunity to conduct the linear control theory. The main objective is maximising the ride comfort while keeping the suspension stroke, tyre dynamic load, and actuator force bounded. To assess the efficiency of the proposed design, simulations are performed on two types of road disturbances. The time and frequency domain simulations show the superiority of the proposed feedback controller in providing ride comfort in comparison with the passive suspension system. Moreover, the proposed design guarantees an agreement between the ride comfort and the other design constraints.

Keywords: active suspension control; feedback linearisation; robust control; mixed sensitivity.


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1 Introduction

A successfully designed suspension system should be able to isolate passengers from road roughness (ride comfort), to maintain uninterrupted contact of wheels to road (vehicle handling and road holding), and to keep suspension strokes within acceptable ranges. These objectives can be achieved mainly by minimising the sprung mass acceleration using suspension system (Qatu et al., 2009; Qatu, 2012).

Passive suspension systems, which are produced by placing a spring and a damper between the vehicle chassis and wheel assembly, have limited capabilities in compromising between these conflicting design requirements. Unlike passive suspensions, active suspension systems can be used to compromise between these conflicting requirements. The active suspension system is constructed by using an external power supply and a feedback controlled actuator. During the last decades, various control approaches have been proposed to improve the performance of active suspension systems (Hrovat, 1997; Tseng and Hrovat, 2015; Zhang et al., 2015), such as optimal linear state feedback theory (Thompson, 1976), linear quadratic regulator (LQR) control (Krtolica and Hrovat, 1990; Sam et al., 2004), linear quadratic Gaussian (LQG) control (Barak and Hrovat, 1988; Elbeheiry and Karnopp, 1996), $H_\infty$ control (Yamashita et al., 1994; Park and Kim, 1999; Chen and Guo, 2001; Du et al., 2005; Chen and Guo, 2005; Zin et al., 2005; Abdalla et al., 2009), $H_\infty$ control for suspension systems with input delay (Li et al., 2014; Choi et al., 2016), mixed $H_\infty/H_2$ control (Abdellari et al., 2000; Gáspár and Bokor, 1998; Lu, 2004), adaptive control schemes (Ramsbottom et al., 1999; Chantranuwathana and Peng, 1999; Fialho and Balas, 2002), sliding mode control (Yoshimura et al., 2001; Guclu and Yagiz, 2004; Yagiz and Sakman, 2005), mixed sliding mode-fuzzy control (Al-Holou et al., 1999), sky-hook damping control (Crosby and Karnopp, 1973; Sannier et al., 2003; Rao and Narayanan, 2009), ground-hook control logic (Valašek et al., 1998), neural networks and fuzzy logic control (FLC) (Vemuri, 1993; Al-Holou et al., 2002; Taskin et al., 2007), linear parameter varying (LPV) schemes (Fialho and Balas, 2002; Poussot-Vassal et al., 2007; Onat et al., 2007, 2009) and backstepping control (Karlsson et al., 2001; Lin and Huang, 2004; Yagiz and Hacioglu, 2008).

In general, vehicle suspension elements are modelled as linear springs and dampers in order to simplify the analysis considerably which is too ideal for real operating conditions. Hasbullah and Faris (2010) designed LQR and FLC controllers for active suspension, which was implemented on half car model and compared to passive suspension. Kaleemullah et al. (2012) proposed LQR and robust $H_\infty$ controllers for quarter car model, both controllers were compared under bump inputs. Sabaneh et al. (2016) developed a neuro-fuzzy and FLC controllers for two and three-axle half-car off-road vehicle models. These controllers were tested against pothole and random road inputs.

In reality, vehicles encounter large suspension strokes and the linear model becomes invalid. Hence, the nonlinear effect must be taken in consideration in designing the controller for practical active suspension systems. Different control approaches are used in active suspensions with nonlinearities. One efficient method to deal with nonlinear suspension is to employ LPV control. Fialho and Balas (2002) designed a road-adaptive active suspension via a combination of LPV and nonlinear backstepping technique. In this method, they considered two levels of adaptation: the first level control design shapes the nonlinear characteristics of the vehicle suspension as a function of road conditions.
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The second level design involves adaptive switching between the different nonlinear characteristics. Zheng and Baz (2007) designed a nonlinear energy sink controller for a full car model, which was more efficient than LQR controller under chirp signal road profiles. Onat et al. (2009) designed an LPV model-based dynamic gain-scheduled output feedback controller for a quarter-car model that takes nonlinearity characteristic of the suspension spring, the bilinear characteristic of the damper and the mass variations of the vehicle into account. In this method, they implemented the $H_\infty$ mixed sensitivity synthesis for loop shaping design. Yagiz and Hacioglu (2008) used backstepping control to design an active suspension of a nonlinear full car model. Backstepping control was used to provide a systematic procedure for the construction of the Lyapunov functions and corresponding feedback control laws. Shi et al. (2009) designed a PID controller based on feedback linearisation of a nonlinear Quarter car model with active hydropneumatic suspension. Abdalla et al. (2009) introduced LMI based $H_\infty$ controller for vehicle’s active suspension, in which they implement gain scheduling controller to cope with spring nonlinear behaviour.

In this work, we are concerned with the controller design problem for a nonlinear quarter car model. The nonlinear suspension spring is assumed to be of a hysteretic nature which is modelled by the Bouc-Wen model (Bouc, 1967; Wen, 1976). The method of feedback linearisation (Monaco, 1981; Slotine et al., 1991; Shaqarin and Abdalla, 2009; Shi et al., 2009; Shaqarin et al., 2014) will be implemented on the nonlinear quarter car model to cancel the nonlinear terms. Then, the weighted $H_\infty$ control will be used to design the controller for the feedback linearised model. The main objective is improving the ride comfort and providing a trade off between ride comfort, vehicle safety and vehicle handling. The rest of the paper is organised as follows: Section 2 introduces the nonlinear quarter car model. The controller design is described in Section 3. To show the efficiency of the proposed method, some simulations are presented in Section 4. Finally, the paper is concluded in Section 5.

2 Nonlinear vehicle model

The nonlinear quarter car model used in this study has two DOF due to the heave of sprung mass and the vertical motion of the unsprung mass (see Figure 1). This model is made up of a sprung mass ($m_s$), unsprung mass ($m_u$), a damper with coefficient $c_s$, and a nonlinear spring. The tyre is modelled by a spring with stiffness coefficient $k_t$. The active input to the suspension system $U(t)$ is normally generated by an actuator placed between the sprung and unsprung masses. In this study, the effect of actuator dynamics is neglected and the actuator is modelled as an ideal force generator. In this model $x_s$ and $x_u$ are the displacements of the sprung and unsprung masses, respectively; and $d$ is the road displacement input. The nonlinear suspension spring is assumed to be of hysteretic nature. The spring nonlinearity is modelled using Bouc-Wen model (Bouc, 1967; Wen, 1976). The equations of the quarter car model shown in Figure 1 are given by Rao and Narayanan (2009).

\begin{align*}
    m_s \ddot{x}_s + c_s (\dot{x}_s - \dot{x}_u) + ak_s (x_s - x_u) + (-\alpha)k_z q - U &= 0 \\
    m_u \ddot{x}_u + c_s (\dot{x}_u - \dot{x}_s) + ak_s (x_u - x_s) - (1-\alpha)k_z q + k_t (x_u - d) + U &= 0
\end{align*}

(1) 

(2)
The Bouc-Wen model is based on a nonlinear ordinary differential equation which contains a memory variable $z_q$, representing the hysteretic displacement. Based on Bouc-Wen model, the spring restoring force consists of a preyielding component and a hysteretic component. The preyielding component is proportional to the relative displacement between the sprung and unsprung masses $kd(x_s - x_u)$. The hysteretic component of spring restoring force is given by $(1 - \alpha)k_z z_q$. The hysteretic displacement $z_q$ is given by:

$$\dot{z}_q = -\gamma_q |\dot{x}_s - \dot{x}_u|^{n-1} - v_q z_q^n (\dot{x}_s - \dot{x}_u) + \beta (\ddot{x}_s - \ddot{x}_u) \quad (3)$$

where $\gamma_q$, $v_q$, $\beta$, and $n$ are loop parameters defining the shape and amplitude of the hysteresis loop.

To quantify ride comfort, the sprung mass displacement and sprung mass acceleration are widely used measures. The ride comfort of passengers is directly related to the root mean square (RMS) value of sprung mass acceleration (Hrovat, 1997). According to ISO2631 standards, the passenger feels highly comfortable if the RMS value of the acceleration is less than 0.315 m/s$^2$ (Griffin, 2012). For vehicle safety, we should ensure a firm uninterrupted contact of wheels to road. For this condition, the dynamic tire load should not exceed the static load, i.e., $kd(x_s(t) - d(t)) < (m_s + m_u)g$. Moreover, the suspension stroke should be constrained to prevent possible structural damage, that is $|x_s(t) - x_d(t)| < SS_{\text{max}}$, where $SS_{\text{max}}$ is the maximum suspension deflection. Due to the limited power of actuator, an additional constrained to be applied, that is $|U(t)| < U_{\text{max}}$.

3 Controller design

The quarter car model in equations (1), (2) and (3) consists of three nonlinear and coupled differential equations. This model structure is an excellent choice for feedback...
linearisation. The feedback linearisation method will be used to cancel the nonlinear terms in equations (1) and (2) and remove the coupling with equation (3). Then, an \( H_\infty \) controller based on the feedback linearised model will be designed.

The feedback linearisation control law is

\[
U = (1 - \alpha)k_z z_q + u
\]

(4)

The control law \( U \) consists of two components; the nonlinear control part \((1 - \alpha)k_z z_q\) and the linear control part \(u\). The nonlinear control part, that is a function of the hysteretic displacement \(z_q\), needs an online solution of the nonlinear equation (3). Therefore, an accurate solution of \(z_q\) needs an accurate online measurement of \(\dot{x}_s\) and \(\dot{x}_u\). The linear control part will be designed based on the linear \( H_\infty \) control theory.

Applying the control law \( U \) to equations (1) and (2), yields the following feedback linearised model:

\[
m_s \ddot{x}_s + c_s (\dot{x}_s - \dot{x}_u) + \alpha k_s (x_s - x_u) - u = 0
\]

(5)

\[
m_u \ddot{x}_u + c_s (\dot{x}_u - \dot{x}_s) + \alpha k_s (x_u - x_s) + k (x_u - d) + u = 0
\]

(6)

The state space representation of the feedback linearised model is

\[
\begin{align*}
\dot{x} &= Ax + B_1 w + B_2 u \\
\dot{z} &= C_1 x + D_{13} w + D_{12} u \\
y &= C_2 x + D_{21} w + D_{22} u
\end{align*}
\]

(7)

where \(x\) is the state vector, which is defined as \(x = [x_s \ \dot{x}_s \ \dot{x}_u \ \dot{x}_u]^T\), \(w\) is the exogenous input \((w = d)\) and \(u\) is the control input. The state space matrices are defined as:

\[
A_p = A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-\alpha k_s & -c_s & \alpha k_s & c_s \\
m_s & m_s & m_s & m_s \\
0 & 0 & 0 & 1 \\
\alpha k_s & c_s & -\alpha k_s - k & -c_s \\
m_u & m_u & m_u & m_u
\end{bmatrix}
\]

\[
B_p = [B_1 B_2] = \begin{bmatrix}
0 & 0 \\
0 & \frac{1}{m_s} \\
0 & 0 \\
\frac{k_s}{m_u} & -1 \\
\frac{1}{m_u}
\end{bmatrix}
\]
The transfer function of the model is

\[ G(s) = C_p \left( sI - A_p \right)^{-1} B_p + D_p \]  \hspace{1cm} (8)

Figure 2 Mixed sensitivity problem

The plant \( G \) has two inputs (\( d \) and \( u \)) and four outputs (\( z \) and \( y \)), where \( y = \dot{x}_c \) is the measured output and \( \tilde{z} \) is the vector of the outputs: \( \tilde{z} = [x, x - \dot{x}, \dot{x}_c] \). The transfer matrix of \( G \) is

\[ G = \begin{bmatrix} G_{zw} & G_{zu} \\ G_{yw} & G_{yu} \end{bmatrix} \]  \hspace{1cm} (9)

The \( H_\infty \) mixed-sensitivity problem is depicted in Figure 2. The road disturbance \( d \) is a single exogenous input, \( z \) is a vector of the regulated outputs, \( y \) is the measured output and \( u \) is the controller output.

The transfer matrix from \( [w \ u]^T \) to \( [z \ y]^T \) is

\[ P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{bmatrix} = \begin{bmatrix} W_{Gzw} & W_{Gzu} \\ 0 & W_{2} \end{bmatrix} \]  \hspace{1cm} (10)
where $P(s)$ is the generalised plant, and where, for example, $P_{zw}$ is the transfer function from $w$ to $z$. $W_d$ is the weighting constant corresponds to road disturbance $d$. $W_2$ is frequency-dependent weighting that corresponds to control input $u$. $W_1$ is frequency dependent weighting that corresponds to system output $\dot{z} = [x_s, x_s - x_v, \dot{x}_v]^T$ and has the following structure:

$$
W_1 = \begin{bmatrix}
W_{11} & 0 & 0 \\
0 & W_{22} & 0 \\
0 & 0 & W_{33}
\end{bmatrix}
$$

Then, the closed loop transfer matrix $T_{zw}$ is defined by:

$$
T_{zw} = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}
$$

where $K(s)$ is the controller. The mixed sensitivity technique computes a controller $K$ that minimises the $H_\infty$ norm of the closed-loop transfer function $T_{zw}$ of the weighted mixed sensitivity problem, such that:

$$
\|T_{zw}\|_\infty < \gamma
$$

where the performance index $\gamma$ is the upper bound of the $H_\infty$ norm of the closed-loop transfer matrix $T_{zw}$.

### Table 1 Properties of the quarter car model with hysteretic suspension

| $m_s$ (Kg) | $m_u$ (Kg) | $k_s$ (N/m) | $k_t$ (N/m) | $c_s$ (Ns/m) | $\gamma_q$ | $v_q$ | $f_q$ | $n$ | $\alpha$
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### 4 Simulation results

In order to evaluate the performance of the suggested control approach, simulations are carried out on the nonlinear passive and active quarter-car model described in Section 2. The quarter-car model parameters have been chosen similar to real vehicle values and are given in Table 1. The open-loop model in equation (10) has an imaginary axis zeros $\pm j60$ that corresponds to the tire-hop frequency (60 rad/s) and $\pm j18.1$ that corresponds to the rattlespace frequency (18.1 rad/s). It is well known that the imaginary zeros of open-loop model limit the performance of the closed-loop system; in particular the closed-loop bandwidth. The weightings $W_1$, $W_2$ and $W_d$ have been selected to satisfy the conflicting constraints, which were discussed in Section 2. Taking into account a wide range of road disturbances, such as bump and limited ramp inputs. The weightings are as follows:

$$
W_1 = \begin{bmatrix}
\frac{11}{s+10} & 0 & 0 \\
0 & \frac{30}{s+10} & 0 \\
0 & 0 & \frac{0.00375(s+150)}{s+0.45}
\end{bmatrix}
$$
Two different road conditions are considered: bump road input and limited ramp road input. The objective of the proposed design is maximising the ride comfort under safety and structural design limitations. For vehicle safety, the relative dynamic tire load \( \frac{k_t(x_u - d)}{g(m_s + m_u)} \) must be less than one. For the proper structural design, the suspension stroke is limited to ±90 mm, and the actuator force is limited to 3 kN.

In the first example, the vehicle is subjected to a bump with a 10 cm height. This bump reflects the most common harmful road irregularity in reality. Using the proposed approach shown previously, an \( H_\infty \) controller based on the model in equation (10) is designed and the achieved performance index \( \gamma \) is 0.941.

**Figure 3** Time-domain simulations for the proposed active suspension and passive suspension systems (bump input) (see online version for colours)
The active suspension responses are shown in Figure 3 together with the passive suspension responses. The figure includes sprung mass displacement and acceleration responses, suspension stroke, and relative dynamic tire load for both the active and passive suspension systems. It can be seen in Figure 3 that the presented active suspension achieves significantly better performance on ride comfort more than the passive suspension. In particular, the reduced sprung mass displacements and accelerations show that the ride comfort is enhanced. Also, the sprung mass displacement and acceleration responses of the active suspension vanish faster than those in the passive suspension.

The active suspension responses based on the combination of the $H_\infty$ controller and feedback linearisation compared with the active suspension responses based on the $H_\infty$ controller alone are shown in Figure 3. The figure depicts the deterioration of the response of $x_s, x_s - x_u$ and $\dot{x}_s$ using $H_\infty$ controller without FBL. That emphasis the fact that the controller will not work well when applied to the actual nonlinear physical model when ignoring the nonlinearities during the controller design.

The RMS values for the sprung mass accelerations for the active suspension with FBL, without FBL and the passive suspension are shown in Figure 6(a). It is clear from the figure that the RMS of the sprung mass acceleration is reduced significantly in the case of active suspension with FBL, which is resulting in better ride comfort. The decrease in the RMS value is 43% and 69% compared to active control without FBL and passive control, respectively. Moreover, the RMS value of the accelerations response is 0.28 m/s$^2$ which is within the highly ride comfortable range according to ISO2631 standards (Griffin, 2012). It’s worth mentioning that all the constraints are respected and this tells the conservatism of the design. The suspension stroke reaches the limits at 0.09 sec. In this case, the controller tends to punish the suspension travel instead of improving the ride comfort.

In the second example, the vehicle is subjected to a limited ramp road input as shown in Figure 5. This road profile represents the sudden change in the road surface elevation. The same controller which is designed for the first example is used in this case. Time histories of the active with FBL, without FBL and passive system responses for the road input with limited ramp are shown in Figure 5. It is clear from the figure that the displacements and accelerations of the sprung mass are reduced significantly using the active suspension with FBL. Also, the vehicle smoothly settles to the final height of the road input. Figure 6(b) shows the RMS values for the sprung mass accelerations for the limited ramp input. The RMS value in the case of active suspension with FBL is decreased by 47% and 77% compared to active control without FBL and passive control, respectively. Figures 5 and 6(b) show that the ride comfort is significantly improved using the proposed controller. The history of the actuator force, produced by the controller also shown in Figure 5.
Figure 4  Comparison between the linear and nonlinear control effort (see online version for colours)

Figure 5  Time-domain simulations for the proposed active suspension and passive suspension systems (ramp input) (see online version for colours)
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**Figure 6** Rms of $\ddot{x}_s$, (a) bump input (b) ramp input (see online version for colours)

**Figure 7** Frequency responses of sprung mass displacement, acceleration and suspension stroke from the road disturbance $d$ (see online version for colours)
Figure 7 depicts the frequency response of the closed-loop system together with the open-loop system. The solid line and dashed line correspond to the closed-loop and open-loop system frequency responses, respectively. It is shown in Figure 7 that the reduction of sprung mass displacement, acceleration and suspension deflection is below the rattle space frequency (18.1 rad/s). This contributes to better ride comfort and road handling.

5 Conclusions

In this study, a new method was adopted to design a quarter-car active suspension system. The quarter-car model has a nonlinear suspension spring with a hysteretic nature. The presented design is based on the combination of feedback linearisation and $H_\infty$ controller. The main objective was maximising the ride comfort while keeping the suspension stroke, tyre dynamic load, and actuator force bounded. To assess the performance of the controller, simulations were performed on two types of road disturbances. The time and frequency domain simulations showed the efficiency of the proposed feedback controller in providing ride comfort in comparison with the passive suspension system. In future work, the efficiency of the proposed approach will be investigated using more complex vehicle models.

References


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