An optimal inventory control in hybrid manufacturing/remanufacturing system with deteriorating and defective items

Ali Khaleel Dhaiban*

The University of Mustansiriyah, Baghdad, Iraq
and
School of Quantitative Science, University Utara Malaysia, Malaysia
Email: ali_alzubiadi@yahoo.com
*Corresponding author

Md. Azizul Baten and Nazrina Aziz

School of Quantitative Sciences, University Utara Malaysia, Malaysia
Email: baten_math@yahoo.com
Email: nazrina@uum.edu.my

Abstract: An optimal control model in a manufacturing/remanufacturing-inventory system, with defective, deteriorating, returned and disposed items was developed. In this model, the time of deterioration, and the percentages of defective, returned and disposed items were assumed random variables that follow the gamma distribution, beta rectangular, four-parameter generalised beta (FPGB) and Kumaraswamy distributions, respectively. The total of returned items from customers and defective items divided into the remanufacturing and disposed items. An optimality conditions were derived from the dynamic of the manufacturing/remanufacturing inventory level. The explicit solution under continuous-review policy was achieved by using the Pontryagin maximum principle. Also, simulation and sensitivity analysis results were illustrated numerically. The numerical results suggested that the model can help firms to maintain a balance between the manufacturing and remanufacturing rates to hedge demand and inventory levels. Also, the manufacturing rate was positively related to the disposal rate.

Keywords: optimal control; inventory system; deteriorating items; defective items; Pontryagin maximum principle; demand function.


Biographical notes: Ali Khaleel Dhaiban works in the Ministry of Higher Education and Scientific Research, Iraq. He is a Lecturer at the Al-Mustansiriyah University, Administration and Economics College, Department of Statistics. His areas of interest are mathematical programming and optimal control.
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Md. Azizul Baten is currently working in the Department of Decision Science, University Utara Malaysia and previously, he had been working in the Department of Statistics as a Full Professor, Shahjalal University of Science and Technology (SUST), Bangladesh.

Nazrina Aziz is a Senior Lecturer of Statistics at the School of Quantitative Sciences (SQS), University Utara Malaysia. She received her PhD (Statistics) from the Victoria University of Wellington, New Zealand, where she specialised in survival and multivariate analysis.

1 Introduction

Currently, remanufacturing is the most important sector in the industrial field. As such, many studies have dealt with returned and disposed items due to their effect on the decision making process with regard to the production rate and the inventory level in the production inventory system. This system deals with returned items from customers and those which are remanufactured into new products or disposed based on the economic benefits. The concept of remanufacturing has spread in many fields, such as in the manufacture of electrical equipment, cellular phones and many others, for various reasons, especially for economic and environmental reasons (Gallo et al., 2009). Remanufacturing includes disassembling, cleaning, refurbishing and reassembling parts, such as automotive parts and heavy equipment parts, to produce ‘just as new’ products. These days, many products in markets are in fact re-manufactured and reused products (Jin et al., 2011). At first, a continuous review policy for a production inventory system with remanufacturing and disposal was considered by Heyman (1977). Muckstadt and Issac (1981) allowed for non-zero manufacturing lead times, stochastic remanufacturing lead times, and finite remanufacturing capacities. Several researchers studied many aspects in relation to this problem such as the life cycle of products, reduction of greenhouse gases and inventory control (García-Alvarado et al., 2015).

Researchers have studied the inventory system with remanufacturing and disposal operations (van der Laan and Salomon, 1997); with deterioration rate (Sharma and Chaudhary, 2013); defective items (Nasri et al., 2013); deterioration and defective rates as a constant (Tai, 2012); with and without a disposal rate as part of the returned items (Kim et al., 2013); with disposed items as a constant percentage of the returned items (Rubio and Corominas, 2008); the returned items as a constant percentage of the demand (Guo and Ya, 2015; Polotski et al., 2015); with the return rate as a random variable that follows a normal distribution (Singh and Saxena, 2012; Lalmazloumian et al., 2014); and with rates of demand and the returned items as random variables that follow Poisson distribution (Nikoofal and Moattar Husseini, 2010; Corum et al., 2014). However, the above literature did not focus on developing an optimal inventory control model for hybrid manufacturing-remanufacturing with rates of deterioration, defective, returned and disposed items together.

Managerially, administration wants to keep the inventory at a specific level, and to hedge demand with the lowest cost, which is happening deviation of the inventory level from its goal. There are many problems that face the administration to achieve their goals, such as defective, deterioration, returned and disposed items. The dynamics of the manufacturing/remanufacturing inventory planning problem was concerned with the case
where units of the product, which are defective and non-defective, are subject to deterioration. The model considered in this study contributes in several ways. First, the remanufacturing items are a percentage of the total number of defective and returned items. Second, an optimal inventory control model is developed following a probability distribution with defective, deteriorating, returned and disposed items together as the random variables. This type of research, which is more realistic, was rarely conducted in the previous studies. Third, the optimum conditions were derived using Pontryagin’s maximum principle. Fourth, several simulation results were provided to investigate the effect of the parameter values on the rates of manufacturing, remanufacturing, return and disposal.

This paper is organised as follows. In Section 2, a description of the literature review is given. In Section 3, the notations and the assumptions involved in the optimal inventory control model were introduced. In Section 4, the rate functions of gamma, beta rectangular, Kumaraswamy and four-parameter generalised beta (FPGB) distributions were estimated. In Section 5, the theoretical solutions along with the optimality conditions for the model were obtained. In Section 6, the simulation and sensitivity analysis results of manufacturing-remanufacturing inventory model were illustrated, with time-varying demand. In the last section, the findings and some suggestions for future research were summarised.

2 Literature review

Many studies in the past dealt with deteriorating and defective items separately or together. Deteriorating inventory model, with assumptions that the deterioration rate is random variable that follows the probability distribution, have been discussed (Covert and Philip, 1973; Chen and Shun Lin, 2003; Ghosh and Chaudhuri, 2004; Sarkar, 2013; Sharma and Chaudhary, 2013; Khanra et al., 2015; Sarkar et al., 2015; Sarkar and Saren, 2015). In contrast, researchers have studied an inventory system with defective items as a random variable having a probability distribution (Tsou and Chen, 2008; Ho, 2009; Nasri et al., 2013). Jaggi and Mittal (2011), Kumar et al. (2011) and Tai (2012) have developed an inventory system with the deterioration rate as a constant, where the percentage of defective items is a stochastic, random variable following a uniform distribution and constant, respectively. van der Laan and Salomon (1997) investigated a stochastic inventory system with remanufacturing and disposal operations. The impact of disposal items on the cost of a system by considering two cases with and without disposal, have been discussed. Jin et al. (2011), Lim et al. (2011) and Kim et al. (2013) have formulated the problem of inventory control and reassembly according to the Markov decision process to minimise costs by finding the optimal policy. They used a continuous-time model with returns following independent Poisson processes. Kim et al. (2013) also discussed a disposal rate as part of the returned item and made comparisons between two models, a production model and a remanufactured model.

Several researchers studied the returned items as a constant percentage of the demand (Dobos, 2003; Rubio and Corominas, 2008; Guo and Liang, 2011; Guo and Ya, 2015; Polotski et al., 2015). Dobos (2003) developed two models; in the first model, the disposal process can occur at any time through the planning horizon, whereas in the second model, the disposal process occurs only at the end of the planning horizon. Meanwhile, Rubio and Corominas (2008) clarified the effect of the used items on the
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system capacity and analysed the capacity of the lines to manufacturing and remanufacturing, which determine the options of the production policy. Moreover, they dealt with disposed items as a constant percentage of returned items. The effects of recycled products, quality and buyback price levels on the recycling rate and the remanufacturing cost were investigated by Guo and Liang (2011). Guo and Ya (2015) investigated the optimal strategy of manufacturing and remanufacturing, where the demand rate is a constant and is greater than or equal to the recycling rate. It was assumed that the costs of buyback and remanufacturing, for a system without shortage and disposed items, were affected by the quality level. Polotski et al. (2015) developed two kinds of systems: the first is a manufacturing system with a low rate of return, while the second is a system with a low use, where the system is for the remanufacturing of raw materials. The two systems are used at the same time by being transformed from one to another to meet demand at a lowest cost.

Hsueh (2010), as well as Singh and Saxena (2012) have handled the return rate as a random variable that follows a normal distribution as the demand rate, with changes in the mean being subject to the lifetime of the products and the linear function of time, respectively. Also, the rates of return and demand, which are random variables that follow a normal distribution, for the optimal quantities of manufacturing, remanufacturing and the reorder point, were derived by Lalmazloumian et al. (2014). Nikoofal and Moattar Husseini (2010) and Corum et al. (2014) have developed a model where the rates of demand and the returned items are random variables that follow Poisson distribution. Nikoofal and Moattar Husseini (2010) studied a periodic ordering, whereas Corum et al. (2014) compared three hybrid systems: traditional, push and pull with lead times of manufacturing and remanufacturing rates. See Table 1 for the contribution of this paper.

Table 1  Comparison with other papers

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3  Notations and assumptions involved in the model

This section introduces notations and assumptions involved in the optimal control model.

3.1  Notations

The main variables and parameters of the model are as follows:

- $T$: the length of the planning horizon (month) ($T > 0$)
- $X_1(t)$: the inventory level at time $t$ (state variable) in the first store
- $X_2(t)$: the inventory level at time $t$ (state variable) in the second store
- $X_3(t)$: the inventory level at time $t$ (state variable) in the third store
- $U_T(t)$: total manufacturing rate at time $t$ (control variable)
- $U_{NT}(t)$: net manufacturing rate at time $t$ (control variable)
- $U_R(t)$: remanufacturing rate at time $t$ (control variable)
- $U_D(t)$: disposal rate at time $t$ (control variable)
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\[ \hat{u}_T(t) \] the total manufacturing goal rate
\[ \hat{u}_N(t) \] the net manufacturing goal rate
\[ \hat{u}_r(t) \] The remanufacturing goal rate
\[ \hat{u}_o(t) \] the disposal goal rate
\[ D(t) \] demand rate for the production at time \( t \)
\[ \theta(t) \] the deterioration rate, where the time of deterioration is a random variable that follows gamma distribution
\[ \mu_\phi \] the percentage of defective items is a random variable that follows beta rectangular distribution
\[ \mu_\delta \] the percentage of disposed items is a random variable that follows Kumaraswamy distribution
\[ \hat{x}_i, i = 1, 2, 3 \] inventory goal levels of the first, second and third stores, respectively
\[ X_i(0), i = 1, 2, 3 \] initial inventory levels of the first, second and third stores, respectively
\[ \tau \] the highest percentage of defective items
\[ R(t) \] return rate at time \( t \)
\[ e \] the percentage of returned items is a random variable that follows FPGB distribution
\[ \tau_e \] the delay of return (month)
\[ h_i > 0, i = 1, 2, 3 \] a penalty is incurred for the inventory levels to deviate from its goal levels in the first, second and third stores, respectively ($/item/month)
\[ k_c > 0 \] a penalty is incurred when the total manufacturing rate deviates from its goal rate ($/item/month)
\[ k_N > 0 \] a penalty is incurred when the net manufacturing rate deviates from its goal rate ($/item/month)
\[ k_r > 0 \] a penalty is incurred when the remanufacturing rate deviates from its goal rate ($/item/month)
\[ k_\delta > 0 \] a penalty is incurred when the disposal rate deviates from its goal rate ($/item/month)
\[ \rho \] constant non-negative (discount rate).

3.2 Assumptions

There are numerous factors that affect the manufacturing/remanufacturing-inventory system, such as defective, deteriorating, disposed and returned items. These factors affect the hedge demand and manufacturing/remanufacturing plan. The total cost could therefore maybe become very high; so a plan of manufacturing and remanufacturing is needed that treats all these factors (Figure 1).
The following assumptions are considered:

1. A firm can manufacture a certain product (total manufacturing) and place it in the first store to check a sample of product items. Practically, any product must check by manufacturer.

2. Other product items and non-defective items (net manufacturing) placed in the second store to hedge demand. There is some deterioration in the items stored in the second store (the gamma deteriorating rate). Practically, deterioration will occur during product storage depending on the lifetime of the product.

3. The percentage of defective items is a random variable that follows beta rectangular distribution. The third store contains the total of returned items from customers and defective items divided into the remanufacturing items, go to the second store as new product, and disposed items. Most of the products are sold with a warranty. Therefore some products will be returned from the customers to the remanufacturer.

4. Demand rate varies with time.

5. The firm has set its inventory goals levels and total manufacturing, net manufacturing, remanufacturing and disposal goals rates without shortage.

6. The use of distributions and values of their parameters was not arbitrary. The gamma distribution is one of the lifetime distributions and is suitable for describing the lifetime of items, especially for specific values of the shape parameter. The percentages of the defective, disposed and returned items needing distribution take values between zero and one and choose parameter values that suit the practical status. For example, the percentage of defective items must take small values, and the reverse would mean a loss.

4 Estimation of the rate functions

4.1 Deterioration rate follows gamma distribution

The rate of deterioration is a random variable that follows a standard Gamma distribution. The probability density function (PDF) is

\[ f(t, \alpha) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \exp(-t), \quad t \geq 0, \quad \alpha > 0 \]
The cumulative distribution function is

\[ F(x, \alpha) = \int_0^t f(x; \alpha)dx = \frac{\gamma(\alpha, t)}{\Gamma(\alpha)}, \quad t \geq 0, \quad \alpha > 0, \]

where \( \gamma(\alpha, t) \) is the lower incomplete gamma function.

The instantaneous rate of deterioration of the standard gamma distribution is represented by the hazard function

\[ \theta(t) = \frac{f(t)}{1 - F(t)} = \frac{1}{\Gamma(\alpha, t)} t^{\alpha-1} \exp(-1). \quad (1) \]

The hazard function curve is increased when \( \alpha > 1 \) and in practice, deterioration increases with time. The shape parameter value represents the lifetime of the product.

### 4.2 Percentages of defective, disposal and returned items

The proportion of returned items follows a FPGB distribution. The PDF of FPGB is (Bowman and Shenton, 2007):

\[ f(e) = \frac{(e-q)^b(p-e)^c}{B(b, c)(p-q)^b} \quad q \leq e \leq p, \quad b > 0, \quad c > 0, \]

where

- \( q, p \) end points (\( q \) and \( p \) represents the lowest and highest percentage of returned items from demand, respectively).
- \( b, c \) shape parameters.

The percentage of returned items is given by

\[ \mu_e = \int_q^p e f(e)de \quad q \leq e \leq p \]

\[ \mu_e = \frac{bp + cq}{b + c} \quad (3) \]

The rate of return is continuous differentiable (Dobos, 2003; Rubio and Corominas, 2008):

\[ R(t) = \mu_e D(t - \tau_e) \quad (4) \]

Substituting equation (3) in equation (4), yields

\[ R(t) = \left( \frac{bp + cq}{b + c} \right) D(t - \tau_e) \quad (5) \]

The proportion of defective items follows a beta rectangular distribution. The probability density function of beta rectangular distribution is

\[ f(t; \alpha, \epsilon) = \frac{\epsilon \alpha t^{\alpha-1}}{\tau^\alpha} \cdot \frac{1}{\tau} e^{\frac{-1 - \epsilon}{\tau}}, \quad 0 \leq t \leq \tau, \quad \alpha > 0, \epsilon \leq 1, \tau > 0, \]
where \( \varepsilon \) (the mixture parameter) takes values between zero and one. The percentage of defective items is given by

\[
\mu_{\varepsilon} = \int_{0}^{1} \psi f(\psi) = \frac{\varepsilon \alpha \tau}{\alpha + 1} + \frac{(1 - \varepsilon)\tau}{2}.
\]  

(6)

The proportion of disposal items follows a Kumaraswamy distribution, also called double-bounded distribution which it was founded as an alternative to the beta distribution (Widemann, 2011). The probability density function of Kumaraswamy distribution is (Garg, 2009):

\[
F(\delta; z, m) = \begin{cases} 2m^z \delta^{z-1} (1 - \delta^z)^{m-1} & z, m > 0, 0 < \delta < 1 \\ 0 & \text{otherwise.} \end{cases}
\]

when \( m = 2 \), yields

\[
f(\delta; z) = 2z\delta^{z-1} (1 - \delta^z), \quad 0 \leq \delta \leq 1.
\]

The percentage of disposal items is given by

\[
U_d(t) = 2z^2 \int_{0}^{1} \delta^z (1 - \delta^z) d\delta, \quad 0 \leq \delta \leq 1
\]

\[
U_d(t) = \frac{2z^2}{(z + 1)(2z + 1)}.
\]  

(7)

The shape parameter values of distributions are that control the lower and upper limits of percentages.

5 Model and theoretical solutions

5.1 Optimal manufacturing/remanufacturing-inventory model

The optimal control for the optimisation is defined by

\[
\text{Min } J(U, x, \dot{u}) = \frac{1}{2} \int_{0}^{T} e^{\alpha t} \left[ h_1 \{ X_1(t) - \hat{x}_1(t) \}^2 + h_2 \{ X_2(t) - \hat{x}_2(t) \}^2 \\
+ h_3 \{ \hat{x}_1(t) \}^2 + k_r \{ U_r(t) - \hat{u}_r(t) \}^2 \\
+ K_N \{ U_N(t) - \dot{u}_N(t) \}^2 + k_f \{ U_f(t) - \dot{u}_f(t) \}^2 \\
+ k_d \{ U_d(t) - \dot{u}_d(t) \}^2 \right] dt,
\]  

(8)

The interpretation of the objective function is we want to keep the inventory, manufacturing, remanufacturing and disposal rates as close as possible to their goals. There are ‘penalties’ representations into the quadratic terms of the aforementioned rates if being far to their corresponding goal rates. This means the equation (8) minimises the sum of mean square deviations of inventory, manufacturing, remanufacturing and disposal of their goals (Ouaret et al., 2011).

Subject to state equations of first, second and third store as
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\[ \frac{d}{dt} X_1(t) = U_T(t) - U_N(t) - \mu_0 X_1(t), \quad (9) \]

\[ \frac{d}{dt} X_2(t) = U_N(t) - U_r(t) - D(t) - \theta(t) X_2(t), \quad (10) \]

\[ \frac{d}{dt} X_3(t) = U_T(t) - U_N(t) + R(t) - U_r(t) - U_d(t) X_3(t), \quad (11) \]

And non-negative constraints given by

\[ U_T(t), U_N(t), U_r(t), U_d(t), X_i(t) \geq 0, \text{ for all } t \in [0, T], i = 1, 2, 3. \quad (12) \]

From equation (11), we have

\[ \frac{d}{dt} X_3(t) + \left( \frac{2z}{z+1} - \frac{2z}{2z+1} \right) X_3(t) = U_T - U_N(t) + R(t) - U_r(t). \quad (13) \]

Assuming that \( X_3(0) = x_3^0 \) is known. Note: the remanufacturing goal rate \( \hat{u}_r(t) \) can be computed using the state equation (13):

\[ \hat{u}_r(t) = \hat{u}_T(t) - \hat{u}_N(t) + R(t) - \left( \frac{2z}{z+1} - \frac{2z}{2z+1} \right) \hat{x}_3. \quad (14) \]

Equation (14) represents the remanufacturing goal rate \( \hat{u}_r(t) \), which is equals to the total of defective rate \( [\hat{u}_T(t) - \hat{u}_N(t)] \) and returned items minus the disposal items.

Substituting equation (5) into equation (14) yields:

\[ \hat{u}_r(t) = \hat{u}_T(t) - \hat{u}_N(t) + \left( \frac{bp + cq}{b + c} \right) D(t - \tau_r) - \left( \frac{2z}{z+1} - \frac{2z}{2z+1} \right) \hat{x}_3 \quad (15) \]

According to equation (7), we can suppose that

\[ \hat{u}_3(t) = \left( \frac{2z}{z+1} - \frac{2z}{2z+1} \right) \hat{x}_3. \quad (16) \]

From equation (10), we have

\[ \frac{d}{dt} X_2(t) + \left( \frac{1}{\Gamma(\alpha, t)} t^{\alpha-1} \exp(-t) \right) X_2(t) = U_N(t) + U_r(t) - D(t). \quad (17) \]

Assuming that \( X_2(0) = x_2^0 \) is known. Note: the net manufacturing goal rate \( \hat{u}_N(t) \) can be computed using the state equation (17):

\[ \hat{u}_N(t) = D(t) + \left( \frac{1}{\Gamma(\alpha, t)} t^{\alpha-1} \exp(-t) \right) \hat{x}_2 - \hat{u}_r(t). \quad (18) \]

Equation (18) represents the net manufacturing goal rate \( \hat{u}_N(t) \) which is equals to the total of demand and deterioration items minus the remanufacturing goal rate.

By substituting equation (15) into equation (18), we get
\[ \dot{u}_T(t) = D(t) + \left\{ \frac{1}{\Gamma(\alpha, t)} t^{\alpha-1} \exp(-t) \right\} \dot{x}_2 - \left( \frac{b p + c q}{b + c} \right) D(t - \tau_c) \]
\[ + \left( \frac{2z}{z+1} - \frac{2z}{2z+1} \right) \dot{x}_3 \]
\[ (19) \]

From equation (9), we have
\[ \frac{d}{dt} x_1(t) + \left\{ \frac{e \alpha \tau}{\alpha + 1} + \frac{(1-\epsilon)\tau}{2} \right\} x_1(t) = U_T(t) - U_N(t). \]
\[ (20) \]

Assuming that \( x_1(0) = x_0 \) is known. Note: the total manufacturing goal rate \( \dot{u}_T(t) \) can be computed using the state equation (20):
\[ \dot{u}_T(t) = \dot{u}_N(t) + \left\{ \frac{e \alpha \tau}{\alpha + 1} + \frac{(1-\epsilon)\tau}{2} \right\} \dot{x}_1. \]
\[ (21) \]

Equation (21) represents the total production goal rate \( \dot{u}_T(t) \), which is equals to the net manufacturing goal rate minus the defective items.

By substituting equation (19) into equation (21), we get
\[ \dot{u}_N(t) = D(t) - \left( \frac{b p + c q}{b + c} \right) D(t - \tau_c) + \left\{ \frac{1}{\Gamma(\alpha, t)} t^{\alpha-1} \exp(-t) \right\} \dot{x}_2 \]
\[ - \left( \frac{e \alpha \tau}{\alpha + 1} + \frac{(1-\epsilon)\tau}{2} \right) \dot{x}_1 + \left( \frac{2z}{z+1} - \frac{2z}{2z+1} \right) \dot{x}_3 \]
\[ (22) \]

5.2 Optimality conditions and model solution

To solve problem, which includes equations (8) through (12), by the Pontryagin maximum principle (Pontryagin et al., 1962; Sethi and Thompson, 2000), there exists an adjoint function \( \lambda(t) \) such that the Hamiltonian functional form is given by
\[ H(t) = \frac{1}{2} e^{t \sigma} \left[ h_1 \{ X_1(t) - \dot{x}_1 \}^2 + h_2 \{ X_2(t) - \dot{x}_2 \}^2 + h_3 \{ X_3(t) - \dot{x}_3 \}^2 \right] \]
\[ + k_r \left\{ U_T(t) - \dot{u}_T(t) \right\}^2 + k_N \left\{ U_N(t) - \dot{u}_N(t) \right\}^2 + k_r \left\{ U_r(t) - \dot{u}_r(t) \right\}^2 \]
\[ + k_b \left\{ U_b(t) - \dot{u}_b(t) \right\}^2 + \lambda_1(t) \left[ U_T(t) - U_N(t) - \left( \frac{e \alpha \tau}{\alpha + 1} + \frac{(1-\epsilon)\tau}{2} \right) X_1(t) \right] \]
\[ + \lambda_3(t) \left[ U_N(t) - U_N(t) + R(t) - U_r(t) - \left( \frac{2z}{z+1} - \frac{2z}{2z+1} \right) X_3(t) \right] \]
\[ (23) \]

To obtain the control variables \( U_T(t) \), \( U_b(t) \), \( U_r(t) \) and \( U_b(t) \), we differentiate the equation (23) with respect to \( U_T \), \( U_N \), \( U_r \) and \( U_b \) respectively, and put zero:
\[ U_T(t) = \dot{u}_T(t) + \frac{1}{k_T} e^{t \sigma} \left\{ \lambda_1(t) + \lambda_3(t) \right\} \]
\[ (24) \]
An optimal inventory control in hybrid manufacturing/ remanufacturing system

\[ U_N(t) = \hat{u}_N(t) + \frac{1}{k_N} e^{\rho t} \{ \dot{\lambda}_2(t) - \dot{\lambda}_1(t) - \dot{\lambda}_3(t) \} \] (25)

\[ U_r(t) = \hat{u}_r(t) + \frac{1}{k_r} e^{\rho t} \{ \dot{\lambda}_2(t) - \dot{\lambda}_3(t) \} \] (26)

\[ U_\delta(t) = \hat{u}_\delta(t) \] (27)

where \( \Omega_i(t) = [0, U_{max}(t)] \), \( i = T, N, r, \delta \).

\( U_{max}(t) \) is the maximum possible total manufacturing, net manufacturing, remanufacturing and disposal rates, respectively.

To get the optimal rates, we substitute boundary conditions \( \dot{\lambda}_i(T) = 0 \) into equations (24) through (26).

\[ \dot{\lambda}_T(t) = \hat{u}_T(t); \dot{U}_N(t) = \hat{u}_N(t); \dot{U}_r(t) = \hat{u}_r(t). \] (28)

To obtain \( \dot{\lambda}_i(t) \) values, the adjoint equation is

\[ \frac{\partial}{\partial X_i(t)} H(t, X(t), U(t), \dot{u}(t), \dot{\lambda}(t)) = - \frac{d}{dt} \dot{\lambda}_i(t), \quad i = 1, 2, 3. \] (29)

Then,

\[ \frac{d}{dt} \dot{\lambda}_1(t) = h_1 e^{-\rho t} \{ X_1(t) - \dot{\lambda}_1 \} + \dot{\lambda}_1 \left( \frac{\varepsilon \alpha t}{\alpha + 1} + \frac{(1 - e^t)}{2} \right), \] (30)

\[ \frac{d}{dt} \dot{\lambda}_2(t) = h_2 e^{-\rho t} \{ X_2(t) - \dot{\lambda}_2 \} + \dot{\lambda}_2 \left( \frac{1}{\Gamma(\alpha + 1)} t^{\alpha-1} \exp(-1) \right), \] (31)

\[ \frac{d}{dt} \dot{\lambda}_3(t) = h_3 e^{-\rho t} \{ X_3(t) - \dot{\lambda}_3 \} + \dot{\lambda}_3 \left( \frac{2z}{z + 1} - \frac{2z}{2z + 1} \right), \] (32)

With boundary conditions \( \dot{\lambda}_i(T) = 0, \quad i = 1, 2, 3 \).

Now, we must find the differential equations that determine the inventory levels of three stores according to \( X_i(t) \) and \( \dot{\lambda}_i(t) \).

We put equations (24 and 25) into the state equation (9), to obtain

\[ \frac{d}{dt} X_1(t) = \dot{u}_T(t) + \frac{1}{k_T} e^{\rho t} \{ \dot{\lambda}_1(t) + \dot{\lambda}_2(t) \} - \dot{u}_N(t) - \frac{1}{k_N} e^{\rho t} \{ \dot{\lambda}_2(t) - \dot{\lambda}_1(t) - \dot{\lambda}_3(t) \} \]

\[ - \left( \frac{\varepsilon \alpha t}{\alpha + 1} + \frac{(1 - e^t)}{2} \right) X_1(t). \] (33)

By substituting equation (21) into equation (33), we get

\[ \frac{d}{dt} X_1(t) = \frac{1}{k_T} e^{\rho t} \{ \dot{\lambda}_1(t) + \dot{\lambda}_2(t) \} - \frac{1}{k_N} e^{\rho t} \{ \dot{\lambda}_2(t) - \dot{\lambda}_1(t) - \dot{\lambda}_3(t) \} \]

\[ - \left( \frac{\varepsilon \alpha t}{\alpha + 1} + \frac{(1 - e^t)}{2} \right) \{ X_1(t) - \dot{\lambda}_1 \}. \]

From equation (34), we have
Substituting equations (30) through (32) in the second derivative of the equation (34), yields

\[
\frac{d^2}{dt^2} X_1(t) = \left( \frac{1}{k_T} + \frac{1}{k_N} \right) \rho e^{\alpha \tau} \left[ \hat{\lambda}_3(t) + \hat{\lambda}_3(t) \right] + \frac{1}{k_T} + \frac{1}{k_N} \left[ h \left\{ X_1(t) - \hat{\lambda}_3 \right\} + \hat{\lambda}_3(t) e^{\alpha \tau} + \frac{1}{\alpha + 1} \right] \left\{ \frac{(1-\varepsilon) \tau}{2} \right\} \left\{ X_1(t) - \hat{\lambda}_3 \right\}.
\]

(35)

Finally, substituting equation (35) into equation (36), yields

\[
\frac{d^2}{dt^2} X_1(t) = \left[ \frac{1}{k_T} + \frac{1}{k_N} \right] \rho e^{\alpha \tau} \left[ \hat{\lambda}_3(t) + \hat{\lambda}_3(t) \right] + \frac{1}{k_T} + \frac{1}{k_N} \left[ h \left\{ X_1(t) - \hat{\lambda}_3 \right\} + \hat{\lambda}_3(t) e^{\alpha \tau} + \frac{1}{\alpha + 1} \right] \left\{ \frac{(1-\varepsilon) \tau}{2} \right\} \left\{ X_1(t) - \hat{\lambda}_3 \right\}
\]

(36)

\[
\frac{d^2}{dt^2} X_1(t) = \left[ \frac{1}{k_T} + \frac{1}{k_N} \right] \rho e^{\alpha \tau} \left[ \hat{\lambda}_3(t) + \hat{\lambda}_3(t) \right] + \frac{1}{k_T} + \frac{1}{k_N} \left[ h \left\{ X_1(t) - \hat{\lambda}_3 \right\} + \hat{\lambda}_3(t) e^{\alpha \tau} + \frac{1}{\alpha + 1} \right] \left\{ \frac{(1-\varepsilon) \tau}{2} \right\} \left\{ X_1(t) - \hat{\lambda}_3 \right\}
\]

(37)

By substituting equations (25 and 26) into the state equation of the second store (10), we obtain

\[
\frac{d}{dt} X_1(t) = \hat{u}_N(t) + \frac{\rho e^{\alpha \tau} \left[ \hat{\lambda}_3(t) - \hat{\lambda}_3(t) - \hat{\lambda}_3(t) \right] + \hat{u}_r(t) + \frac{\rho e^{\alpha \tau} \left[ \hat{\lambda}_3(t) - \hat{\lambda}_3(t) \right] - d(t)}{k_T} - \frac{1}{\Gamma(\alpha, t)} e^{\alpha \tau} \left[ \hat{\lambda}_3(t) - \hat{\lambda}_3(t) \right]
\]

(38)

Substituting equation (18) into equation (38), yields
An optimal inventory control in hybrid manufacturing/remanufacturing system

\[
\frac{d}{dt} X_2(t) = -\left\{1 - \frac{1}{\Gamma(\alpha, t)} t^{\alpha-1} \exp(-t)\right\} \{X_2(t) - \hat{x}_2\} + \frac{1}{k_N} e^{\lambda t} \{\hat{\lambda}_2(t) - \lambda(t) - \lambda_3(t)\} \\
+ \frac{1}{k_r} e^{\lambda t} \{\hat{\lambda}_2(t) - \lambda_3(t)\}.
\] (39)

From equation (39), we have

\[
\left\{\frac{1}{k_N} + \frac{1}{k_r}\right\} e^{\lambda t} \{\hat{\lambda}_2(t) - \lambda_3(t)\} = \left\{1 - \frac{1}{\Gamma(\alpha, t)} t^{\alpha-1} \exp(-t)\right\} \{X_1(t) - \hat{x}_2\} \\
+ \frac{1}{k_N} e^{\lambda t} \lambda_3(t) + \frac{d}{dt} X_2(t).
\] (40)

By substituting equations (30) through (32) in the second derivative of equation (39), we get

\[
\frac{d^2}{dt^2} X_2(t) = \left\{\frac{1}{k_N} + \frac{1}{k_r}\right\} \rho e^{\lambda t} \{\hat{\lambda}_2(t) - \lambda_3(t)\} - \frac{1}{k_N} \rho e^{\lambda t} \lambda_3(t) + \left\{\frac{1}{k_N} + \frac{1}{k_r}\right\} \\
\left[h_2 \{X_2(t) - \hat{x}_2\} + \hat{\lambda}_2(t) e^{\lambda t} \left\{1 - \frac{1}{\Gamma(\alpha, t)} t^{\alpha-1} \exp(-t)\right\}\right] \\
- h_3 \{X_3(t) - \hat{x}_3\} - \lambda_3(t) e^{\lambda t} \left\{\frac{2z}{z + 1} - \frac{2z}{2z + 1}\right\} \\
- \frac{1}{k_N} \left[h_2 \{X_1(t) - \hat{x}_1\} + \lambda_3(t) e^{\lambda t} \left\{\frac{\epsilon \alpha r}{\alpha + 1} + \frac{(1 - e)r}{2}\right\}\right] \\
- \frac{1}{\Gamma(\alpha, t)} e^{\lambda t} \left\{\frac{2z}{z + 1} - \frac{2z}{2z + 1}\right\} \frac{d}{dt} X_2(t) \\
- \{X_2(t) - \hat{x}_2\} \left\{\frac{e^{-z t} \rho^2 - e^{\lambda t} (-\alpha + t + 1)(\alpha, t)}{\Gamma(\alpha, t)^2}\right\}. 
\] (41)

Substituting equation (40) into equation (41), yields

\[
\frac{d^2}{dt^2} X_2(t) = \left[\frac{\rho}{\Gamma(\alpha, t)} t^{\alpha-1} \exp(-t)\right] \frac{d}{dt} X_2(t) + \{X_2(t) - \hat{x}_2\} \left[\frac{1}{k_N} + \frac{1}{k_r}\right] h_2 \\
+ \frac{1}{\Gamma(\alpha, t)} e^{\lambda t} \left\{\frac{2z}{z + 1} - \frac{2z}{2z + 1}\right\} \\
+ \frac{1}{\Gamma(\alpha, t)} \left[h_2 \{X_1(t) - \hat{x}_1\} + \lambda_3(t) e^{\lambda t} \left\{\frac{\epsilon \alpha r}{\alpha + 1} + \frac{(1 - e)r}{2}\right\}\right] \\
- h_3 \{X_3(t) - \hat{x}_3\} - \lambda_3(t) e^{\lambda t} \left\{\frac{2z}{z + 1} - \frac{2z}{2z + 1}\right\} \\
- \frac{1}{k_N} \left[h_2 \{X_1(t) - \hat{x}_1\} + \lambda_3(t) e^{\lambda t} \left\{\frac{\epsilon \alpha r}{\alpha + 1} + \frac{(1 - e)r}{2}\right\}\right]. 
\] (42)
Substituting equations (24) through (26) into the state equation of the third store (11), yields

$$\frac{d}{dt}X_3(t) = \hat{u}_T(t) + \frac{1}{k_T} e^{\rho t} \left\{ \hat{\lambda}_2(t) + \hat{\lambda}_3(t) \right\} - \frac{1}{k_N} e^{\rho t} \left\{ \hat{\lambda}_2(t) - \hat{\lambda}_3(t) \right\}$$

$$- \hat{\lambda}_3(t) - \frac{1}{k_T} e^{\rho t} \left\{ \hat{\lambda}_2(t) - \hat{\lambda}_3(t) \right\} - \left( \frac{2z}{z+1} - \frac{2z}{2z+1} \right) X_3(t).$$

(43)

Substituting equations (18 and 21) into equation (43), yields

$$\frac{d}{dt}X_3(t) = \left( \frac{1}{k_N} + \frac{1}{k_T} \right) e^{\rho t} \left\{ \hat{\lambda}_3(t) - \hat{\lambda}_2(t) \right\} + \frac{1}{k_T} e^{\rho t} \left\{ \hat{\lambda}_1(t) + \hat{\lambda}_3(t) \right\} + \frac{1}{k_N} e^{\rho t} \hat{\lambda}_1(t)$$

$$- \left( \frac{2z}{z+1} - \frac{2z}{2z+1} \right) - \{X_3(t) - \hat{\lambda}_3\}.$$

Substituting equations (18 and 21) into equation (43), yields

$$\frac{d}{dt}X_3(t) = \left( \frac{1}{k_N} + \frac{1}{k_T} \right) e^{\rho t} \left\{ \hat{\lambda}_3(t) - \hat{\lambda}_2(t) \right\} + \frac{1}{k_T} e^{\rho t} \left\{ \hat{\lambda}_1(t) + \hat{\lambda}_3(t) \right\} + \frac{1}{k_N} e^{\rho t} \hat{\lambda}_1(t)$$

$$- \left( \frac{2z}{z+1} - \frac{2z}{2z+1} \right) - \{X_3(t) - \hat{\lambda}_3\}.$$

(44)

From equation (44), we have

$$\left( \frac{1}{k_N} + \frac{1}{k_T} \right) e^{\rho t} \left\{ \hat{\lambda}_3(t) - \hat{\lambda}_2(t) \right\} = \frac{d}{dt}X_3(t) - \frac{1}{k_T} e^{\rho t} \left\{ \hat{\lambda}_1(t) + \hat{\lambda}_3(t) \right\} - \frac{1}{k_N} e^{\rho t} \hat{\lambda}_1(t)$$

$$- \left( \frac{2z}{z+1} - \frac{2z}{2z+1} \right) \{X_3(t) - \hat{\lambda}_3\}.$$

(45)

By substituting equations (30) through (32) in the second derivative of equation (44), we get

$$\frac{d^2}{dt^2}X_3(t) = \left( \frac{1}{k_N} + \frac{1}{k_T} \right) \rho e^{\rho t} \left\{ \hat{\lambda}_3(t) - \hat{\lambda}_2(t) \right\} + \left( \frac{1}{k_N} + \frac{1}{k_T} \right) \left\{ h_3 \{X_3(t) - \hat{\lambda}_3\} \right\}$$

$$+ \hat{\lambda}_3(t) e^{\rho t} \left( \frac{2z}{z+1} - \frac{2z}{2z+1} \right) - h_2 \{X_2(t) - \hat{\lambda}_2\}$$

$$+ \hat{\lambda}_2(t) e^{\rho t} \left( \frac{1}{k_T} \right) e^{\rho t} \left\{ \hat{\lambda}_1(t) + \hat{\lambda}_3(t) \right\}$$

$$+ \frac{1}{k_T} \left[ h_1 \{X_1(t) - \hat{\lambda}_1\} + \hat{\lambda}_1(t) e^{\rho t} \left( \frac{\alpha \tau}{\alpha + 1} + \frac{(1 - \epsilon) \tau}{2} \right) + h_3 \{X_3(t) - \hat{\lambda}_3\} \right]$$

$$+ \hat{\lambda}_3(t) e^{\rho t} \left( \frac{2z}{z+1} - \frac{2z}{2z+1} \right) + \frac{1}{k_N} \rho e^{\rho t} \hat{\lambda}_1(t) + \frac{1}{k_N} \left[ h_1 \{X_1(t) - \hat{\lambda}_1\} \right]$$

$$+ \hat{\lambda}_1(t) e^{\rho t} \left( \frac{\alpha \tau}{\alpha + 1} + \frac{(1 - \epsilon) \tau}{2} \right) - \left( \frac{2z}{z+1} - \frac{2z}{2z+1} \right) \frac{d}{dt} \{X_3(t) - \hat{\lambda}_3\}.$$

(46)

Finally, substituting equation (45) into equation (46), yields
An optimal inventory control in hybrid manufacturing/ remanufacturing system

\[
\frac{d^2}{dt^2} X_3(t) = \left[ \frac{1}{k_N} + \frac{1}{k_r} \right] + \frac{\lambda_3(t)e^{\rho t}}{1} \left( \frac{2z}{z+1} \right) + \frac{\lambda_3(t)e^{\rho t}}{1} \left( \frac{2z}{z+1} \right) - h_2 \{ X_2(t) - \hat{x}_2 \} \\
- \lambda_2(t)e^{\rho t} \left( \frac{1}{\Gamma(\alpha, t)} + 1 \right) + \frac{1}{k_r} \left[ h_1 \{ X_1(t) - \hat{x}_1 \} \right] \\
+ \lambda_1(t)e^{\rho t} \left( \frac{1}{\alpha + 1} + \frac{(1-\varepsilon)t}{2} \right) + \frac{1}{k_N} \left[ \lambda_1(t)e^{\rho t} \left( \frac{2z}{z+1} \right) \right] \\
+ \left[ \frac{2z}{z+1} \right] \frac{d}{dt} X_3(t) \right] \tag{47}
\]

The equations system (30, 31, 32, 37, 42 and 47) is used to describe the time evolution of inventory levels, as well as total manufacturing, net manufacturing, remanufacturing and disposal rates.

The analytical solution of this system is very difficult. This system is nonlinear; we therefore solve it numerically with the initial condition \( X_i(0) = x^0_i \) and the terminal condition \( \lambda_i(T) = 0 \).

6 Numerical solution and sensitivity analysis

6.1 Numerical solution

In this sub-section, we present and illustrate the solutions of optimal control of manufacturing/remanufacturing-inventory model with time-varying demand numerically.

Consider an inventory system, with the following parameter values, in proper units

\[
T = 9; \rho = 0.001; h_1 = 1; h_2 = 2; h_3 = 3; k_r = 2; k_N = 3; k_3 = 4; \alpha = 3; \alpha = 6; \tau = 0.4; \\
\varepsilon = 0.3; \lambda_1 = 40; \lambda_2 = 30; \lambda_3 = 30; x^0_1 = 30; x^0_2 = 20; x^0_3 = 10; z = 0.6; \tau_x = 1; \rho = 0.5; \\
q = 0.2
\]

The numerical solutions of the system are discussed for three different types of demand rates:

1. an increasing function \( \{ D(t) = 30 + 5t \} \)
2. a decreasing function \( \{ D(t) = 50 - 2t \} \)
3. a fluctuating function \( \{ D(t) = 10[3 + \cos(e^{0.5t})] \} \).

By using MatLab software (version 8.5), we find the solution of the system:
From Figure 2, we can deduce the following:

1. optimal inventory levels (I.L) in the three stores are converging to its goals (I.G.L) in all cases of demand.
2. the inventory level of the first store is the fastest to reach its goal than other stores, despite it has the lowest penalty cost.

The left part of Figure 3 represents the optimal rates of total manufacturing (T.M.R), net manufacturing (N.M.R), remanufacturing (R.R) and disposal (D.R), whereas, their goal rates appear in the right side of figure in the case of an increasing demand.

The left part of Figure 4 represents the optimal rates of total manufacturing, net manufacturing, remanufacturing and disposal. Whereas, their goal rates appear in the right side of figure in the case of a decreasing demand.
The left part of Figure 5 represents the optimal rates of total manufacturing, net manufacturing, remanufacturing and disposal. Whereas, their goal rates appear in the right side of figure in the case of a fluctuating demand.

From Figures 3, 4 and 5, the left part (optimal rates) of figures similar to the right part (goal rates) of figures, that means the optimal total manufacturing, net manufacturing and remanufacturing rates are converging to their goals for different kinds of demand. Whereas, optimal and goal disposal rates take same values. As a result, objective function value up to the optimum value.

6.2 Sensitivity analysis

This sub-section explains the effect of the values of the Kumaraswamy and FPGB parameters on the rates of total manufacturing, net manufacturing, remanufacturing and disposal, addition to inventory levels. A sensitivity analysis is performed by increasing the Kumaraswamy and FPGB parameters, while maintaining the other parameter values as unchanged. Moreover, the effect of the type of demand on the model is clarified.
A.K. Dhaiban et al.

Table 2  Optimal solution with $D(t) = 30 + 5t$ and $z = 0.6$

<table>
<thead>
<tr>
<th>Time</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$U_T$</th>
<th>$U_N$</th>
<th>$U_r$</th>
<th>$U_\delta$</th>
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<td>29.98</td>
<td>19.96</td>
<td>80.75</td>
<td>80.75</td>
<td>71.04</td>
<td>71.04</td>
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Table 3  Optimal solution with $D(t) = 50 + 2t$ and $z = 0.6$

<table>
<thead>
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<th>$x_3$</th>
<th>$U_T$</th>
<th>$U_N$</th>
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<td>49.27</td>
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</table>

Table 4  Optimal solution with $D(t) = 10[3 + \cos(0.5t)]$ and $z = 0.6$

<table>
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<tr>
<th>Time</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$U_T$</th>
<th>$U_N$</th>
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Table 5  Inventory levels with $D(t) = 30 + 5t$

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The following can be deduced from Tables 2 through 5:

1. Inventory levels are insensitive towards a change in the demand function, Kumaraswamy parameter and FPGB parameter.

2. The manufacturing and remanufacturing rates are highly sensitive with respect to the demand functions.

The above conclusions go back to the fact that the manufacturing and remanufacturing rates are control variables, whereas the inventory levels are state variables.
Table 6  
Sensitivity analysis of the values of the Kumaraswamy and FPGB parameter with  
\[ D(t) = 30 + 5t \] and \( z = 0.3 \)

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The following can be deduced from Table 6:
1. Increasing the b value leads to a decrease in the total manufacturing rate and an increase in the remanufacturing rate by the same number of items at the end of the planning period. This means an increase in the returned rate.

2. Increasing the c value leads to an increase in the total manufacturing rate and a decrease in the remanufacturing rate by the same number of items at the end of the planning period. This means a decrease in the returned rate.

3. Increasing the b and c values together by the same percentage lead to the same result as without increasing. This means one of them cancels another impact.

Table 7  
Sensitivity analysis of the values of the Kumaraswamy and FPGB parameter with $D(t) = 30 + 5t$ and $z = 0.45$

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An optimal inventory control in hybrid manufacturing/ remanufacturing system

Table 8  Sensitivity analysis of the values of the Kumaraswamy and FPGB parameter with 
\( D(t) = 30 + 5t \) and \( z = 0.6 \)

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<th>( b )</th>
<th>( c )</th>
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As a result, the shape parameters give the administration greater flexibility in the decision making process according to previous data to increase or decrease the return rate within the upper and lower limits.

The following can be deduced from Tables 6 through 8:

1. The disposal rate is highly sensitive (increasing) and the remanufacturing rate is slightly sensitive (decreasing) towards an increase in the Kumaraswamy parameter value. This is because increasing the shape parameter value gives a curve that is skewed to left, thereby increasing the disposal rate, and vice versa.

2. Increasing the \( z \) and \( b \) values together decreases the total manufacturing rate and increases the remanufacturing rate by the same number of items at the end of the planning period to meet demand.
Increasing the $z$ and $c$ values together lead to an increase in the total manufacturing rate and a decrease in the remanufacturing rate by the same number of items at the end of the planning period, converging on the inventory goal level of the second store.

The biggest effect on the manufacturing rate (increasing) happens with an increase in $z$ and $c$ together as this has the same effect on the manufacturing rate (increasing).

7 Conclusions and recommendations

We studied an optimal manufacturing-remanufacturing inventory control model with defective, deterioration, returned and disposed items. The explicit solution of the model using the Pontryagin maximum principle was derived. The simulation and sensitivity analysis results, for the optimal control model with different demand patterns, were illustrated numerically to investigate the effect of the values of the Kumaraswamy and FPGB parameter on the inventory levels, manufacturing rate, remanufacturing rate and disposal rate. From the results, the optimal inventory levels in the three stores were converging to their goals in all cases of demand but the manufacturing and remanufacturing rates were highly sensitive with respect to the demand functions. The Kumaraswamy and FPGB parameter values had an effect on the rates of manufacturing, remanufacturing and disposal. The disposal rate was highly sensitive (increasing) and the remanufacturing rate was slightly sensitive (decreasing) towards the increase in the Kumaraswamy parameter value. The remanufacturing rate increased with an increase in the $b$ value, as a result of the increase in returned items, and decreased with an increase in the $c$ or $z$ value. The manufacturing rate was inversely related to the remanufacturing rate and positively related to the disposal rate. The inventory levels were insensitive toward the change in the demand and the Kumaraswamy and FPGB parameter values. This means that the model was efficient in controlling inventory levels. This model can be recommended for use in industries that have a high percentage of defective and deteriorating items and that depend on the remanufacture of some returned items to hedge demand. In practical terms, the use of probability distributions is a more realistic and flexible way of dealing with many situations. The model can be discussed in terms of the stochastic demand rate or stochastic holding cost.

Acknowledgements

The authors express heartfelt gratitude to the referees for their several useful comments and valuable suggestions.

References


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