Adaptive optimal control for integrated active front steering and direct yaw moment based on approximate dynamic programming

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Abstract: In this paper, a novel adaptive optimal control algorithm based on approximate dynamic programming (ADP) approach is proposed for integrated active front steering (AFS) and direct yaw moment control (DYC). The corrective yaw moment and active steering angle are generated online without knowing system dynamics, which is realised by using a neural network (NN) identifier to identify the unknown system dynamics and a critic NN to calculate the optimal control action, respectively. Control commands are executed via active steering angle on front wheels and proper brake torque distribution on the effective wheels. Computer simulations under three different driving manoeuvres, i.e., lane change manoeuvre, step steer manoeuvre and sine with dwell manoeuvre, are carried out to evaluate the proposed control method. Simulation results show that the proposed ADP-based control method demonstrates improved tracking performance in terms of enhancing vehicle handling and stability performance when encountering the varying longitudinal velocity, the uncertain cornering stiffness and the different road/tyre friction coefficients. Model-free and self-adaptive properties of the proposed method provide a new solution to vehicle stability controller design instead of the commonly used model-based methods.

Keywords: approximation dynamic programming; ADP; vehicle stability control; active front wheel steering; direct yaw moment control; DYC; optimal tracking control.

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1 Introduction

Many advanced chassis control systems have been developed aiming to improve vehicle handling and stability, such as anti-lock braking system (ABS) (Imine et al., 2012; Do et al., 2014), active front steering (AFS) (Lian et al., 2015) and electronic stability program (ESP) (Yim et al., 2012). AFS can apply an additional steering angle to the driver’s steering command and influence the vehicle lateral dynamic behaviour directly by modulating the tyre lateral forces. However, AFS is not able to produce enough tyre lateral forces when tyres enter the nonlinear regions and hence the performance of AFS is limited in linear handling region. On the other hand, direct yaw moment control (DYC) based on active braking is very effective for vehicle stability in both linear and nonlinear handling regions (Zhou et al., 2010). With DYC technique, a proper amount of corrective yaw moment is produced to keep vehicle stability through differential braking between the left and right sides of the vehicle (Hsu and Chen, 2013; Fallah et al., 2013). DYC, however, is only desirable in limited handling because of tyre wear and significant deceleration caused by braking. It can be seen that AFS and DYC techniques are designed and optimised individually to achieve a specific safety performance. Recently, integrated AFS/DYC control has become a very active field in vehicle stability control studies (Ahn et al., 2012; Aripina et al., 2014; Zhang and Wang, 2015; Yim et al., 2015). The main objective of integrated AFS/DYC control method is to track the desired yaw rate and sideslip angle with the aim of achieving the satisfactory stability performance under different driving manoeuvres. The tyre cornering stiffness parameter and longitudinal speed are usually assumed to be constant in most of the existing integrated AFS/DYC controller designs (Canale et al., 2008; Li et al., 2016; Karbalaei et al., 2007). To overcome the influence of the uncertain parameters, some control methods have been used for integrated AFS and DYC controller design. Doumiati et al. (2013) and Çaglar et al. (2011) design a robust AFS and differential braking controller based on LMI theory to reduce the influence of the tyre cornering stiffness uncertainty on the system performance. The fuzzy logic controllers adopted by Li et al. (2010) and Fan et al. (2016)
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can effectively control the sideslip angle and yaw rate and the vehicle trajectory following performance is satisfied. He and Ji (2012) design a robust nonlinear global chassis controller based on robust theory to reduce the influence of the external uncertainty and disturbance on the system performance. Considering the tyre cornering stiffness uncertainty, Ding and Taheri (2010) propose an adaptive integrated algorithm for AFS and direct yaw moment. The control methods discussed above mainly focus on ensuring the robust stability of the system to a certain extent but they cannot guarantee the optimal performance in terms of the minimisation of trajectory error and energy consumption. The quadratic optimal control method (QOC) (Katrinio et al., 2013; Waschi et al., 2014; Tavan et al., 2015) not only can realise the tracking control of the desired yaw rate and sideslip angle but also can realise the minimisation of the specified cost function with predetermined weighting coefficients. However, the main drawback of the conventional QOC method lies in that the two degrees of freedom (2-DOF) single track vehicle model has to be known for solving Hamilton-Jacobi-Bellman (HJB) equation in an off-line manner to find the optimal control law. Once the feedback gains of the controller are obtained, it cannot be changed with the uncertain parameters, which may influence the stability control performance due to the system uncertainties in the real environment (Yang et al., 2009; Wang et al., 2016).

Recent research has shown that approximate dynamic programming (ADP)-based controller design can overcome the requirement for an exact model and meanwhile achieve the optimal control (Lewis and Liu, 2013). It is well known that the ‘curse of dimensionality’ problem and offline learning property of the dynamic programming (DP) make it inefficient for the existing DP-based optimal control strategies. ADP, originated from biological systems, can enormously increase the computational power of resolving optimal control problem, while not introducing prominent approximation inaccuracies. Since Werbos (1992) introduced the general actor-critic (AC) framework for ADP-based control design, various modifications to ADP have been proposed, such as heuristic dynamic programming (HDP) (Miller et al., 1990), dual heuristic programming (DHP) (Fairbank et al., 2012), action dependent heuristic dynamic programming (ADHDP) (Zhu et al., 2015) and Q-learning (Wei et al., 2015). To extend the existed ADP studies from the discrete-time domain to the continuous-time domain, an online optimal control algorithm with the actor-critic-identifier (ACI) architecture is proposed to solve the nonlinear continuous-time infinite horizon optimal control problem without the knowledge of system drift dynamics, but the knowledge of the input dynamics is required (Bhasin et al., 2013). Then, adaptive optimal control methods with the simplified ACI structure are proposed (Yang et al., 2014; Lv et al., 2016), where just one critic NN instead of the action-critic dual network is used to approximate the solution of the HJB equation and calculate the optimal control action, which greatly simplifies the algorithm architecture. It should be noted that most of the aforementioned ADP-based optimal control studies focus on dealing with the regulation issues rather than optimal tracking control problem. As far as we know, only recent research proposed by Kamalapurkar et al. (2015) addressed the optimal tracking control problem, where approximate optimal tracking control consisting of a steady-state control and an optimal feedback control are designed. However, it is based on the complex ACI architecture which increases the online computation. In addition, the knowledge of the input dynamic is also required, which is a rather stringent requirement for practical implementation.
In this paper, a novel ADP-based adaptive optimal tracking control algorithm with a critic-identifier framework (a simplified form of ACI) is proposed for the integrated AFS/DYC controller design. First, a recurrent NN identifier with the online updating laws for the weights and the linear part matrices is used to identify the unknown system dynamics by using the available inputs/outputs data. Then, with the obtained identification model, the optimal tracking control law is obtained by using the ADP approach. The learning laws of the identifier and the critic NN of the developed ADP algorithm are online updated simultaneously and the stability of the overall closed-loop system is proved via Lyapunov approach. The model-free and self-adaptive properties of the proposed ADP-based optimal tracking control method makes it possible for developing a high performance integrated AFS/DYC controller which is robust to the unknown system dynamics. Simulation results from a nonlinear vehicle model considering the varying longitudinal velocity, the uncertain cornering stiffness and the different road conditions under three different driving manoeuvres verify the improved performance of the proposed ADP-based control method compared with the model-based QOC method.

The rest of the paper is organised as follows. Section 2 presents a 7-DOF nonlinear vehicle model with Dugoff tyre model. The problem formulation is given in Section 3. ADP-based optimal tracking control method for integrated AFS/DYC system is developed in Section 4. Simulation results with the parameter varying 7-DOF nonlinear vehicle model under different manoeuvres are presented in Section 5. Finally, the conclusions are drawn in Section 6.

### 2 Vehicle model

In this paper, a 7-DOF nonlinear vehicle model incorporating longitudinal and lateral tyre forces calculated from Dugoff tyre model is used to verify the effectiveness of the proposed control algorithm, as shown in Figure 1 (Osborn and Shim, 2006). This model ignores heave, roll and pitch motions but considers the lateral and longitudinal load transfers. With the assumption that the required yaw moment can be realised through the distribution of brake torques and the steering angles of both front wheels are equal, the motion equations consisting of the external forces acting on the vehicle body in the longitudinal, lateral axes and the moments acting around the vertical axis can be written as

\[
\begin{align*}
\dot{v}_x &= (F_{s1} + F_{s2}) \cos(\delta_f) + F_{s3} + F_{s4} - (F_{s1} + F_{s2}) \sin(\delta_f) + m v_y \\
\dot{v}_y &= F_{s3} + F_{s4} + (F_{s1} + F_{s2}) \sin(\delta_f) + (F_{s1} + F_{s2}) \cos(\delta_f) - m v_x \\
I_{yy} &= I_f (F_{s1} + F_{s2}) \sin(\delta_f) + l_f (F_{s1} + F_{s2}) \cos(\delta_f) - l_s (F_{s1} + F_{s2}) \\
&+ \frac{l_f}{2} (F_{s2} + F_{s3}) + \frac{l_f}{2} (F_{s4} + F_{s3}) + \frac{l_s}{2} (F_{s2} - F_{s3}) \sin(\delta_f) \\
\end{align*}
\]  

(1)

The vertical load acting on each wheel can be expressed as
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\[ F_{x1} = \frac{mgl_i}{2(l_f + l_i)} - \frac{ma_h}{2(l_f + l_i)} + \frac{ma_h}{2r} \]
\[ F_{x2} = \frac{mgl_i}{2(l_f + l_i)} - \frac{ma_h}{2(l_f + l_i)} + \frac{ma_h}{2r} \]
\[ F_{x3} = \frac{mgl_i}{2(l_f + l_i)} - \frac{ma_h}{2(l_f + l_i)} + \frac{ma_h}{2r} \]
\[ F_{x4} = \frac{mgl_i}{2(l_f + l_i)} - \frac{ma_h}{2(l_f + l_i)} + \frac{ma_h}{2r} \] (2)

Magic formula tyre model (Bakker et al., 1987) is one of the well-known and widely used tyre models to reflect the highly nonlinear and complex tyre force generation in vehicle simulation. It is a semi-empirical model in which the tyre coefficients are determined from a curve that is fitted to experimental data. Although this model yields realistic tyre behaviour, it requires many tyre coefficients that need to be determined by experiments. Unlike this model, Dugoff tyre model requires relatively few parameters and is capable of simulating the tyre forces under pure cornering, pure braking and combined braking/cornering manoeuvres (Dugoff et al., 1969). The significant advantages of Dugoff tyre model in the controller’s development stage are reported by Tankut (2009) and Farzad et al. (2015). In this study, the Dugoff tyre model is selected to calculate the longitudinal tyre forces \( F_{xwi} \) and lateral tyre forces \( F_{ywi} \), such that

\[ F_{xwi} + C_{\sigma_i} \frac{\sigma_i}{1 + \sigma_i} f(\lambda) \]
\[ F_{ywi} + C_{\mu} \tan \alpha_i \frac{\sigma_i}{1 + \sigma_i} f(\lambda) \] (3)

where
\[ \lambda = \frac{\mu F_{\sigma}}{2 \left( C_{\sigma_i} (1 + \sigma_i) \right)^2 + (C_{\mu} \tan \alpha_i)^2} \], \((i = 1, 2, 3, 4); f(\lambda) = \begin{cases} (2 - \lambda)^{\lambda} & (\lambda < 1) \\ 1 & (\lambda > 1) \end{cases} \]

The slip angle at each tyre can be defined as
\[ \alpha_1 = \delta_f - \tan^{-1} \frac{v_x + l_f \gamma}{v_x - 0.5 \gamma} , \alpha_2 = -\tan^{-1} \frac{v_x - l_f \gamma}{v_x - 0.15 \gamma} \]
\[ \alpha_3 = \delta_f - \tan^{-1} \frac{v_x + l_f \gamma}{v_x + 0.5 \gamma} , \alpha_4 = -\tan^{-1} \frac{v_x - l_f \gamma}{v_x + 0.15 \gamma} \] (5)

The wheel slip ratio at each tyre is described as
\[ \sigma_i = \frac{R_o \omega_{rot} - v_x}{\text{max}(R_o \omega_{rot}, V_o)} \] (6)

The wheel rotation dynamics can be given as
\[ J_{rot} \dot{\omega}_{rot} = T_{des} - T_{hub} - R_o F_{x} \] (7)
The tyre forces $F_{xi}$ and $F_{yi}$ developed along the body fixed axis, as shown in Figure 1, are calculated using the following transformation

$$
\begin{bmatrix}
F_{xi} \\
F_{yi}
\end{bmatrix} =
\begin{bmatrix}
\cos \delta_i & -\sin \delta_i \\
\sin \delta_i & \cos \delta_i
\end{bmatrix}
\begin{bmatrix}
F_{xoi} \\
F_{yoi}
\end{bmatrix}
$$

(8)

where a front steering vehicle is considered, i.e., $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$.

The parameter notations mentioned above are described in Appendix 1.

Figure 1 7-DOF nonlinear vehicle model (see online version for colours)

3 Problem formulation

Integrated AFS/DYC control is usually based on the two-level control structure (Yang et al., 2009; Katriniok et al., 2013; Waschi et al., 2014; Tavan et al., 2015). The upper level control aims to make the actual yaw rate and sideslip angle follow the desired handling performance determined by a 2-DOF reference vehicle model and produce the control command. The appropriate brake torque distribution scheme designed in the lower level control is used to execute the control command from the upper level control. This paper mainly focuses on the upper level control design. According to the previous studies (Li and Yu, 2010; Zhang and Wang, 2015; Li et al., 2016), 2-DOF linear vehicle model is usually selected as the reference model to generate the desired yaw rate and sideslip angle. The reference model can be written in the state-space format, such that

$$
\dot{x}_r = A_r x_r + E_r \delta_f
$$

(9)

where

$$
E_r =
\begin{bmatrix}
1 - \frac{m l_f}{2(l_f + l_r)} l_C \nu_s^2 & \frac{v_s}{l_f + l_r} & v_s \\
\frac{m}{(l_f + l_r)} \left( \frac{l_f}{2C_r} - \frac{l_r}{2C_f} \right) \nu_s^2 & \frac{1}{l_f + l_r} & \left( \frac{l_f}{2C_r} - \frac{l_r}{2C_f} \right) \nu_s
\end{bmatrix}^T,
$$
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\[ x_r = [\beta, \gamma] , A_r = \begin{bmatrix} \frac{1}{\tau_{\beta}} & 0 \\ 0 & -\frac{1}{\tau_{\gamma}} \end{bmatrix} , \]

\( \tau_r \) and \( \tau_{\beta} \) are the designed time constants for yaw rate and sideslip angle, respectively.

Most of the control models for the existing integrated AFS/DYC control designs are based on the prior known linear 2-DOF vehicle model (Li and Yu, 2009; Yang et al., 2009; Katriniok et al., 2013; Waschi et al., 2014; Zhang and Wang, 2015; Tavan et al., 2015). However, the deviation between the control model (linear 2-DOF vehicle model) and the real vehicle model (nonlinear vehicle model) may affect the control accuracy. Moreover, the feedback control law is usually obtained by solving Riccati equation or certain LMI offline. Once it is obtained, the control law cannot be updated online when subjected to the model uncertainties. Thus, the controller may lose stability when the tire cornering stiffness, vehicle speed and road surface adhesion coefficients vary in a large range under some critical driving manoeuvres (Aripina et al., 2014). Therefore, more advanced control method that guarantee the performance in terms of accurate tracking and tolerance to various operating conditions should be developed for the integrated AFS/DYC controller design.

According to the literature reviews in the first section, ADP-based control approach provides a novel solution for online optimal control of the model uncertain nonlinear system, which inspires the study in this paper. The proposed ADP-based optimal control structure for the integrated AFS/DYC control system is showed in Figure 2.

**Figure 2** Schematic diagram of the integrated AFS/DYC control

The reference model in equation (9) is used to produce the desired yaw rate \( \gamma_r \) and the desired sideslip angle \( \beta_r \) based on the sensor signals of the driver command \( \delta_f \) and longitudinal velocity \( v_x \). ADP-based optimal controller is used to generate the active steering angle \( \delta_c \) and corrective yaw moment \( M_c \), such that the actual yaw rate \( \gamma \) and the
actual sideslip angle $\beta$ can track the desired yaw rate $\gamma$, and the desired sideslip angle $\beta_r$, respectively. The active steering angle $\delta_c$ together with the front wheel steering angle $\delta_f$ are regarded as the total steering angle acting on the front wheels. The generated corrective yaw moment $M_c$ from the ADP-based optimal controller will be implemented via the proper distribution of brake torques on four wheels. The brake torque distribution scheme, directly obtained from Yang et al., (2009) is employed, considering different situations corresponding to potential yaw instability denoted as understeer (yaw rate error $> 0$) and oversteer (yaw rate error $< 0$). This paper mainly focuses on the upper level control design, i.e., ADP-based optimal controller design which is introduced in detail in the following section.

4 ADP-based adaptive integrated AFS/DYC control algorithm

In this section, an adaptive integrated AFS/DYC control method is realised by designing an ADP-based optimal tracking control method with the identifier-critic structure in two steps: First, establish an adaptive recurrent NN identifier to identify the uncertain nonlinear system; Second, design the adaptive optimal tracking controller based on the recurrent NN identifier and the ADP approach.

4.1 Identifier design

Consider the following general nonlinear system

$$\dot{x} = f(x, u, t) + \xi_0$$  (10)

where $x \in \mathbb{R}^n$, $f(x, u, t) : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ is the unknown nonlinear dynamic, $u \in \mathbb{R}^p$ is the control input and $\xi_0$ is the unknown disturbance. In this paper, $x = [\beta \gamma]^T$, $u = [\delta_c M_c]^T$.

Then, equation (10) can be identified via the following recurrent NN model

$$\dot{x}(t) = \tilde{A} \dot{x}(t) + \tilde{W}_1 \sigma(\tilde{V}_1 \tilde{x}(t)) + \tilde{W}_2 \psi(\tilde{V}_2 \tilde{x}(t))u(t)$$  (11)

where $\tilde{x}(t) \in \mathbb{R}^n$ is the state of the recurrent NN, $\tilde{W}_1 \in \mathbb{R}^{n \times n}$, $\tilde{W}_2 \in \mathbb{R}^{n \times n}$ are the weights in the output layers, $\tilde{V}_1 \in \mathbb{R}^{n \times n}$, $\tilde{V}_2 \in \mathbb{R}^{n \times n}$ are the weights in the hidden layer, $\tilde{A} \in \mathbb{R}^{n \times n}$ is the matrix for the linear part of neural networks (NNs), $u = (\delta_c, M_c, 0, 0)^T \in \mathbb{R}^p$ is the control input, the active functions $\sigma(\cdot)$ (as well as $\psi(\cdot)$) is sigmoidal vector function which is defined as $\sigma(\cdot) = a / (1 + e^{-b \cdot \cdot})$, where $a, b, c$ are designed constants.

Li and Yu (2002) prove that recurrent NN with the form in equation (11) can approximate model uncertain system in equation (10) to any degree of accuracy if the hidden layers are large enough. To simplify the analysis process and reduce online computing time, the simplest structure reported by Fu et al. (2013) is considered here, i.e., $\tilde{V}_{1,2} = I$, $\psi(\cdot) = I$. Then, we know that there definitely exists a nominal recurrent NN model, which can approximate (10) and is defined as

$$\dot{x}(t) = A' x(t) + W'_1 \sigma(x(t)) + W'_2 u(t) + \xi_1$$  (12)
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A, W\(^1\), W\(^2\) are the nominal unknown matrices, W\(^1\), W\(^2\) are bounded as W\(^1\)\(\Lambda^1\)W\(^1\)\(^T\) \(\leq\) \(\overline{W}\), W\(^2\)\(\Lambda^2\)W\(^2\)\(^T\) \(\leq\) \(\overline{W}\) (\(\Lambda^1\), \(\Lambda^2\) are any positive definite symmetric matrices), \(\xi_1\) is regarded as model error and assumed to be bounded.

**Assumption 1:** The difference of the activation function \(\tilde{\sigma} = \sigma(x(t)) - \sigma(\hat{x}(t))\) satisfies the generalised Lipshitz condition, \(\tilde{\sigma}^T \Lambda \tilde{\sigma} < \Delta x^T D \Delta x\), where \(\Delta x = x(t) - \hat{x}(t)\) is the identification error, \(D = D^T > 0\) is the known normalising matrix.

Then, from equations (11) and (12), one can obtain the error dynamic equation as below,

\[\Delta \dot{x} = A^T \Delta x + \tilde{A} \tilde{x}(t) + W_1^\prime \tilde{\sigma} + \tilde{W}_1 \sigma(\hat{x}(t)) + \tilde{W}_2 u(t) + \xi_1\]

(13)

where \(\tilde{A} = A^\prime - \hat{A}\), \(\tilde{W}_1 = W_1^\prime - \hat{W}_1\), \(\tilde{W}_2 = W_2^\prime - \hat{W}_2\).

**Theorem 1:** Based on Lyapunov’s direct method, we have the following updating law

\[
\begin{align*}
\dot{\hat{A}} &= -q_1 \Delta x \tilde{x}^T, \\
\dot{\hat{W}}_1 &= -q_2 \Delta x \sigma_1^T(\hat{x}), \\
\dot{\hat{W}}_2 &= -q_3 \Delta x u^T
\end{align*}
\]

(14)

where \(q_1, q_2, q_3\) are positive constants. Considering the recurrent NN identifier equation (11) for equation (10) with equation (14) can guarantee the model error to be uniformly asymptotically stable without modelling error and uniformly ultimately bounded (UUB) for bounded modelling error.

Proof of Theorem 1 is given in Appendix 2.

### 4.2 Optimal tracking controller design

According to the identifier design above, we know that model uncertain system in equation (10) can be represented by the recurrent NN with the updating laws (14), such that

\[
\dot{x} = \hat{A} \tilde{x} + \hat{W}_1 \sigma(\hat{x}) + \hat{W}_2 u + \xi_2
\]

(15)

where the modelling error \(\xi_2\) is still assumed to be bounded, i.e., \(\xi_2 \leq \overline{\xi}_2\).

To achieve optimal tracking control, the control action \(u\) is designed as, \(u = u_r + u_e\), where \(u_r = [\hat{\delta}_x M_{cx}]^T\), \(u_e = [\hat{\delta}_e M_{ce}]^T\). The steady state control \(u_r\) can ensure that the steady state tracking error converge to zero and the adaptive optimal control \(u_e\) is used to stabilise the tracking error dynamics in an optimal manner. Steady state control \(u_r\) should be designed to compensate for the nonlinear dynamic in equation (15). Hence, let \(u_r\) be

\[
u_r = \left[\hat{\delta}_x M_{cx}\right]^T = \hat{W}_2^\oplus \left[\dot{x}_d - \hat{A} \tilde{x} - \hat{W}_1 \sigma(\hat{x}) - Ke\right]
\]

(16)

where \(e = x_d - x\) denotes the state tracking error, \(K\) is the feedback gain and \(\hat{W}_2^\oplus\) denotes the generalised inverse of \(\hat{W}_2\).

By substituting equation (16) into equation (15), one can obtain the error dynamic equation

\[
\dot{e} = -Ke + \hat{W}_2 u_e + \xi_2
\]

(17)
In this case, the tracking problem with equation (15) is transferred to the regulator problem of equation (17). Hence, the infinite horizon performance cost function is defined as

\[ V(e(t)) = \int_{e}^{\tau} r(e(\tau), u_e(e(\tau)))dt \]  

(18)

where \( r(e, u_e) = e^T Q e + u_e^T R u_e \) is the utility function with adaptive optimal control \( u_e \).

According to the optimal regulator problem design (Khalaf and Lewis, 2005), an admissible control policy \( u_e \) should be designed to ensure that infinite horizon cost function in equation (18) related to equation (17) is minimised. The Hamiltonian of equation (17) is thus given by

\[ H(e, u_e, V) = V_e^T \left[ -Ke + \hat{W}_2 u_e + \xi_2 \right] + e^T Q e + u_e^T R u_e \]  

(19)

where \( V_e = \frac{\partial V(e)}{\partial e} \) is the partial derivative of the value function with respect to \( e \).

Then, the optimal cost function is defined as

\[ V^*(e(t)) = \min_{u_e \in \Omega} \left( \int_{e}^{\tau} r(e(\tau), u_e(e(\tau)))dt \right) \]  

(20)

and it satisfies the following HJB equation

\[ \min_{u_e \in \Omega} \left[ H(e, u^*_e, V^*) \right] = 0 \]  

(21)

Last, the feedback optimal control value \( u^*_e \) for equation (17) can be obtained by solving

\[ \frac{\partial H(e, u^*_e, V^*)}{\partial u_e} = 0 \]  

from equations (19) and (20)

\[ u^*_e = -\frac{1}{2} R^{-1} \hat{W}_2 \frac{\partial V^*(e)}{\partial e} \]  

(22)

From equation (22), one can learn that optimal control value \( u^*_e \) is based the optimal value function \( V^*(e) \). However, it is difficult to solve the nonlinear partial differential HJB equation (19) to obtain \( V^*(e) \). The commonly used method is to get the approximate solution via a critic NN (Fairbank et al., 2012; Bhasin et al., 2013; Yang et al., 2014; Kamalapurkar et al., 2015; Zhu et al., 2015; Wei et al., 2015; Lv et al., 2016). Hence, a single layer NN is be used to approximate the optimal value function

\[ V^*(e) = W^*_2^T \phi(e) + \xi_3 \]  

(23)

and its derivative is

\[ \frac{\partial V^*(e)}{\partial e} = \nabla \phi(e) W^*_2 + \nabla \xi_3 \]  

(24)
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where $W^*_i \in \mathbb{R}^I$ is the nominal weight vector, $\phi(e) \in \mathbb{R}^I$ is the active function and is $\xi_3$ is the approximation error, $I$ is the number of neurons. $\nabla(e) = \frac{\partial \phi(e)}{\partial e}$ and $\nabla \xi_3 = \frac{\partial \xi_3}{\partial e}$ are the partial derivative of $\phi(e)$ and $\xi_3$ with respect to $e$, respectively.

Assumption 2: The nominal weight vector $W^*_i$, the active function $\phi(e)$ and its derivative $\nabla \phi(e)$ are all bounded, i.e.$\|W^*_i\| \leq \overline{W}, \|\phi(e)\| \leq \overline{\phi}, \|\nabla \phi(e)\| \leq \overline{\nabla \phi}, \|\nabla \xi_3\| \leq \overline{\xi}$.

Then, substituting equation (24) into equation (22), one obtains

$$u^*_i = -\frac{1}{2} R^{-1} \hat{W}^T \nabla \phi(e)W^*_i + \nabla \xi_3 \tag{25}$$

In practical implementation, the critic NN is represented as

$$V(e) = \hat{W}^T \phi(e) \tag{26}$$

where $\hat{W}_i$ is the estimation of nominal $W^*_i$.

Then, the approximate optimal control can be obtained from equations (25) and (26)

$$u_e = \left[ \delta_e \ M_e \right] - \frac{1}{2} R^{-1} \hat{W}^T \nabla \phi(e) \hat{W}_i \tag{27}$$

In order to obtain the optimal control action from equation (27), we need to design the updating law of $\hat{W}_i$. Thus, substituting equation (23) into equation (19), one obtains

$$0 = W^*_i \nabla \phi(e) \left[ -Ke + \hat{W}^T \delta_{s} + e^T \xi_3 \right] + e^T \xi_3 \left[ -Ke + \hat{W}^T \delta_{s} + e^T \xi_3 \right] + \xi_{HJB}^T \tag{28}$$

where $\xi_{HJB} = W^*_i \nabla \phi(e) \hat{W}_i + \nabla \xi_3 \left[ -Ke + \hat{W}^T \delta_{s} + e^T \xi_3 \right]$ is the residual HJB equation error due to the recurrent NN identifier error $\hat{W}_i$ and NN approximation error $\nabla \xi_3$.

From the basic online parameter identification procedure (Ioannou and Sun, 1996), equation (28) can be expressed in the compact form below

$$Y = -W^*_i \nabla \phi(e) \left[ -Ke + \hat{W}^T \delta_{s} \right] + e^T \xi_3 \left[ -Ke + \hat{W}^T \delta_{s} + e^T \xi_3 \right] \tag{29}$$

where $X = \nabla \phi(e) \left[ -Ke + \hat{W}^T \delta_{s} \right], Y = e^T \xi_3 \left[ -Ke + \hat{W}^T \delta_{s} + e^T \xi_3 \right]$.

Note that equation (29) is a general parametric model. There are many ways to estimate parameter $*3W$ with the general estimation methods such as least squares (Kamalapurkar et al., 2015; Sardarmehni and Heydari, 2015) and gradient method (Bhasin et al., 2013). In those methods, the estimation error between the outputs of the estimation model and parametric model is used to drive the adaptive law that generates $\hat{W}_i$ online. They are designed by minimising the estimation error, but cannot guarantee the convergence of the estimation parameter $\hat{W}_i$ to its nominal value $W^*_i$. However, it has been proved that the convergence of the critic NN weight is important for the convergence of the ADP-based optimal controller (Modares et al., 2013; Lv et al., 2016). Meanwhile, it has been well-recognised in the control research area that the parameter adaptive laws should include some information on the parameter estimation error to
improve the convergence performance (Bechlioulis et al., 2008; Adetola et al., 2014; Li et al., 2014; Na et al., 2015). Inspired by these results, a novel robust estimation method of $W^*$. Considering the parameter error information in this paper is presented for the critic NN instead of the commonly used least squares or gradient method.

Thus, we define the auxiliary regression matrix $E \in \mathbb{R}^{n \times l}$ and vector $F \in \mathbb{R}^l$ as

$$
\begin{align*}
\dot{E}(t) &= -\eta E(t) + XX^T, \quad E(0) = 0 \\
\dot{F}(t) &= -\eta F(t) + XY, \quad F(0) = 0
\end{align*}
$$

where $\eta$ is a positive constant.

Another auxiliary vector $M$ based on $E$ and $F$ is defined as

$$M = E(t)\hat{W}_3 + F(t)$$

Then, the adaptive law for updating $\hat{W}_3$ is provided by

$$\dot{\hat{W}}_3 = -\rho M$$

where $\rho$ is the learning rate.

**Figure 3** Flowchart of the proposed control algorithm
Theorem 2: For system (10) with the adaptive optimal control u from equations (16) and (27) and adaptive laws in equations (14) and (32), the tracking error e is UUB and the optimal control \( u_e \) in equation (27) converges to a small bound around its ideal optimal solution \( u_e^* \) in equation (25). The overall closed-loop signals are all UUB during the whole learning process. Please refer to Appendix 2 for the detailed proof of Theorem 2. Based on the above analysis, the flowchart of the proposed control algorithm is depicted in Figure 3 and can be summarised as follows.

Step 1 Select the proper initial values of active functions \( \sigma(\cdot) \) in equation (11) and updating gains \( q_1, q_2, q_3 \) in equation (14) for the identifier. \( \sigma(\cdot) \) is usually selected as the sigmoidal function \( \sigma(\cdot) = a / (1 + e^{-bx}) - c \). Where \( a, b, c \) are the designed constants. \( \hat{A}, \hat{W}_1, \hat{W}_2 \) and \( \hat{W}_3 \) are tuned online according equations (14) and (32). Hence, there is no need to select the initial values of \( \hat{A}, \hat{W}_1, \hat{W}_2 \) and \( \hat{W}_3 \). Meanwhile, select the proper function \( \phi(\cdot) \) in equation (26) and the updating gain \( \rho \) in equation (32) for the critic NN. \( \phi(\cdot) \) is usually selected as a smooth function consisting with the different combination between states tracking error.

Step 2 The inputs/outputs data is used to train the NN including the recurrent NN identifier in equation (11) and critic NN in equation (26) and produce the reference model in equation (9). Here, the driver command \( \delta_f \), the actual yaw rate \( \gamma \) and the actual sideslip angle \( \beta \) are used to train the identifier.

Step 3 Adaptive optimal tracking control law consisting of the steady-state control law in equation (16) and the optimal feedback control law in equation (27) is obtained based on the first two steps.

5 Simulation results

Simulations are carried out on a 7-DOF nonlinear vehicle model platform presented in the second section to evaluate the effectiveness of the proposed ADP-based adaptive integrated AFS/DYC control method. The main vehicle parameter set is listed in Table 1 (Yang et al., 2009).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>1,704</td>
<td>Kg</td>
</tr>
<tr>
<td>( I_z )</td>
<td>3,048</td>
<td>Kgm²</td>
</tr>
<tr>
<td>( I_y )</td>
<td>1.035</td>
<td>m</td>
</tr>
<tr>
<td>( I_r )</td>
<td>1.655</td>
<td>m</td>
</tr>
<tr>
<td>( C_f )</td>
<td>105,850</td>
<td>N/rad</td>
</tr>
<tr>
<td>( C_r )</td>
<td>79,030</td>
<td>N/rad</td>
</tr>
<tr>
<td>( t )</td>
<td>1.535</td>
<td>m</td>
</tr>
</tbody>
</table>
The aim of vehicle stability control is to minimise the vehicle sideslip angle $\beta_r$, which is usually set to zero, together with tracking the desired yaw rate $\gamma_r$ which is calculated from equation (11). The model-based QOC control law based on the linear 2-DOF vehicle model was derived by solving a set of LMI offline (Yang et al., 2009), such that

$$u = \begin{bmatrix} -0.6624 & 0.7342 \\ -5172.25 & 5464.00 \end{bmatrix} \epsilon.$$

It should be noted that, once the control law is obtained, the gains of control law cannot be updated online with the changing of external environment for the QOC method. While the proposed ADP-based adaptive control method (ADP for short) can successfully overcome this shortcoming with the online adaptive control law in equations (16) and (27), which is realised using an online recurrent NN identifier to identify the vehicle model and a critic NN to identify the optimal value function, respectively.

Sigmoid function is the most popular and suitable activation function for dynamic NN owing to its flexibility on the shape and the smoothness in the en tyre domain. The parameters $a$, $b$, $c$ in the sigmoid function are selected according to the property of the sigmoid function (Poznyak et al., 2001). In general, $a$, $b$, $c$ should not be too large to achieve smooth nonlinear approximation effect. The inputs/outputs data of the nonlinear vehicle model under different conditions is used to train the NN identifier and obtain the proper parameters $q_1$, $q_2$, $q_3$ and $a$, $b$, $c$. In this paper, by applying the different steer inputs to the nonlinear vehicle model, one can get the corresponding sideslip angle $\beta$ and the yaw rate $\gamma$. In order to show the convergence property of the proposed NN identifier, the identification results of the sideslip angle $\beta$ and the yaw rate $\gamma$ of the vehicle with a sinusoidal steering input $\delta_s = 5\sin(\pi/2t)$ at the longitudinal velocity 16.67 m/s are shown in Figure 4. One can see that the proposed identifier demonstrates faster convergence rate in less 0.5 s. No matter how much the initial weights are the weights can be tuned online in this paper. Here, the initial weights are set to zero.

**Figure 4** Identification results (see online version for colours)

From ADP theory, we know that the critic NN basis set is usually selected as the quadratic vector in the state components $\phi = [e_1^2, \epsilon e_1, \epsilon_1^2]$. The feedback control gain $\rho$ is determined using trial-and-error methods as in the general adaptive control design. Here, we select $\rho = 100$. 
Three different manoeuvres, i.e., lane change manoeuvre, step steer manoeuvre and sine with dwell manoeuvre, are performed in this paper to validate the proposed ADP method. Comparisons between ADP method and QOC method are given in terms of improving vehicle handling and stability performance. Moreover, time varying parameters of the cornering stiffness and longitudinal velocity are also considered in the simulations.

5.1 Lane change manoeuvre

In the lane change manoeuvre, the vehicle is simulated with a sinusoidal steering input ($\delta_s = 5\sin(\pi / 2t)$) at the initial longitudinal velocity ($22.22 \text{ m/s}$). The road friction coefficient is 0.8 (dry road). The uncertainties of tyre cornering stiffness are expressed as follows:

\[
\begin{align*}
C_f &= C_{f0} (1 + \Delta_f \rho_f), & \|\rho_f\| \leq 1 \\
C_r &= C_{r0} (1 + \Delta_r \rho_r), & \|\rho_r\| \leq 1
\end{align*}
\]

where $C_{f0}$, $C_{r0}$ and $C_f$, $C_r$ are the nominal and actual cornering stiffness of the front and rear axle tyres, respectively, $\Delta_f$ and $\Delta_r$ are the deviation magnitudes, $\rho_f$ and $\rho_r$ are the perturbations. Figure 5 shows the comparison results at different uncertainty deviation magnitude ($\Delta_f = \Delta_r = 0$ and $\Delta_f = \Delta_r = 0.5$) with the band-limited white noise ($\rho_f = \rho_r = 0.2$) as the perturbations. One can see from Figure 5 that the sideslip angle and yaw rate of the uncontrolled vehicle become smaller with the increasing of deviation magnitude of cornering stiffness which demonstrates the effects of uncertain cornering stiffness on the vehicle dynamic response. Meanwhile, compared with the uncontrolled vehicle, the vehicle performance is improved noticeably with the two control methods (ADP and QOC). As shown in Figures 5(a) and 5(b), the sideslip angle response is suppressed effectively with the ADP and the QOC methods, but a much smaller sideslip angle for the proposed ADP method. Meanwhile, compared with the QOC method, the yaw rate response with the proposed ADP method shown in Figures 5(c) and 5(d) presents better tracking performance to the desired yaw rate when subject to uncertain cornering stiffness.

The vehicle control performance with the QOC method and the proposed ADP method at two different longitudinal vehicle speeds ($27.78 \text{ m/s}$ and $33.33 \text{ m/s}$) and the same deviation magnitudes of cornering stiffness ($\Delta_f = \Delta_r = 0.5$) is compared in Figure 6. One can easily find that the proposed ADP method still has better tracking performance with smaller tracking error and faster convergence rate to the steady state value when subject to different longitudinal speeds compared with the QOC method. That means the vehicle can be brought back to the desired path faster from runaway with the proposed ADP method. Note that the uncontrolled vehicle is unstable in this case. The control inputs for the ADP and the QOC methods under this manoeuvre with the longitudinal vehicle speed $27.78 \text{ m/s}$ are presented in Figure 7. It can be seen that the magnitude of the control input for the proposed ADP method is a little larger than that for the QOC method. It is reasonable in actual situation as long as the control effort does not exceed the actuator’s limit. In fact, there are also many optimal control methods to handle the actuator’s saturation problem (Liu et al., 2015; Yang et al., 2016). Furthermore, the
performance index in terms of root mean square (RMS) for the tracking errors in Figure 5 and Figure 6 are listed in Table 2. The RMS values of the proposed ADP method are smaller than that with the QOC method, which further shows the improved performance of the proposed ADP method.

Figure 5 Comparisons of sideslip angle and yaw rate response under lane change manoeuvre with uncertain cornering stiffness (see online version for colours)

Table 2 The RMS values for tracking errors (x10–4)

<table>
<thead>
<tr>
<th></th>
<th>ADP</th>
<th>QOC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_f = \Delta_r = 0$ Slip angle</td>
<td>0.01338</td>
<td>0.09088</td>
</tr>
<tr>
<td>Yaw rate</td>
<td>0.01742</td>
<td>2.52600</td>
</tr>
<tr>
<td>$\Delta_f = \Delta_r = 0.5$ Slip angle</td>
<td>0.01218</td>
<td>0.09233</td>
</tr>
<tr>
<td>Yaw rate</td>
<td>1.24200</td>
<td>3.18000</td>
</tr>
<tr>
<td>$V_x = 27.78$  Slip angle</td>
<td>0.03829</td>
<td>5.46700</td>
</tr>
<tr>
<td>Yaw rate</td>
<td>1.34500</td>
<td>9.30900</td>
</tr>
<tr>
<td>$V_x = 33.33$  Slip angle</td>
<td>0.08218</td>
<td>9.23200</td>
</tr>
<tr>
<td>Yaw rate</td>
<td>1.38700</td>
<td>14.4100</td>
</tr>
</tbody>
</table>
5.2 Step steer manoeuvre

Similarly, a 5-degree step steer manoeuvre with two different initial speeds (27.78 m/s and 33.33 m/s) and the same deviation magnitudes of cornering stiffness ($\Delta_f = \Delta_r = 0.5$) is simulated for further verifying the proposed ADP method. As shown in Figure 8, the proposed ADP method still demonstrates strong robustness and self-adaptive
performance, i.e., less tracking error for both yaw rate and sideslip angle, when encountering time varying longitudinal speed and cornering stiffness in step steer manoeuvre.

Figure 8  Comparisons results of sideslip angle and yaw rate response under step steer maneuver
(see online version for colours)

5.3 Sine with dwell manoeuvre

In this simulation scenario, the sine with dwell steer input as shown in Figure 9 and \( \mu \)-split surface situation are used to study the robustness properties of the proposed method. The vehicle runs at an initial velocity of 27.78 m/s on \( \mu \)-split road, i.e. On the left side \( \mu = 0.5 \) which denotes the wet road surface, on the right side \( \mu = 0.2 \) which denotes the ice road surface. The simulation results for sideslip angle and yaw rate response are given in Figure 10. It can be seen that the uncontrolled vehicle has lost stability in this scenario. The sideslip angle and the yaw rate of the QOC method deviate largely from the reference response values under this scenario, whereas, with the proposed ADP method, the sideslip angle and the yaw rate can still follow the reference response values well.

It should be mentioned that there are two main reasons for the performance improvements with the proposed ADP method in the simulation results above. First, the control law of the QOC method is based on the accurate linear 2-DOF vehicle model, the control accuracy cannot be guaranteed when vehicle parameters vary. However, the
Adaptive optimal control for integrated active front steering

The proposed ADP method is not limited to a mathematical model, but based on the inputs/outputs data of vehicle system. Second, the feedback control law of the proposed ADP method can be updated online with the time varying parameters like longitudinal speed and cornering stiffness, whereas for the QOC method (Yang et al., 2009) and many other existed optimal control methods (Katriniok et al., 2013; Waschi et al., 2014; Tavan et al., 2015), the feedback control law is fixed in advance. Therefore, it could be concluded that self adaptive property of the proposed ADP method provides a more effective solution for the integrated AFS/DYC controller design and can greatly enhance the vehicle handling and stability performances.

Figure 9  Sine with dwell steer input (see online version for colours)

![Figure 9](image)

Figure 10  Comparisons results of sideslip angle and yaw rate response under sine with dwell and μ-split manoeuvre (see online version for colours)

![Figure 10](image)

6 Conclusions

The main contribution of this paper is that a novel adaptive ADP-based optimal control algorithm based on integration of AFS/DYC is proposed for improving vehicle stability performance. The proposed method does not require the complete knowledge of system
dynamics, which demonstrates great superiority compared with the commonly used model-based controller design. Moreover, the optimal control law of the proposed method can be updated online with the external environment, whereas in many existed AFS/DYC control methods, the feedback control laws are obtained offline. Simulations under three different manoeuvres are performed on a nonlinear vehicle model with the varying longitudinal velocity, the cornering stiffness and the different road/tyre friction coefficients to verify the performance of the proposed control method. Simulation results demonstrate that the proposed ADP-based control method can achieve improved performance compared with the commonly used model-based QOC control method. The proposed method in this paper is also expecting to be extended to other active control systems. In the future, the driver-vehicle-road closed-loop system will be used to develop a more advanced adaptive optimal integrated control method for vehicle stability enhancement considering the states’ constraints and actuator saturation limits.

Acknowledgements

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References

Adaptive optimal control for integrated active front steering


Appendix 1

Nomenclature

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>Vehicle mass</td>
</tr>
<tr>
<td>I_z</td>
<td>Yaw moment of inertia</td>
</tr>
<tr>
<td>l_f, l_r</td>
<td>Distance from CG to front axle and rear axle</td>
</tr>
<tr>
<td>C_yi</td>
<td>Tyre lateral stiffness</td>
</tr>
<tr>
<td>C_xi</td>
<td>Tyre longitudinal stiffness</td>
</tr>
<tr>
<td>h</td>
<td>CG height</td>
</tr>
<tr>
<td>v_x, v_y</td>
<td>Vehicle longitudinal and lateral speed</td>
</tr>
<tr>
<td>a_x, a_y</td>
<td>Vehicle longitudinal and lateral acceleration</td>
</tr>
<tr>
<td>F_yf, F_yr</td>
<td>Combined front and rear tyre lateral force</td>
</tr>
<tr>
<td>F_zi</td>
<td>Normal force of ( i^{th} ) wheel</td>
</tr>
<tr>
<td>g</td>
<td>Gravity acceleration</td>
</tr>
<tr>
<td>R_w, J_w</td>
<td>Wheel rolling radius, moment of inertia</td>
</tr>
<tr>
<td>\omega_{ai}</td>
<td>Wheel angular speed</td>
</tr>
<tr>
<td>T_{bui}</td>
<td>Active brake torque</td>
</tr>
<tr>
<td>T_{dwi}</td>
<td>Driving torque</td>
</tr>
<tr>
<td>t</td>
<td>Wheel track width</td>
</tr>
<tr>
<td>\mu</td>
<td>Friction coefficient between tyre and road</td>
</tr>
<tr>
<td>\gamma</td>
<td>Yaw rate about z axis</td>
</tr>
<tr>
<td>\alpha_{i}, \sigma_{i}</td>
<td>The ( i^{th} ) wheel slip angle, slip ratio</td>
</tr>
<tr>
<td>\delta_f</td>
<td>Front wheel steering angle</td>
</tr>
</tbody>
</table>
Appendix 2

Stability analysis

The stability of the proposed control algorithm is proved using the Lyapunov function $L$ given by

$$L = L_I + L_o + L_c$$

where $L_I$, $L_o$, $L_c$ are defined as

$$L_I = \Delta x^T P \Delta x + q_1^1 tr \{ \hat{\mathbf{W}}_1^T P \hat{\mathbf{W}}_1 \} + q_2^1 tr \{ \hat{\mathbf{W}}_2^T \dot{P} \hat{\mathbf{W}}_2 \} + q_3^1 tr \{ \hat{\mathbf{W}}_3^T \dot{\mathbf{A}} \}$$

$$L_o = 2 \mathbf{W}_1^T \rho^{-1} \hat{\mathbf{W}}_3$$

$$L_c = \Gamma \mathbf{e}^T \mathbf{e} + \kappa \mathbf{V}^T (\mathbf{e})$$

where $q_1, q_2, q_3, \rho, \Gamma, \kappa > 0$ are positive constants, $P$ is the solution to the matrix differential Riccati equation.

From equation (13), the time derivative of $L$ is obtained as

$$\dot{L}_I = \Delta x^T ( A^T P + PA^T ) \Delta x + 2 \Delta x^T P \Delta \hat{x} + 2 \Delta x^T \hat{P} \Delta \hat{x} + 2 \Delta x^T \hat{P} \sigma ( \hat{x} ) + 2 \Delta x^T \mathbf{P} \hat{\mathbf{W}}_3^T \mathbf{u} + 2 \Delta x^T \mathbf{P} \hat{\mathbf{W}}_3^T \sigma ( \mathbf{e} ) + q_1^1 tr \{ ( \mathbf{A}^T P \mathbf{A} ) \} + q_2^1 tr \{ \hat{\mathbf{W}}_2^T \dot{P} \hat{\mathbf{W}}_2 \} + q_3^1 tr \{ \hat{\mathbf{W}}_3^T \dot{\mathbf{A}} \}$$

By using the updating laws (14) and taking the facts $\dot{\mathbf{A}} = - \hat{\mathbf{A}}, \dot{\mathbf{W}}_{1,2} = - \hat{\mathbf{W}}_{1,2}$ into consideration, equation (34) becomes

$$\dot{L}_I = \Delta x^T ( A^T P + PA^T ) \Delta x + 2 \Delta x^T \mathbf{P} \hat{\mathbf{W}}_3^T \sigma + 2 \Delta x^T P \sigma_1$$

The following matrix inequality is established (Han et al., 2011)

$$X^T Y + ( X^T Y )^T \leq X^T \Lambda^{-1} X + Y^T \Lambda Y$$

where $X, Y \in \mathbb{R}^{n \times k}$ are any matrices and $\Lambda \in \mathbb{R}^{n \times k}$ is any positive definite matrix. Based on equation (36) and assumption 1, one obtains

$$2 \Delta x^T \mathbf{P} \hat{\mathbf{W}}_3^T \sigma \leq \Delta x^T \mathbf{P} \hat{\mathbf{W}}_3^T \mathbf{P} \Delta x + \Delta x^T D \Delta x$$

$$2 \Delta x^T P \sigma_1 \leq \Delta x^T \mathbf{P} \Lambda \sigma_1 \mathbf{P} \Delta x + \sigma_1^T \Lambda \sigma_1$$

Then, substituting (37) into (34), one has

$$\dot{L}_I \leq \Delta x^T ( A^T P + PA^T + \mathbf{P} \hat{\mathbf{W}}_3^T \mathbf{P} + D + Q_o ) \Delta x$$

$$\quad - \Delta x^T Q_o \Delta x + \Delta x^T \mathbf{P} \Lambda \sigma_1 \mathbf{P} \Delta x + \sigma_1^T \Lambda \sigma_1$$

For known $\hat{\mathbf{W}}_3$, $D$ and proper selection of $Q_o$, there exists positive solution $P$ to the following matrix differential Riccati equation

$$A^T P + PA^T + \mathbf{P} \hat{\mathbf{W}}_3^T \mathbf{P} + D + Q_o = 0$$

Hence, equation (38) becomes

$$\dot{L}_I \leq - \Delta x^T Q_o \Delta x + \Delta x^T \mathbf{P} \Lambda \sigma_1 \mathbf{P} \Delta x + \sigma_1^T \Lambda \sigma_1$$
For precise identification case, i.e., $\xi = 0$, equation (40) becomes
\[
\dot{L}_f \leq -\Delta x^T Q_0 \Delta x \leq -\lambda_{\min}(Q_0) \|\Delta x\|_2^2 \leq 0
\]  
(41)

From equation (41), we have $\Delta x, \dot{\tilde{W}}_{i,2}, \dot{A} \in L_\infty$. Furthermore, from the error dynamics equation (13), one gets $\Delta x \in L_2$. By integrating equation (41) on both sides from 0 to $\infty$, one has
\[
\int_0^\infty [-\lambda_{\min}(Q_0) \|\Delta x\|_2^2] \leq [L_1(0) - L_1(\infty)] < \infty, \quad \text{which implies that } \Delta x \in L_2.
\]
Since $\Delta x \in L_2 \cap L_\infty$ and $\dot{A} \in L_\infty$, we get $\lim_{t \to \infty} \Delta x = 0$ using Barbalat’s Lemma.

For bounded modelling error, i.e., $\xi_i \leq \xi_i$, equation (40) can be represented as
\[
\dot{L}_f \leq -\Delta x^T Q_0 \Delta x + \Delta x^T P \Lambda^{-1}_0 P \Delta x + \Delta x^T \Lambda^{-1}_0 \Delta x_i \leq -\alpha(\|\Delta x\|) + \beta(\|\xi\|)
\]  
(42)

where $\alpha(\|\Delta x\|) = (\lambda_{\min}(Q_0) - \lambda_{\min}(P \Lambda^{-1}_0 P)) \|\Delta x\|$, $\beta(\|\xi\|) = \lambda_{\min}(\Lambda^{-1}_0) \|\xi\|$. Since $\alpha, \beta$ are $K_\infty$ function, $L_1$ is input to state Lyapunov function, the dynamics of the identification error equation (13) is thus input to state stability, which implies $\Delta x, \dot{\tilde{W}}_{i,2}, \dot{A} \in L_\infty$. This completes the proof of Theorem 1.

By substituting equation (30) into equation (31), one obtains
\[
M = E(\dot{\rho}) + F(t) = -E(t) \dot{\tilde{W}}_3 + \zeta_f
\]  
(43)

where $\zeta_f = \int_0^\infty e^{-\eta^{-1}r} X_{\text{bias}} dr$ is bounded as $\|\zeta_f\| \leq \zeta_f$.

It can be proved that the persistently excited (PE) for $X$ can guarantee the positive define matrix in equation (30) (Na et al., 2015), i.e., $\lambda_{\min}(E) > \sigma > 0$. Then, according to the fact $\dot{\tilde{W}}_3 = -\tilde{W}_3$, the time derivative of $L_o$ is calculated as
\[
\dot{L}_o = \dot{\tilde{W}}_3^T \rho \dot{\tilde{W}}_3 = -E(t) \dot{\tilde{W}}_3 \tilde{W}_3 + \tilde{W}_3^T \zeta_f \leq -\|\tilde{W}_3\| \|\rho\| \|\tilde{W}_3\| - \zeta_f
\]  
(44)

From the basic inequality $ab \leq a^2 \delta / 2 + b^2 / 2 \delta$ with $\delta > 0$, one can rewrite equation (44) as
\[
\dot{L}_o \leq -\left(\sigma - \frac{1}{2 \delta}\right) \|\tilde{W}_3\|^2 + \frac{\delta \zeta_f^2}{2}
\]  
(45)

The time derivation of $L_c$ can be deduced from the third equation of (33), such that
\[
\dot{L}_c = 2\Gamma e^T e + \kappa \left(-e^T Q e - u_c^T R u_c\right)
\]
\[
= 2\Gamma e^T \left(-K e + \frac{1}{2} \dot{W}_3 R^{-1} \dot{W}_3 \nabla \phi^T \tilde{W}_3 + \tilde{W}_3 u_c^T + \frac{1}{2} \dot{W}_3 R^{-1} \tilde{W}_3 \nabla \phi^T \tilde{W}_3 + \xi_2 \right)
\]
\[
+ \kappa \left(-e^T Q e - u_c^T R u_c\right) \leq -\Gamma \left(\|\tilde{W}_3\|^2 R^{-1} \|\tilde{W}_3\|^2 + \|\xi_2\|^2 \right) + \frac{1}{2} \|\tilde{W}_3\|^2 R^{-1} \|\tilde{W}_3\|^2 \nabla \xi_2^T \nabla \xi_2
\]  
(46)
Then, from equations (42), (44) and (46), the time derivative of $L$ is

$$
\dot{L} \leq -\left(\lambda_{\text{min}}(Q_h) - \lambda_{\text{max}}(P A_{\lambda}^* P)\right)\|\Delta x\|^2 \\
- \left[\Gamma K + \kappa \lambda_{\text{min}}(Q) - \Gamma \left(\|\hat{\mathcal{J}}_T^{-1} \hat{\mathcal{W}}_T^{-1} \| \|\nabla \phi\| + 1\right)\right]\|e\|^2 \\
- \left[\lambda_{\text{max}}(R) - \Gamma \|\hat{\mathcal{W}}_T\|^2 + \Gamma \zeta_1 \zeta_2\right] \\
- \left[\sigma - \frac{1}{2\Delta} - \frac{1}{4}\Gamma \left(\|\hat{\mathcal{J}}_T^{-1} \hat{\mathcal{W}}_T^{-1} \| \|\nabla \phi\| + 1\right)\right]\|\hat{\mathcal{W}}_T\|^2 + \lambda_{\text{max}}(\Lambda_{\lambda}^*)\|\zeta_1\|^2 \\
+ \frac{1}{2}\Gamma \|\hat{\mathcal{J}}_T^{-1} \hat{\mathcal{W}}_T^{-1} \| \nabla \zeta_1 \nabla \zeta_1 + \frac{\sigma}{2}\right]
$$

(47)

By choosing the appropriate parameters, such that

$$
\lambda_{\text{min}}(Q_h) > \lambda_{\text{max}}(P A_{\lambda}^* P), \Gamma < \frac{4\sigma\Delta - 2}{\sigma} \left(\|\hat{\mathcal{J}}_T^{-1} \hat{\mathcal{W}}_T^{-1} \| \|\nabla \phi\| + 1\right) \\
\kappa > \max \left\{\frac{\Gamma \|\hat{\mathcal{W}}_T\|^2}{\lambda_{\text{max}}(R)}, \frac{\Gamma \left(\|\hat{\mathcal{J}}_T^{-1} \hat{\mathcal{W}}_T^{-1} \| \|\nabla \phi\| + 1\right)}{\lambda_{\text{min}}(Q)}\right\}
$$

(48)

With equation (48), equation (47) can be simplified as below

$$
\dot{L} \leq -h_1 \|\Delta x\|^2 - h_2 \|\hat{\mathcal{W}}_T\|^2 - h_3 \|e\|^2 + \vartheta
$$

(49)

where

$$
h_1 = \lambda_{\text{min}}(Q_h) - \lambda_{\text{max}}(P A_{\lambda}^* P) \\
h_2 = \sigma - \frac{1}{2\Delta} - \frac{1}{4}\Gamma \left(\|\hat{\mathcal{J}}_T^{-1} \hat{\mathcal{W}}_T^{-1} \| \|\nabla \phi\| + 1\right) \\
h_3 = \Gamma K + \kappa \lambda_{\text{min}}(Q) - \Gamma \left(\|\hat{\mathcal{J}}_T^{-1} \hat{\mathcal{W}}_T^{-1} \| \|\nabla \phi\| + 1\right) + 2 \\
h_4 = \Gamma K + \kappa \lambda_{\text{min}}(Q) - \Gamma \left(\|\hat{\mathcal{J}}_T^{-1} \hat{\mathcal{W}}_T^{-1} \| \|\nabla \phi\| + 1\right) + 2 \\
\vartheta = \lambda_{\text{max}}(\Lambda_{\lambda}^*)\|\zeta_1\|^2 + \frac{1}{2}\Gamma \|\hat{\mathcal{J}}_T^{-1} \hat{\mathcal{W}}_T^{-1} \| \nabla \zeta_1 \nabla \zeta_1 + \Gamma \zeta_1 \zeta_2 + \frac{\sigma}{2}\right]
$$

are all positive constants based on condition (48).

Then, $\dot{L} < 0$ if the following inequities hold

$$
\|\Delta x\| > \sqrt{\vartheta / h_1}, \|\hat{\mathcal{W}}_T\| > \sqrt{\vartheta / h_2}, \|e\| > \sqrt{\vartheta / h_3}
$$

(50)

which means the identification error $\|\Delta x\|$, tracking error $e$ and NN weights error $\|\hat{\mathcal{W}}_T\|$ are all bounded.

Moreover, we have
When $t \to \infty$, the upper bound of control input is

\[ \| \dot{u}_c - u_c^* \| \leq \frac{1}{2} R^{-1} \hat{W}_f \| \nabla \phi \| \hat{W}_f + R^{-1} \hat{W}_f \nabla \xi_3 \]

where $\xi$ depends on the approximate errors of the NN identifier and critic NN. This completes the proof of Theory 2.