Development of IFDEA models for IF input-oriented mix efficiency: case of hospitals in India

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Abstract: In conventional input-oriented mix efficiency (IOME), the input-output data are crisp numbers. But these data fluctuate in real world applications. Intuitionistic fuzzy set (IFS) theory can be used to solve such problem. In this paper, models are proposed to determine intuitionistic fuzzy input-oriented mix-efficiency (IFIOME) with IF input and IF output data. For determining IFIOME, intuitionistic fuzzy input-oriented CCR (IFIOCCR) model and intuitionistic fuzzy input-oriented slack-based measure (IFIOSBM) model are proposed with IF input-output data. These models are solved by using expected values of intuitionistic fuzzy numbers (IFNs). Based on IFIOME, a ranking method is developed to rank the DMUs. Also, the intuitionistic fuzzy correlation coefficient (IFCC) between IF variables is proposed to validate the proposed models. To validate the proposed models, an illustrative example and a health sector application are presented.

Keywords: intuitionistic fuzzy input-oriented CCR model; intuitionistic fuzzy input-oriented SBM model; intuitionistic fuzzy input-oriented mix-efficiency; IFIOME; hospital efficiency.

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1 Introduction

Data envelopment analysis (DEA) is a non-parametric linear programming (LP)-based technique to measure the relative efficiencies of homogeneous organisations or decision making units (DMUs). Charnes et al. (1978) proposed the constant returns to scale (CRS) model, known as CCR model to measure the relative efficiencies of homogeneous DMUs. It neglects the slacks in the calculation of efficiencies. Tone (2001) proposed the slack-based measure (SBM) model to solve this neglection. Tone (1998) proposed the mix efficiency (ME) from both the CCR and the SBM models. The ME is a measure to estimate how well the set of inputs are used (or outputs are produced) together (Herrero et al., 2006; Asbullah and Jaafar, 2010; Puri and Yadav, 2013). Tone (1998) proposed input-oriented mix efficiency (IOME) by using input-oriented CCR and input-oriented SBM models and output-oriented ME (OOME) by using output-oriented CCR and output-oriented SBM models. Gulati and Kumar (2016) determined the productivity growth of Indian banks. Valiakos and Charles (2017) proposed a statistical sample technique of large datasets to determine the frontiers. Khalfallah and Nabli (2018) proposed a hybrid approach to determine the bicriteria flow shop problem. Shekarian et al. (2017) proposed a fuzzy VIKOR method to determine the performance of healthcare. Ji et al. (2017) proposed a fuzzy decision making framework for treatment selection.

The conventional DEA is restricted to crisp input and output data. But real world applications have some input and/or output data which possess some degree of fluctuation, imprecision or uncertainties such as quality of input resources, quality of treatment, the satisfaction level of patients, quality of medicines etc. in health sector. The fluctuation can be represented as an interval/fuzzy number/intuitionistic fuzzy number (IFN), etc.

Fuzzy set (FS) theory (Zadeh, 1965) is an important tool to handle the fluctuations/uncertainties in real world problems. FDEA models are developed to determine the performance efficiencies of DMUs with fuzzy data (see Sengupta, 1992a, 1992b; Hatami-Marbini et al., 2011; Arya and Yadav, 2018a; Soudi et al., 2019).

In FS theory, the sum of the membership (acceptance) value and non-membership (rejection) value of an element being in the set is equal to one. But in real life applications, the sum of the acceptance and rejection values of an element being in the set may be one or it may be less than one. Thus, there is some degree of hesitation. Intuitionistic fuzzy set (IFS) theory deals with such a situation. IFS theory, proposed by Atanassov (1986), is an extension of the FS theory. So, IFS theory is more suitable to deal with such type of situation rather than FS theory. IFDEA models are developed to determine the performance efficiencies of DMUs with IF data (see De et al., 2001; Puri and Yadav, 2015; Arya and Yadav, 2018b, 2018c, Otay et al., 2017; Hajiagha et al., 2013).

In health sector, the managers/other authorities reorganise the facilities: beds, non-medical staff and medical staff time to time. Therefore, the uncertainty comes in beds, non-medical staff and medical staff, i.e., uncertainty in input data. Some patients leave the hospital without treatment due to insufficient resources: such as available beds for hospitalisation and impossibility to provide appropriate care etc. Therefore, there is uncertainty in output data. Due to this, in order to deal with uncertainty in health sector, the input data and output data are taken as IFNs.

For the present study, let us consider two IF inputs:

1 number of doctors

2 number of pharmacists.

two IF outputs:

1 number of inpatients

2 number of outpatients.

In this piece of work, a crisp input-oriented CCR (CIOCCR) model and crisp input-oriented SBM (CIOSBM) model are extended to fuzzy input-oriented CCR (FIOCCR) model and fuzzy input-oriented SBM (FIOSBM) model respectively by using FNs, then to intuitionistic fuzzy input-oriented CCR (IFIOCCR) model and intuitionistic fuzzy input-oriented SBM (IFIOSBM) model respectively by using IFNs. Further, the results of these models are applied to determine intuitionistic fuzzy input-oriented mix-efficiency (IFIOME). An intuitionistic fuzzy correlation coefficient (IFCC) between IF variables is also proposed and defuzzify it. All the proposed approaches are then applied to an example and health sector application.

The rest of the paper is organised as follows: Section 2 presents preliminary. Section 3 presents the extension of fuzzy input-oriented mix efficiency (FIOME) to IFIOME. Section 4 presents the proposed IFCC. Section 5 presents an illustrative example and a case study (health sector) to illustrate and validate the proposed models. Section 6 presents the conclusion of the paper.

2 Preliminary

Definition 1 (performance efficiency): The performance efficiency of a DMU is defined as the ratio of the weighted sum of outputs (called virtual output) to the weighted sum of inputs (called virtual input) (Arya and Yadav, 2017). Thus,

$$Performance\ efficiency = \frac{virtual\ output}{virtual\ input}.$$

The relative performance efficiency of a DMU is defined as the ratio of its performance efficiency to the largest performance efficiency. The relative performance efficiency of a DMU lies in (0, 1]. DEA evaluates the relative performance efficiency of a set of homogeneous DMUs.

Definition 2 (IFS): Let $\mathbb{X}$ be the universe of discourse. Then an IFS (Atanassov, 1986) is denoted by $\tilde{A}^I$ and defined by $\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x))\}$, where $\mu_{\tilde{A}^I} : \mathbb{X} \to [0,1]$ and $\nu_{\tilde{A}^I} : \mathbb{X} \to [0,1]$ represent the membership and non-membership functions respectively. The values $\mu_{\tilde{A}^I}(x)$ and $\nu_{\tilde{A}^I}(x)$ represent the membership and non-membership values of x being in $\tilde{A}^I$ with the condition that $0 \leq \mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) \leq 1$, $\mu_{\tilde{A}^I}(x) \in [0,1]$ and $\nu_{\tilde{A}^I}(x) \in [0,1]$. The hesitation (indeterminacy) degree of an element x being in $\tilde{A}^I$ is denoted by $\pi_{\tilde{A}^I}$ and is defined by $\pi_{\tilde{A}^I}(x) = 1 - \mu_{\tilde{A}^I}(x) - \nu_{\tilde{A}^I}(x)\ \forall x \in \mathbb{X}$. Obviously $0 \leq \pi_{\tilde{A}^I}(x) \leq 1$. If $\pi_{\tilde{A}^I}(x) = 0\, \forall x \in \mathbb{X}$, then $\tilde{A}^I$ is reduced to an FS.

Definition 3 (normal IFS): Let $\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)); x \in \mathbb{X}\}$ be an IFS. Then $\tilde{A}^I$ is called the normal IFS (Grzegorzewski, 2003), if $\exists\ x_1, x_2 \in \mathbb{X}$ such that $\mu_{\tilde{A}^I}(x_1) = 1$

and $\nu_{\tilde{A}^I}(x_2) = 1$.

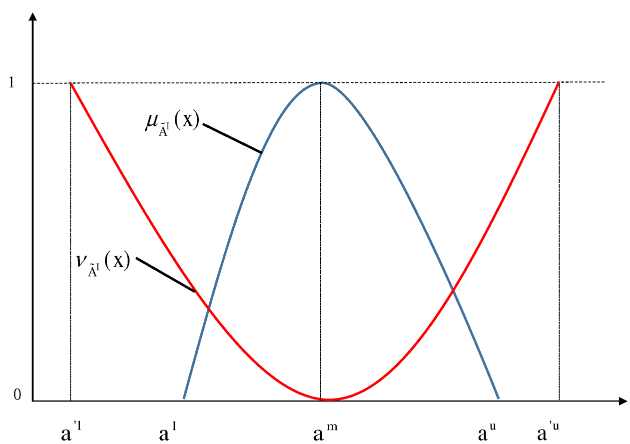
Definition 4 (convex IFS): Let $\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)); x \in \mathbb{X}\}$ be an IFS. Then $\tilde{A}^I$ is called the convex IFS if

- $\min(\mu_{\tilde{A}^I}(x), \mu_{\tilde{A}^I}(y)) \leq \mu_{\tilde{A}^I}(\lambda x + (1-\lambda)y)$, $\forall x, y \in \mathbb{X}$ and $\lambda \in [0,1]$, i.e., $\mu_{\tilde{A}^I}$ is a quasi-concave function over $\mathbb{X}$.
- $\max(\nu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(y)) \geq \nu_{\tilde{A}^I}(\lambda x + (1-\lambda)y)$, $\forall x, y \in \mathbb{X}$ and $\lambda \in [0,1]$, i.e., $\nu_{\tilde{A}^I}$ is a quasi-convex function over $\mathbb{X}$.

Definition 5 (IFN): Let $\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)) : x \in \mathbb{R}\}$ be an IFS with its membership function $\mu_{\tilde{A}^I}$ and non-membership function $\nu_{\tilde{A}^I}$, where $\mathbb{R}$ is the set of real numbers. Then $\tilde{A}^I$ is called an IFN (Mahapatra and Roy, 2009) if the following conditions hold:

- $\exists$ a unique $x_0 \in \mathbb{R}$ such that $\mu_{\tilde{A}^I}(x_0) = 1$, i.e., $\tilde{A}^I$ is normal. x_0 is called the mean value of $\tilde{A}^I$.
- $\tilde{A}^I$ is convex IFS over $\mathbb{R}$.

Figure 1 Membership and non-membership functions of IFN $\tilde{A}^I$ (see online version for colours)



Mathematically, an IFS $\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)) : x \in \mathbb{R}\}$ is an IFN if $\mu_{\tilde{A}^I}$ and $\nu_{\tilde{A}^I}$ are piecewise continuous functions from $\mathbb{R}$ to $[0,1]$ and $0 \leq \mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) \leq 1$ $\forall x \in \mathbb{R}$, where

$$\mu_{\tilde{A}^I}(x) = \begin{cases} g_1(x), a^l \leq x < a^m, \\ 1, \quad x = a^m, \\ g_2(x), a^m < x \leq a^u, \\ 0, \quad \text{elsewhere.} \end{cases} \qquad \nu_{\tilde{A}^I}(x) = \begin{cases} h_1(x), a'^l \leq x < a^m, \\ 0, \quad x = a^m, \\ h_2(x), a^m < x \leq a'^u, \\ 1, \quad \text{elsewhere.} \end{cases}$$

where $a'^l \leq a^l \leq a^m \leq a^u \leq a'^u$; a^m is called the mean or modal value of $\tilde{A}^I$; $a^m - a^l$ and $a^u - a^m$ are called the left and right hand spreads of the membership function

$\mu_{\tilde{A}^I}$ respectively; $a^m - a'^l$ and $a'^u - a^m$ are called the left and right hand spreads of the hesitation function $\pi_{\tilde{A}^I}$ respectively; g_1 and g_2 are piecewise continuous, strictly increasing and strictly decreasing functions in $[a^l, a^m)$ and $(a^m, a^u]$ respectively; h_1 and h_2 are piecewise continuous, strictly decreasing and strictly increasing functions in $[a'^l, a^m)$ and $(a^m, a'^u]$ respectively. The IFN $\tilde{A}^I$ is represented by $\tilde{A}^I = (a^m; a^m - a^l, a^u - a^m; a^m - a'^l, a'^u - a^m)$. Its graphical representation is given in Figure 1.

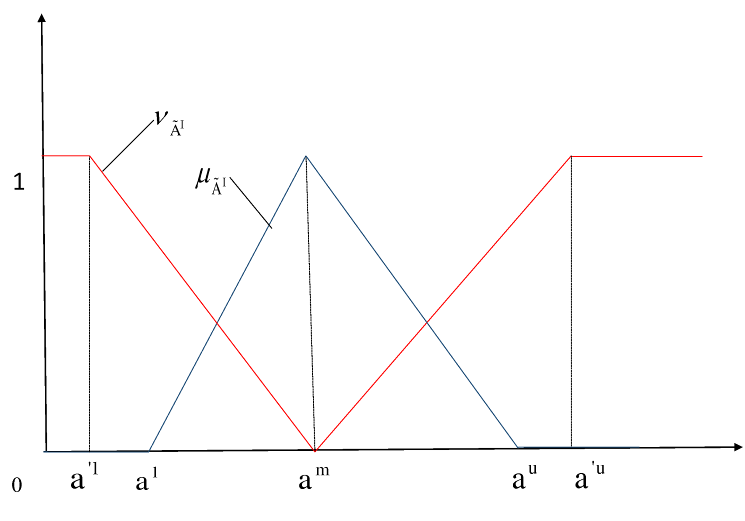
Definition 6 (positive IFN): Let $\tilde{A}^I = (a^m; a^m - a^l, a^u - a^m; a^m - a'^l, a'^u - a^m)$ be an IFN, where $a'^l \le a^l \le a^m \le a^u \le a'^u$. Then $\tilde{A}^I$ is called a positive IFN if $a'^l > 0$.

Definition 7 [triangular intuitionistic fuzzy number (TIFN) (Mahapatra and Roy, 2009)]: Let $\tilde{A}^I = (a^l, a^m, a^u; a'^l, a^m, a'^u)$ is an IFN with the membership function $\mu_{\tilde{A}^I}$ and non-membership function $\nu_{\tilde{A}^I}$. Then $\tilde{A}^I$ is called a TIFN if $\mu_{\tilde{A}^I}$ and $\nu_{\tilde{A}^I}$ are given by

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \dfrac{x - a^l}{a^m - a^l}, & a^l < x \le a^m, \\ \dfrac{a^u - x}{a^u - a^m}, & a^m \le x < a^u \\ 0, & \text{elsewhere.} \end{cases} \qquad \nu_{\tilde{A}^I}(x) = \begin{cases} \dfrac{x - a^m}{a'^l - a^m}, & a'^l < x \le a^m, \\ \dfrac{a^m - x}{a^m - a'^u}, & a^m \le x < a'^u, \\ 1, & \text{elsewhere.} \end{cases}$$

where $a^l, a^m, a^u, a'^l, a'^u \in \mathbb{R}$ such that $a'^l \le a^l \le a^m \le a^u \le a'^u$. The TIFN is represented by $(a^l, a^m, a^u; a'^l, a^m, a'^u)$ Its graphical representation is given in Figure 2.

Figure 2 TIFN $\tilde{A}^I = (a^l, a^m, a^u; a'^l, a^m, a'^u)$ (see online version for colours)



Definition 8 (arithmetic operations on TIFNs): Let $\tilde{A}^I = (a^l, a^m, a^u; a'^l, a^m, a'^u)$ and $\tilde{B}^I = (b^l, b^m, b^u; b'^l, b^m, b'^u)$ be two TIFNs. Then the arithmetic operations on TIFNs (Mahapatra and Roy, 2009) are defined as follows:

- *Addition:* $\tilde{A}^I \oplus \tilde{B}^I = (a^l + b^l, a^m + b^m, a^u + b^u; a'^l + b'^l, a^m + b^m, a'^u + b'^u)$.
- *Multiplication:* $\tilde{A}^I \otimes \tilde{B}^I \simeq (a^l b^l, a^m b^m, a^u b^u; a'^l b'^l, a^m b^m, a'^u b'^u)$, where $\tilde{A}^I, \tilde{B}^I > 0$.

- *Scalar multiplication:* If $\lambda \in \mathbb{R}$, then

$$\lambda \tilde{A}^I = \begin{cases} (\lambda a^l, \lambda a^m, \lambda a^u; \lambda a'^l, \lambda a^m, \lambda a'^u), \text{for } \lambda \geq 0, \\ (\lambda a^u, \lambda a^m, \lambda a^l; \lambda a'^u, \lambda a^m, \lambda a'^l), \text{for } \lambda < 0. \end{cases}$$

Definition 9 (expected value of IFNs and their ordering): Let $\tilde{A}^I = (a^m; a^m - a^l, a^u - a^m; a^m - a'^l, a'^u - a^m)$ be an IFN with membership and non-membership functions $\mu_{\tilde{A}^I}(x)$ and $\nu_{\tilde{A}^I}(x)$ respectively. Then the expected interval (Grzegorzewski, 2003) of $\tilde{A}^I$ is defined as the crisp interval $EI(\tilde{A}^I) = [E^L(\tilde{A}^I), E^U(\tilde{A}^I)]$, where

$$E^L(\tilde{A}^I) = \frac{a'^l + a^m}{2} + \frac{1}{2}\int_{a'^l}^{a^m} h_1(x)dx - \frac{1}{2}\int_{a^l}^{a^m} g_1(x)dx, \tag{2.1}$$

$$E^U(\tilde{A}^I) = \frac{a^m + a'^u}{2} + \frac{1}{2}\int_{a^m}^{a^u} g_2(x)dx - \frac{1}{2}\int_{a^m}^{a'^u} h_2(x)dx. \tag{2.2}$$

The expected value of an IFN $\tilde{A}^I$ is defined by $EV(\tilde{A}^I)$ and is defined by:

$$EV(\tilde{A}^I) = \frac{E^L(\tilde{A}^I) + E^U(\tilde{A}^I)}{2}. \tag{2.3}$$

Theorem 1: Let $\tilde{A}^I = (a^l, a^m, a^u; a'^l, a^m, a'^u)$ be a TIFN. Then $EV(\tilde{A}^I) = \frac{a'^l + a^l + 4a^m + a^u + a'^u}{8}$.

Proof: Using equation (2.1), we have $E^L(\tilde{A}^I) = \frac{a^l + 2a^m + a'^l}{4}$.

Using equation (2.2), we have $E^U(\tilde{A}^I) = \frac{a^u + 2a^m + a'^u}{4}$.

Therefore, $EV(\tilde{A}^I) = \frac{a'^l + a^l + 4a^m + a^u + a'^u}{8}$. ■

Definition 10 (ordering of TIFNs): Let $\tilde{A}^I$ = (a^l, a^m, a^u; a'^l, a^m, a'^u) and $\tilde{B}^I$ = (b^l, b^m, b^u; b'^l, b^m, b'^u) be two TIFNs. Then

1. $\tilde{A}^I \geq \tilde{B}^I \iff EV(\tilde{A}^I) \geq EV(\tilde{B}^I)$
2. $\tilde{A}^I \leq \tilde{B}^I \iff EV(\tilde{A}^I) \leq EV(\tilde{B}^I)$
3. $\tilde{A}^I = \tilde{B}^I \iff EV(\tilde{A}^I) = EV(\tilde{B}^I)$
4. $\min(\tilde{A}^I, \tilde{B}^I) = \tilde{A}^I \; if \tilde{A}^I \leq \tilde{B}^I$
5. $\max(\tilde{A}^I, \tilde{B}^I) = \tilde{A}^I \; if \; \tilde{A}^I \geq \tilde{B}^I$.

3 Extension of FIOME to IFIOME

3.1 IOCCR and IOSBM models

Charnes et al. (1978) proposed the IOCCR model. Tone (2001) proposed the IOSBM model. Suppose that the performance of a set of n homogeneous DMUs (DMU_j; j = 1, 2, 3, ..., n) is to be determined. The performance efficiency of DMU_j is characterised by a production process of m inputs x_{ij} (i = 1, 2, 3, ..., m) to produce s outputs y_{rj} (r = 1, 2, 3, ..., s). Assume x_{ik} be the amount of the ith input used and y_{rk} be the amount of the rth output produced by the DMU_k. Let input data and output data be positive. Let s_{ik}^{-}, s_{rk}^{+} be the slacks of the IOCCR model and S_{ik}^{-}, S_{rk}^{+} be the slacks of the IOSBM model. Let θ_k and ρ_k be the IOCCR efficiency and IOSBM efficiency of DMU_k respectively. The IOCCR and IOSBM models for DMU_k are given in Models 1(a) and 1(b) respectively.

Model 1(a) IOCCR model

$$\begin{aligned}
&\min\ \theta_k \\
&\text{subject to} \\
&x_{ik}\theta_k = \sum_{j=1}^{n} x_{ij}\lambda_{jk} + s_{ik}^{-}\ \forall i, \\
&y_{rk} = \sum_{j=1}^{n} y_{rj}\lambda_{jk} - s_{rk}^{+}\ \forall r, \\
&s_{ik}^{-} \geq 0,\ s_{rk}^{+} \geq 0,\ \lambda_{jk} \geq 0, \\
&\theta_k \text{ unrestricted in sign.}
\end{aligned}$$

Model 1(b) IOSBM model

$$\begin{aligned}
&\min\ \rho_k = 1 - \frac{1}{m}\sum_{i=1}^{m} S_{ik}^{-}/x_{ik}, \\
&\text{subject to} \\
&x_{ik} = \sum_{j=1}^{n} x_{ij}\eta_{jk} + S_{ik}^{-}\ \forall i, \\
&y_{rk} = \sum_{j=1}^{n} y_{rj}\eta_{jk} - S_{rk}^{+}\ \forall r, \\
&\eta_{jk} \geq 0\,\forall j, S_{ik}^{-} \geq 0\,\forall i\,, S_{rk}^{+} \geq 0\,\forall r.
\end{aligned}$$

3.2 Input-oriented mix-efficiency

Let θ_k^* and ρ_k^* be the optimal values of IOCCR and IOSBM efficiencies respectively. Then the ratio of IOSBM efficiency ρ_k^* and IOCCR efficiency θ_k^* of DMU_k is called the IOME of DMU_k and is denoted by χ_k. Thus,

$$\chi_k = \frac{\rho_k^*}{\theta_k^*} 0 < \rho_k^* \leq 1 \text{ and } 0 < \theta_k^* \leq 1.$$

Obviously, $0 < \chi_k^* \leq 1$ iff $\rho_k^* \leq \theta_k^*$; and $\chi_k^* = 1$ iff $\rho_k^* = \theta_k^*$. DMU_k is said to have the most efficient combination of inputs if $\chi_k^* = 1$.

Definition 11: DMU_k is called the *IOCCR-efficient* if θ_k^* = 1 and all slacks are zero, i.e., s_{ik}^{-*} = 0, i = 1, 2, 3, ..., m and s_{rk}^{+*} = 0, r = 1, 2, 3, ..., s. Otherwise, DMU_k is called the *IOCCR-inefficient*.

Definition 12 (peer group or reference set): Let DMU_k be IOCCR-inefficient DMU, i.e., $\theta_k^* < 1$ or $(s_{ik}^{-*}, s_{rk}^{+*}) \neq (0,0)$. Then the set R_k of the IOCCR-efficient DMUs for which $\lambda_{jk}^* > 0$ is called the peer group or reference set of DMU_k. Thus, $R_k = \{DMU_j : \lambda_{jk}^* > 0\}$ is called the peer group or reference set of DMU_k.

For an IOCCR-inefficient DMU_k, we have the following expressions:

$$x_{ik}\theta_k^* = \sum_{j=1}^{n} x_{ij}\lambda_{jk}^* + s_{ik}^{-*} \ \forall i,$$

$$y_{rk} = \sum_{j=1}^{n} y_{rj}\lambda_{jk}^* - s_{rk}^{+*} \ \forall r,$$

The IOCCR-inefficient DMU_k can be improved and become IOCCR-efficient by using the following operation:

$$x_{ik} \leftarrow x_{ik}\theta_k^* - s_{ik}^{-*} \ \forall i, \tag{3.1}$$

$$y_{rk} \leftarrow y_{rk} + s_{rk}^{+*} \ \forall r, \tag{3.2}$$

This operation is called the *IOCCR-projection*.

Definition 13 (Tone, 1998): DMU_k is called the *IOSBM-efficient* if $\rho_k^* = 1$ and $S_{rk}^{+*} = 0 \ \forall r$. Otherwise, DMU_k is called the *IOSBM-inefficient*.

Theorem 2: $\rho_k^* = 1$ iff $S_{ik}^{-*} = 0 \ \forall i$.

Proof: $\rho_k^* = 1 - \frac{1}{m}\sum_{i=1}^{m} S_{ik}^{-*}/x_{ik} \implies \rho_k^* = 1$ iff $S_{ik}^{-*} = 0 \ \forall i$. ■

Definition 14 (peer group or reference set): Let DMU_k be IOSBM-inefficient DMU, i.e., $\rho_k^* < 1$ or $S_{ik}^{-*} \neq 0$. Then the set R_k of the IOSBM-efficient DMUs for which $\eta_{jk}^* > 0$ is called the peer group or reference set of DMU_k. Thus, $R_k = \{DMU_j : \eta_{jk}^* > 0\}$ is called the peer group or reference set of DMU_k.

For an IOSBM-inefficient DMU_k, we have the following expressions (Tone, 2001):

$$x_{ik} = \sum_{j=1}^{n} x_{ij}\eta_{jk}^* + S_{ik}^{-*} \ \forall i,$$

$$y_{rk} = \sum_{j=1}^{n} y_{rj}\eta_{jk}^* - S_{rk}^{+*} \ \forall r,$$

The IOSBM-inefficient DMU_k can be improved and become IOSBM-efficient by using the following operation:

$$x_{ik} \leftarrow x_{ik} - S_{ik}^{-*} \ \forall i, \tag{3.3}$$

$$y_{rk} \leftarrow y_{rk} + S_{rk}^{+*} \ \forall r, \tag{3.4}$$

This operation is called the *IOSBM-projection*.

The reference set of multiple optimal solutions is not unique (Tone, 2001). In this case we can choose any one of them for our purpose.

Let $I_k = \{j | DMU_j \in R_k\}$.

Using I_k, we can express (x_{ik}, y_{rk}) by

$$x_{ik} = \sum_{j \in I_k} x_{ij} \eta_{jk}^* + S_{ik}^{-*} \ \forall i, \tag{3.5}$$

$$y_{rk} = \sum_{j \in I_k} y_{rj} \eta_{jk}^* - S_{rk}^{+*} \ \forall r, \tag{3.6}$$

Since the IOSBM ρ_k^* depends only on slack S_{ik}^{-*}, ρ_k^* is not affected by the values attributed to other DMUs not in R_k.

Now, we shall prove the following theorem:

Theorem 3 (Tone, 2001): The optimal IOSBM ρ_k^* is less than or equal to the optimal IOCCR θ_k^*.

Proof: Let $(\theta_k^*, \lambda_{jk}^*, s_{ik}^{-*}, s_{rk}^{+*})$ be an optimal solution of IOCCR model. Then the following holds

$$x_{ik} = \sum_{j=1}^{n} x_{ij} \lambda_{jk}^* + s_{ik}^{-*} + x_{ik}(1 - \theta_k^*) \ \forall i, \tag{3.7}$$

$$y_{rk} = \sum_{j=1}^{n} y_{rj} \lambda_{jk}^* - s_{rk}^{+*} \ \forall r. \tag{3.8}$$

Let $\eta_{jk} = \lambda_{jk}^*$, $S_{ik}^- = s_{ik}^{-*} + (1 - \theta_k^*) x_{ik}$ and $S_{rk}^+ = s_{rk}^{+*}$. Then $(\eta_{jk}, S_{ik}^-, S_{rk}^+)$ is feasible solution for the IOSBM and the objective value of the IOSBM at this solution is

$$\rho_k = 1 - \frac{1}{m} \sum_{i=1}^{m} [s_{ik}^{-*} + x_{ik}(1 - \theta_k^*)]/x_{ik}$$

$$= \theta_k^* - \frac{1}{m} \sum_{i=1}^{m} s_{ik}^{-*}/x_{ik} \tag{3.9}$$

$$\leq \theta_k^* \tag{3.10}$$

Since $\rho_k^* \leq \rho_k$,

$$\rho_k^* \leq \theta_k^* \tag{3.11}$$

■

Conversely, let $(\rho_k^*, \eta_{jk}^*, S_{ik}^{-*}, S_{rk}^{+*})$ be IOSBM optimal solution. Then the following holds:

$$x_{ik}\theta_k = \sum_{j=1}^{n} x_{ij}\eta_{jk}^* + S_{ik}^{-*} + x_{ik}(\theta_k - 1) \ \forall i. \tag{3.12}$$

$$y_{rk} = \sum_{j=1}^{n} y_{rj}\eta_{jk}^* - S_{rk}^{+*} \ \forall r. \tag{3.13}$$

Let us assume that

$$S_{ik}^{-*} + x_{ik}(\theta_k - 1) \geq 0. \tag{3.14}$$

Then $(\theta_k, \lambda_{jk} = \eta_{jk}^*, s_{ik}^{-} = S_{ik}^{-*} + x_{ik}(\theta_k - 1), s_{rk}^{+} = S_{rk}^{+*})$ is a feasible solution for IOCCR model.

Theorem 4 (Tone, 2001): A DMU_k is IOCCR-efficient iff it is IOSBM-efficient.

Proof: Let DMU_k be IOCCR-inefficient. Then $\theta_k^* < 1$ or $\theta_k^* = 1$ and $(s_{ik}^{-*}, s_{rk}^{+*}) \neq (0, 0)$. From equation (3.9), $\rho_k < 1$ in both cases for a feasible solution of IOSBM model. Consequently, $\rho_k^* < 1$. Therefore, DMU_k is IOSBM-inefficient.

Conversely, let DMU_k be IOSBM-inefficient. Then $\rho_k^* < 1$ or $S_{rk}^{+*} \neq 0$.

1. Let $\rho_k^* < 1$. Then from Theorem 2, $S_{ik}^{-*} \neq 0$ for some i. We know that $(\theta_k, \lambda_{jk} = \eta_{jk}^*, s_{ik}^{-} = S_{ik}^{-*} + x_{ik}(\theta_k - 1), s_{rk}^{+} = S_{rk}^{+*})$ is a feasible solution for IOCCR model, provided $S_{ik}^{-*} + x_{ik}(\theta_k - 1) \geq 0$. There are two cases:

   Case 1 Let $\theta_k = 1$. Then $s_{ik}^{-} = S_{ik}^{-*} \neq 0$. Therefore, DMU_k is IOCCR-inefficient.

   Case 2 Let $\theta_k < 1$. Then $\theta_k^* < 1$ therefore DMU_k is IOCCR-inefficient.

2. Let $S_{rk}^{+*} \neq 0$. Then $s_{rk}^{+} = S_{rk}^{+*} \neq 0$. Therefore, DMU_k is IOCCR-inefficient.

Thus, IOCCR inefficiency is equivalent to IOSBM inefficiency. Since definitions of inefficient and efficient are mutually exclusive, DMU_k is IOCCR-efficient iff it is IOSBM-efficient. ■

3.3 FIOCCR and FIOSBM models

To describe FIOCCR and FIOSBM efficiency evaluations, the fuzzy efficiency of DMU_j (j = 1, 2, ..., n) is characterised by a production process of m fuzzy inputs $\tilde{x}_{ij}$; i = 1, 2, 3, ..., m to yield s fuzzy outputs $\tilde{y}_{rj}$; r = 1, 2, 3, ..., s (Arya and Yadav, 2017). Suppose that the input and output data are positive FNs. The FIOCCR and FIOSBM fuzzy efficiencies of the DMU_k are denoted by θ_k and $\tilde{\rho}_k$ respectively. Models 1(a) and 1(b) are extended to the Models 2(a) and 2(b) respectively.

Model 2(a) FIOCCR model

$$\begin{aligned}
&\min\ \theta_k \\
&\text{subject to} \\
&\tilde{x}_{ik}\theta_k = \sum_{j=1}^{n} \tilde{x}_{ij}\lambda_{jk} + \tilde{s}_{ik}^{-}\ \forall i, \\
&\tilde{y}_{rk} = \sum_{j=1}^{n} \tilde{y}_{rj}\lambda_{jk} - \tilde{s}_{rk}^{+}\ \forall r, \\
&\tilde{s}_{ik}^{-} \geq \tilde{0}\ \forall i,\ \tilde{s}_{rk}^{+} \geq \tilde{0}\ \forall r, \\
&\theta_k \text{ unrestricted in sign}
\end{aligned}$$

Model 2(b) FIOSBM model

$$\begin{aligned}
&\min\ \tilde{\rho}_k = 1 - \frac{1}{m}\sum_{i=1}^{m} \tilde{S}_{ik}^{-}/\tilde{x}_{ik}, \\
&\text{subject to} \\
&\tilde{x}_{ik} = \sum_{j=1}^{n} \tilde{x}_{ij}\eta_{jk} + \tilde{S}_{ik}^{-}\ \forall i, \\
&\tilde{y}_{rk} = \sum_{j=1}^{n} \tilde{y}_{rj}\eta_{jk} - \tilde{S}_{rk}^{+}\ \forall r, \\
&\eta_{jk} \geq 0\ \forall j, \tilde{S}_{ik}^{-} \geq \tilde{0}\ \forall i\ , \tilde{S}_{rk}^{+} \geq \tilde{0}\ \forall r,
\end{aligned}$$

Definition 15: Let θ_k^* be the optimal value of θ_k. Then DMU_k is said to be *FIOCCR-efficient* if $\theta_k^* = 1$ and all slacks are zero, i.e., $\tilde{s}_{ik}^{-*} = \tilde{0}\ \forall i$ and $\tilde{s}_{rk}^{+*} = \tilde{0}\ \forall r$. Otherwise, DMU_k is said to be *FIOCCR-inefficient*.

Definition 16 (peer group or reference set): Let DMU_k be FIOCCR-inefficient DMU, i.e., $\theta_k^* < 1$ or $(\tilde{s}_{ik}^{-*}, \tilde{s}_{rk}^{+*}) \neq (\tilde{0}, \tilde{0})$. Then the set R_k of the FIOCCR-efficient DMUs for which $\lambda_{jk}^* > 0$ is called the peer group or reference set of DMU_k. Thus, $R_k = \{DMU_j : \lambda_{jk}^* > 0\}$ is called the peer group or reference set of DMU_k.

For an FIOCCR-inefficient DMU_k, we have the following expressions:

$$\tilde{x}_{ik}\theta_k^* = \sum_{j=1}^{n} \tilde{x}_{ij}\lambda_{jk}^* + \tilde{s}_{ik}^{-*}\ \forall i,$$

$$\tilde{y}_{rk} = \sum_{j=1}^{n} \tilde{y}_{rj}\lambda_{jk}^* - \tilde{s}_{rk}^{+*}\ \forall r.$$

The FIOCCR-inefficient DMU_k can be improved and become FIOCCR-efficient by using the following operation:

$$\tilde{x}_{ik} \leftarrow \tilde{x}_{ik}\theta_k^* - \tilde{s}_{ik}^{-*}\ \forall i, \tag{3.15}$$

$$\tilde{y}_{rk} \leftarrow \tilde{y}_{rk} + \tilde{s}^{+*}_{rk} \ \forall r. \tag{3.16}$$

This operation is called the *FIOCCR-projection*.

Definition 17: Let $\tilde{\rho}^*_k$ be the optimal value of $\tilde{\rho}_k$. Then DMU_k is said to be *FIOSBM-efficient* if $\tilde{\rho}^*_k = \tilde{1}$ and $\tilde{S}^{+*}_{rk} = \tilde{0} \ \forall r$. Otherwise, DMU_k is said to be *FIOSBM-inefficient*.

Theorem 5: $\tilde{\rho}^*_k = \tilde{1}$ iff $\tilde{S}^{-*}_{ik} = \tilde{0} \ \forall i$.

Proof: $\tilde{\rho}^*_k = 1 - \dfrac{1}{m} \sum\limits_{i=1}^{m} \tilde{S}^{-*}_{ik} / \tilde{x}_{ik} \implies \tilde{\rho}^*_k = \tilde{1}$ iff $\tilde{S}^{-*}_{ik} = \tilde{0} \ \forall i$. ■

Definition 18 (peer group or reference set): Let DMU_k be FIOSBM-inefficient DMU, i.e., $\tilde{\rho}^*_k < \tilde{1}$ or $\tilde{S}^{+*}_{rk} \neq \tilde{0}$. Then the set R_k of the FIOSBM-efficient DMUs for which $\eta^*_{jk} > 0$ is called the peer group or reference set of DMU_k. Thus, $R_k = \{DMU_j : \eta^*_{jk} > 0\}$ is called the peer group or reference set of DMU_k.

For an FIOSBM-inefficient DMU_k, we have the following expressions:

$$\tilde{x}_{ik} = \sum_{j=1}^{n} \tilde{x}_{ij}\eta^*_{jk} + \tilde{S}^{-*}_{ik} \ \forall i,$$

$$\tilde{y}_{rk} = \sum_{j=1}^{n} \tilde{y}_{rj}\eta^*_{jk} - \tilde{S}^{+*}_{rk} \ \forall r.$$

The FIOSBM-inefficient DMU_k can be improved and become FIOSBM-efficient by using the following operation:

$$\tilde{x}_{ik} \leftarrow \tilde{x}_{ik} - \tilde{S}^{-*}_{ik} \ \forall i, \tag{3.17}$$

$$\tilde{y}_{rk} \leftarrow \tilde{y}^{+}_{rk} + \tilde{S}^{+*}_{rk} \ \forall r. \tag{3.18}$$

This operation is called the *FIOSBM-projection*.

The reference set of multiple optimal solutions is not unique. In this case we can choose any one of them for our purpose.

Let $I_k = \{j | DMU_j \in R_k\}$.

Using I_k, we can express $(\tilde{x}_{ik}, \tilde{y}_{rk})$ by

$$\tilde{x}_{ik} = \sum_{j \in I_k} \tilde{x}_{ij}\eta^*_{jk} + \tilde{S}^{-*}_{ik} \ \forall i, \tag{3.19}$$

$$\tilde{y}_{rk} = \sum_{j \in I_k} \tilde{y}_{rj}\eta^*_{jk} - \tilde{S}^{+*}_{rk} \ \forall r, \tag{3.20}$$

Since the FIOSBM $\tilde{\rho}^*_k$ depends only on slack $\tilde{S}^{-*}_{ik}$, $\tilde{\rho}^*_k$ is not affected by the values attributed to other DMUs not in R_k.

3.4 Proposed IFIOME

The IF efficiency of DMU_j (j = 1, 2,..., n) is characterised by a production process of m IF inputs $\tilde{x}^I_{ij}$; i = 1, 2, 3, ..., m to yield s IF outputs $\tilde{y}^I_{rj}$; r = 1, 2, 3, ..., s. Suppose that the IF input and IF output data are positive IFNs. The IFIOCCR and intuitionistic IFIOSBM efficiencies of the DMU_k are denoted by θ_k and $\tilde{\rho}^I_k$ respectively. Models 2(a) and 2(b) are extended to the Models 3(a) and 3(b) respectively as follows:

Model 3(a) IFIOCCR model

$$\begin{aligned}
&\min\ \theta_k \\
&\text{subject to} \\
&\tilde{x}^I_{ik}\theta_k = \sum_{j=1}^{n} \tilde{x}^I_{ij}\lambda_{jk} + \tilde{s}^{-I}_{ik}\ \forall i, \\
&\tilde{y}^I_{rk} = \sum_{j=1}^{n} \tilde{y}^I_{rj}\lambda_{jk} - \tilde{s}^{+I}_{rk}\ \forall r, \\
&\tilde{s}^{-I}_{ik} \geq \tilde{0}^I\ \forall i,\ \tilde{s}^{+I}_{rk} \geq \tilde{0}^I\ \forall r, \\
&\theta_k \text{ unrestricted in sign}
\end{aligned}$$

Model 3(b) IFIOSBM model

$$\begin{aligned}
&\tilde{\rho}^I_k = \min 1 - \frac{1}{m}\sum_{i=1}^{m} \tilde{S}^{-^I}_{ik} / \tilde{x}^I_{ik}, \\
&\text{subject to} \\
&\tilde{x}^I_{ik} = \sum_{j=1}^{n} \tilde{x}^I_{ij}\eta_{jk} + \tilde{S}^{-^I}_{ik}\ \forall i, \\
&\tilde{y}^I_{rk} = \sum_{j=1}^{n} \tilde{y}^I_{rj}\eta_{jk} - \tilde{S}^{+^I}_{rk}\ \forall r, \\
&\eta_{jk} \geq 0\ \forall j, \tilde{S}^{-^I}_{ik} \geq \tilde{0}^I\ \forall i\ , \tilde{S}^{+^I}_{rk} \geq \tilde{0}^I\ \forall r.
\end{aligned}$$

Model 3(a) is the proposed IFIOCCR (PIFIOCCR) model and Model 3(b) is the proposed IFIOSBM (PIFIOSBM) model.

Definition 19: Let θ^*_k be the optimal value of PIFIOCCR efficiency θ_k of Model 3(a). Then DMU_k is called the *PIFIOCCR-efficient* if $\theta^*_k = 1$ and all slacks are zero, i.e., $\tilde{s}^{-I*}_{ik} = \tilde{0}^I\ \forall i$ and $\tilde{s}^{+I*}_{rk} = \tilde{0}^I\ \forall r$. Otherwise, DMU_k is called the *PIFIOCCR-inefficient*.

Definition 20 (peer group or reference set): Let DMU_k be PIFIOCCR-inefficient DMU, i.e., $\theta^*_k < 1$ or $(\tilde{s}^{-I*}_{ik}, \tilde{s}^{+I*}_{rk}) \neq (\tilde{0}^I, \tilde{0}^I)$. Then the set R_k of the PIFIOCCR-efficient DMUs for which $\lambda^*_{jk} > 0$ is called the peer group or reference set of DMU_k. Thus, $R_k = \{DMU_j : \lambda^*_{jk} > 0\}$ is called the peer group or reference set of DMU_k.

For an PIFIOCCR-inefficient DMU_k, we have the following expressions:

$$\tilde{x}_{ik}^{I}\theta_k^* = \sum_{j=1}^{n} \tilde{x}_{ij}^{I}\lambda_{jk}^* + \tilde{s}_{ik}^{-I*} \ \forall i.$$

$$\tilde{y}_{rk}^{I} = \sum_{j=1}^{n} \tilde{y}_{rj}^{I}\lambda_{jk}^* - \tilde{s}_{rk}^{+I*} \ \forall r.$$

The PIFIOCCR-inefficient DMU_k can be improved and become PIFIOCCR-efficient by using the following operation:

$$\tilde{x}_{ik}^{I} \leftarrow \tilde{x}_{ik}^{I}\theta_k^* - \tilde{s}_{ik}^{-I*} \ \forall i. \tag{3.21}$$

$$\tilde{y}_{rk}^{I} \leftarrow \tilde{y}_{rk}^{I} + \tilde{s}_{rk}^{+I*} \ \forall r. \tag{3.22}$$

This operation is called the *PIFIOCCR-projection*.

Definition 21: Let $\tilde{\rho}_k^{I*}$ be the optimal value of PIFIOSBM efficiency $\tilde{\rho}_k^{I}$ of Model 3(b). Then DMU_k is called the *PIFIOSBM-efficient* if $\tilde{\rho}_k^{I*} = \tilde{1}^I$ and $\tilde{S}_{rk}^{+I*} = \tilde{0}^I \ \forall r$. Otherwise, DMU_k is called the *PIFIOSBM-inefficient*.

Definition 22 (peer group or reference set): Let DMU_k be PIFIOSBM-inefficient DMU, i.e., $\tilde{\rho}_k^{I*} < \tilde{1}^I$ or $\tilde{S}_{ik}^{-I*} \neq \tilde{0}^I$. Then the set R_k of the PIFIOSBM-efficient DMUs for which $\eta_{jk}^* > 0$ is called the peer group or reference set of DMU_k. Thus, $R_k = \{DMU_j : \eta_{jk}^* > 0\}$ is called the peer group or reference set of DMU_k.

Theorem 6: $\tilde{\rho}_k^{I*} = \tilde{1}^I$ iff $\tilde{S}_{ik}^{-I*} = \tilde{0}^I \ \forall i$.

Proof: $\tilde{\rho}_k^{I*} = 1 - \frac{1}{m}\sum_{i=1}^{m} \tilde{S}_{ik}^{-I*}/\tilde{x}_{ik}^{I} \implies \tilde{\rho}_k^{I*} = \tilde{1}^I$ iff $\tilde{S}_{ik}^{-I*} = \tilde{0}^I \ \forall i$. ■

For an PIFIOSBM-inefficient DMU_k, we have the following expressions:

$$\tilde{x}_{ik}^{I} = \sum_{j=1}^{n} \tilde{x}_{ij}^{I}\eta_{jk}^* + \tilde{S}_{ik}^{-I*} \ \forall i,$$

$$\tilde{y}_{rk}^{I} = \sum_{j=1}^{n} \tilde{y}_{rj}^{I}\eta_{jk}^* - \tilde{S}_{rk}^{+I*} \ \forall r.$$

The PIFIOSBM-inefficient DMU_k can be improved and become PIFIOSBM-efficient by using the following operation:

$$\tilde{x}_{ik}^{I} \leftarrow \tilde{x}_{ik}^{I}\theta_k^* - \tilde{S}_{ik}^{-I*} \ \forall i, \tag{3.23}$$

$$\tilde{y}_{rk}^{I} \leftarrow \tilde{y}_{rk}^{I} + \tilde{S}_{rk}^{+I*} \ \forall r. \tag{3.24}$$

This operation is called the *PIFIOSBM-projection*.

The reference set of multiple optimal solutions is not unique. In this case we can choose any one of them for our purpose.

Let $I_k = \{j|DMU_j \in R_k\}$.
Using I_k, we can express $(\tilde{x}^I_{ik}, \tilde{y}^I_{rk})$ by

$$\tilde{x}^I_{ik} = \sum_{j \in I_k} \tilde{x}^I_{ij}\eta^*_{jk} + \tilde{S}^{-I*}_{ik} \ \forall i, \tag{3.25}$$

$$\tilde{y}^I_{rk} = \sum_{j \in I_k} \tilde{y}^I_{rj}\eta^*_{jk} - \tilde{S}^{+I*}_{rk} \ \forall r, \tag{3.26}$$

Since the PIFIOSBM $\tilde{\rho}^{I*}_k$ depends only on slack $\tilde{S}^{-I*}_{ik}$, $\tilde{\rho}^{I*}_k$ is not affected by the values attributed to other DMUs not in R_k.

Definition 23: The ratio of PIFIOSBM efficiency and PIFIOCCR efficiency of the DMU_k is called the proposed IF IOME (PIFIOME) of the DMU_k. It is denoted by $\tilde{\chi}^I_k$. Thus,

$$\tilde{\chi}^I_k = \frac{\tilde{\rho}^{*I}_k}{\theta^*_k}, \ \theta^*_k > 0.$$

3.4.1 Methodology for solving IFIOCCR and IFIOSBM models

Assume that IF input, IF output data and slacks in Model 3(a) are TIFNs. Let $\tilde{x_{ij}}^I = ({x_{ij}}^l, {x_{ij}}^m, {x_{ij}}^u; {x_{ij}}^{'l}, {x_{ij}}^m, {x_{ij}}^{'u})$, $\tilde{y_{rj}}^I = ({y_{rj}}^l, {y_{rj}}^m, {y_{rj}}^u; {y_{rj}}^{'l}, {y_{rj}}^m, {y_{rj}}^{'u})$, $\tilde{s}^{-I*}_{ik} = (s^{-l}_{ik}, s^{-m}_{ik}, s^{-u}_{ik}; s^{-'l}_{ik}, s^{-m}_{ik}, s^{-'u}_{ik})$ and $\tilde{s}^{+I*}_{rk} = (s^{+l}_{rk}, s^{+m}_{rk}, s^{+u}_{rk}, s^{+'l}_{rk}, s^{+m}_{rk}, s^{+'u}_{rk})$. Then the IFIOCCR model is reduced to Model 4:

Model 4

$$\begin{aligned}
&\min \theta_k \\
&\text{subject to} \\
&(x^l_{ik}, {x_{ik}}^m, {x_{ik}}^u; {x_{ik}}^{'l}, {x_{ik}}^m, {x_{ik}}^{'u})\theta_k \\
&= \sum_{j=1}^{n} ({x_{ij}}^l, {x_{ij}}^m, {x_{ij}}^u; {x_{ij}}^{'l}, {x_{ij}}^m, {x_{ij}}^{'u})\lambda_{jk} \\
&+(s^{-l}_{ik}, s^{-m}_{ik}, s^{-u}_{ik}; s^{-'l}_{ik}, s^{-m}_{ik}, s^{-'u}_{ik}) \ \forall i, \\
&({y_{rk}}^l, {y_{rk}}^m, {y_{rk}}^u; {y_{rk}}^{'l}, {y_{rk}}^m, {y_{rk}}^{'u}) \\
&= \sum_{j=1}^{n} ({y_{rj}}^l, {y_{rj}}^m, {y_{rj}}^u; {y_{rj}}^{'l}, {y_{rj}}^m, {y_{rj}}^{'u})\lambda_{jk} \\
&-(s^{+l}_{rk}, s^{+m}_{rk}, s^{+u}_{rk}, s^{+'l}_{rk}, s^{+m}_{rk}, s^{+'u}_{rk}) \ \forall r, \\
&s^{-'l}_{ik} \geq 0 \ \forall i, \ \ s^{+'l}_{rk} \geq 0 \ \forall r, \ \ \theta_k \text{ unrestricted in sign.}
\end{aligned}$$

Assume that IF input, IF output data and slacks in Model 3(b) are TIFNs. Let $\tilde{x_{ij}}^I = ({x_{ij}}^l, {x_{ij}}^m, {x_{ij}}^u; {x_{ij}}^{'l}, {x_{ij}}^m, {x_{ij}}^{'u})$, $\tilde{y_{rj}}^I = ({y_{rj}}^l, {y_{rj}}^m, {y_{rj}}^u; {y_{rj}}^{'l}, {y_{rj}}^m, {y_{rj}}^{'u})$, $\tilde{S}^{-I}_{ik} = (S^{-l}_{ik}, S^{-m}_{ik}, S^{-u}_{ik}; S^{-'l}_{ik}, S^{-m}_{ik}, S^{-'u}_{ik})$ and $\tilde{S}^{+I}_{rk} = (S^{+l}_{rk}, S^{+m}_{rk}, S^{+u}_{rk}; S^{+'l}_{rk}, S^{+m}_{rk}, S^{+'u}_{rk})$. Then the IFIOSBM model is reduced to the following model (Model 5):

Model 5

$$
\begin{aligned}
&\min\ (\rho_k^l, \rho_k^m, \rho_k^u; \rho_k^{'l}, \rho_k^m, \rho_k^{'u}) = 1 \\
&-\frac{1}{m}\sum_{i=1}^{m} \frac{(S_{ik}^{-l}, S_{ik}^{-m}, S_{ik}^{-u}; S_{ik}^{-'l}, S_{ik}^{-m}, S_{ik}^{-'u})}{(x_{ik}^l, {x_{ij}}^m, {x_{ik}}^u; {x_{ik}}^{'l}, {x_{ik}}^m, {x_{ik}}^{'u})}, \\
&\text{subject to} \\
&(x_{ik}^l, {x_{ik}}^m, {x_{ik}}^u; {x_{ik}}^{'l}, {x_{ik}}^m, {x_{ik}}^{'u}) \\
&= \sum_{j=1}^{n} (x_{ij}^l, {x_{ij}}^m, {x_{ij}}^u; {x_{ij}}^{'l}, {x_{ij}}^m, {x_{ij}}^{'u}) \otimes \eta_{jk} \\
&+(S_{ik}^{-l}, S_{ik}^{-m}, S_{ik}^{-u}; S_{ik}^{-'l}, S_{ik}^{-m}, S_{ik}^{-'u})\ \forall i, \\
&({y_{rk}}^l, {y_{rk}}^m, {y_{rk}}^u; {y_{rk}}^{'l}, {y_{rk}}^m, {y_{rk}}^{'u}) \\
&= \sum_{j=1}^{n} ({y_{rj}}^l, {y_{rj}}^m, {y_{rj}}^u; {y_{rj}}^{'l}, {y_{rj}}^m, {y_{rj}}^{'u}) \otimes \eta_{jk} \\
&-(S_{rk}^{+l}, S_{rk}^{+m}, S_{rk}^{+u}; S_{rk}^{+'l}, S_{rk}^{+m}, S_{rk}^{+'u})\ \forall r, \\
&\eta_{jk} \geq 0\ \forall j\ \ S_{ik}^{-'l} \geq 0\ \forall i \\
&S_{rk}^{+'l} \geq 0\ \forall r,
\end{aligned}
$$

Using expected values of IFNs, Model 4 is reduced to following model (Model 6):

Model 6

$$
\begin{aligned}
&\min\ \theta_k \\
&\text{subject to} \\
&(x_{ik}^l + 4{x_{ik}}^m + {x_{ik}}^u + {x_{ik}}^{'l} + {x_{ik}}^{'u})\theta_k \\
&= \sum_{j=1}^{n} ({x_{ij}}^l + 4{x_{ij}}^m + {x_{ij}}^u + {x_{ij}}^{'l} + {x_{ij}}^{'u})\lambda_{jk} \\
&+(s_{ik}^{-l} + 4s_{ik}^{-m} + s_{ik}^{-u} + s_{ik}^{-'l} + s_{ik}^{-'u})\ \forall i, \\
&({y_{rk}}^l + 4{y_{rk}}^m + {y_{rk}}^u + {y_{rk}}^{'l} + {y_{rk}}^{'u}) \\
&= \sum_{j=1}^{n} ({y_{rj}}^l + 4{y_{rj}}^m + {y_{rj}}^u + {y_{rj}}^{'l} + {y_{rj}}^{'u})\lambda_{jk} \\
&-(s_{rk}^{+l} + 4s_{rk}^{+m} + s_{rk}^{+u} + s_{rk}^{+'l} + s_{rk}^{+'u})\ \forall r, \\
&s_{ik}^{-'l} \geq 0\ \forall i\ s_{rk}^{+'l} \geq 0\ \forall r, \\
&\theta_k \text{ unrestricted in sign.}
\end{aligned}
$$

Using expected values of IFNs, Model 5 is reduced to following model (Model 7).

Model 7

$$
\min\ \rho_k^1 = 1 - \frac{1}{m}
$$

$$\sum_{i=1}^{m}(S_{ik}^{-\,l} + 4S_{ik}^{-\,m} + S_{ik}^{-\,u} + S_{ik}^{-\,'l} + S_{ik}^{-\,'u})/(x_{ik}^{l} + 4{x_{ik}}^{m} + {x_{ik}}^{u} + {x_{ik}}^{'l} + {x_{ik}}^{'u}),$$

subject to

$$(x_{ik}^{l} + 4{x_{ik}}^{m} + {x_{ik}}^{u} + {x_{ik}}^{'l} + {x_{ik}}^{'u})$$
$$= \sum_{j=1}^{n}(x_{ij}^{l} + 4{x_{ij}}^{m} + {x_{ij}}^{u} + {x_{ij}}^{'l} + {x_{ij}}^{'u})\eta_{jk}$$
$$+(S_{ik}^{-\,l} + 4S_{ik}^{-\,m} + S_{ik}^{-\,u} + S_{ik}^{-\,'l} + S_{ik}^{-\,'u})\ \forall i,$$
$$({y_{rk}}^{l} + 4{y_{rk}}^{m} + {y_{rk}}^{u} + {y_{rk}}^{'l} + {y_{rk}}^{'u}) =$$
$$\sum_{j=1}^{n}({y_{rj}}^{l} + 4{y_{rj}}^{m} + {y_{rj}}^{u} + {y_{rj}}^{'l} + {y_{rj}}^{'u})\eta_{jk}$$
$$-(S_{rk}^{+\,l} + 4S_{rk}^{+\,m} + S_{rk}^{+\,u} + S_{rk}^{+\,'l} + S_{rk}^{+\,'u})\ \forall r,$$
$$\eta_{jk} \geq 0\ \forall j\ \ S_{ik}^{-\,'l} \geq 0\ \forall i$$
$$S_{rk}^{+\,'l} \geq 0\ \forall r,$$

Model 6 is the proposed IOCCR (PIOCCR) model and Model 7 is the proposed IOSBM (PIOSBM) model.

Definition 24: If $\theta_k^* = 1$ and all slacks are zero, then DMU_k is said to be *PIOCCR-efficient*. Otherwise, DMU_k is said to be *PIOCCR-inefficient*.

Definition 25: Let ρ_k^{1*} be optimal value of ρ_k^1. If $\rho_k^{1^*} = 1$, then DMU_k is said to be *PIOSBM-efficient*. Otherwise, DMU_k is said to be *PIOSBM-inefficient*.

Definition 26: The ratio of PIOSBM efficiency and PIOCCR efficiency of DMU_k is called the proposed IOME (PIOME) of DMU_k. It is denoted by $\chi_k^{1^*}$. Thus

$$\chi_k^{1^*} = \frac{\rho_k^{1^*}}{\theta_k^*}\ \theta_k^* > 0 \tag{3.27}$$

DMU_k has the most efficient combination of inputs if $\chi_k^{1^*} = 1$; otherwise inefficient combination.

Definition 27: If $\chi_k^{1^*} = 1$, then DMU_k is said to be *PIOM-efficient*. Otherwise, DMU_k is said to be *PIOM-inefficient*.

4 Proposed IFCC between two IF variables using expected value

Definition 28 (correlation coefficient (CC) between two crisp variables): Let x_i and y_r be two crisp variables and (x_i, y_r) assume the values (x_{ij}, y_{rj}), j = 1, 2, 3, ..., n. Then the CC between x_i and y_r is denoted by $C_\rho(x_i, y_r)$ and is defined by

$$C_\rho(x_i, y_r) = \frac{n\sum_{j=1}^{n} x_{ij}y_{rj} - \sum_{j=1}^{n} x_{ij}\sum_{j=1}^{n} y_{rj}}{\sqrt{n\sum_{j=1}^{n} {x_{ij}}^2 - \left(\sum_{j=1}^{n} x_{ij}\right)^2} \cdot \sqrt{n\sum_{j=1}^{n} {y_{rj}}^2 - \left(\sum_{j=1}^{n} y_{rj}\right)^2}}. \tag{4.1}$$

Let $\tilde{x}_i^I$ and $\tilde{y}_r^I$ be two IF variables and $(\tilde{x}_i^I, \tilde{y}_r^I)$ assume the values $(\tilde{x}_{ij}^I, \tilde{y}_{rj}^I)$, j = 1, 2, 3, ..., n. Then the IFCC between $\tilde{x}_i^I$ and $\tilde{y}_r^I$ is denoted by $\tilde{C}_\rho^I(\tilde{x}_i^I, \tilde{y}_r^I)$ and is defined by

$$\tilde{C}_\rho^I(\tilde{x}_i^I, \tilde{y}_r^I) = \frac{n\sum_{j=1}^{n} \tilde{x_{ij}}^I \tilde{y_{rj}}^I - \sum_{j=1}^{n} \tilde{x_{ij}}^I \sum_{j=1}^{n} \tilde{y_{rj}}^I}{\sqrt{n\sum_{j=1}^{n} (\tilde{x_{ij}}^I)^2 - \left(\sum_{j=1}^{n} \tilde{x_{ij}}^I\right)^2} \cdot \sqrt{n\sum_{j=1}^{n} (\tilde{y_{rj}}^I)^2 - \left(\sum_{j=1}^{n} \tilde{y_{rj}}^I\right)^2}}. \tag{4.2}$$

This is the proposed IFCC (PIFCC) between two IF variables. But, it is difficult to apply equation (4.2) if data is large. So, we proceed as follows:

We find a real number which will act as an representative of the PIFCC $\tilde{C}_\rho^I(\tilde{x}_i^I, \tilde{y}_r^I)$. For this we propose a new method using the EV of TIFN as described below: assume that $\tilde{x}_{ij}^I = (x_{ij}^l, x_{ij}^m, x_{ij}^u; x_{ij}^{'l}, x_{ij}^m, x_{ij}^{'u})$ and $\tilde{y}_{rj}^I = (y_{rj}^l, y_{rj}^m, y_{rj}^u; y_{rj}^{'l}, y_{rj}^m, y_{rj}^{'u})$, j = 1, 2, 3, ..., n. Then

$$EV(\tilde{x}_{ij}^I) = \frac{1}{8}(x_{ij}^l + 4x_{ij}^m + x_{ij}^u + x_{ij}^{'l} + x_{ij}^{'u}) = a_{ij} \text{ (say)},$$
$$EV(\tilde{y}_{rj}^I) = \frac{1}{8}(y_{rj}^l + 4y_{rj}^m + y_{rj}^u + y_{rj}^{'l} + y_{rj}^{'u}) = b_{rj} \text{ (say)} \tag{4.3}$$

Now we find the CC between a_i and b_r using equation (4.1), where (a_i, b_r) assume values (a_{ij}, b_{rj}) j = 1, 2, 3, ..., n given by equation (4.3).

Remark 1: Let $\tilde{x}_i^I$ and $\tilde{y}_r^I$ be the IF inputs and IF outputs; and $(\tilde{x}_i^I, \tilde{y}_r^I)$ assume values $(\tilde{x}_{ij}^I, \tilde{y}_{rj}^I)$ corresponding to the jth DMU. Let $a_{ij} = EV(\tilde{x}_{ij}^I)$ and $b_{rj} = EV(\tilde{y}_{rj}^I)$. If $C_\rho(a_i, a_p), C_\rho(a_i, b_q), C_\rho(b_q, b_r)$ are all positive, where i, p = 1, 2, 3, ..., m; q, r = 1, 2, 3, ..., s, then the proposed model is consistent and inclusion of IF input-output data is justified.

4.1 Ranking approach

Step 1 Find the PIOCCR and PIOSBM efficiencies for each DMU using Models 6 and 7 respectively.

Step 2 Find the PIOME for each DMU using equation (3.27).

Step 3 DMUs are ranked according to the decreasing values of PIOMEs.

4.2 Advantages of the proposed approaches

1 The proposed models and IOME for IF data.

2 Herrero et al. (2006), Asbullah and Jaafar (2010) and Tone (1998) proposed the input mix-efficiency for crisp data, Puri and Yadav (2013) proposed the fuzzy IOME for fuzzy data, but did not develop the models for intuitionistic fuzzy data.

5 Numerical examples

In this section we have taken two numerical examples. The first is illustrative example and the second is application to the health sector taken from Meerut District, Uttar Pradesh, India.

5.1 Illustrative example

Let there be five DMUs having two IF inputs and two IF outputs which are represented as TIFNs (see Table 1). The CCs between EVs of IF input and IF output data are determined using equation (4.1) and listed in Table 2. Since all the CCs listed in Table 2 are positive, the inclusion of the IF input and IF output data is justified and the PIOCCR, PIOSBM models are consistent.

The PIOCCR efficiencies, PIOSBM efficiencies and PIOMEs of the DMUs are calculated using Models 6, 7 and equation (3.27) respectively, and the results are shown in Table 3. Using the ranking approach described in Subsection 4.1, the DMUs are ranked as $DMU2 = DMU4 = DMU5 > DMU3 > DMU1$. The results show that DMU2, DMU4 and DMU5 are PIOCCR-efficient, PIOSBM-efficient and also PIOM-efficient. There are two DMUs (DMU1 and DMU3) which are PIOM-inefficient.

Table 1 IF input and IF output data for five DMUs

DMUs	*Inputs*		*Outputs*	
	$\tilde{x}^I_{1j}$	$\tilde{x}^I_{2j}$	$\tilde{y}^I_{1j}$	$\tilde{y}^I_{2j}$
1	(3.5, 4.0, 4.5; 3.2, 4.0, 4.7)	(1.9, 2.1, 2.3; 1.7, 2.1, 2.5)	(2.4, 2.6, 2.8; 2.2, 2.6, 3)	(3.8, 4.1, 4.4; 3.6, 4.1, 4.6)
2	(2.9, 2.9, 2.9; 2.9, 2.9, 2.9)	(1.4, 1.5, 1.6; 1.3, 1.5, 1.8)	(2.2, 2.2, 2.2; 2.2, 2.2, 2.2)	(3.3, 3.5, 3.7; 3.1, 3.5, 3.9)
3	(4.4, 4.9, 5.4; 4.2, 4.9, 5.6)	(2.2, 2.6, 3.0; 2.1, 2.6, 3.2)	(2.7, 3.2, 3.7; 2.5, 3.2, 3.9)	(4.3, 5.1, 5.9; 4.1, 5.1, 6.2)
4	(3.4, 4.1, 4.8; 3.1, 4.1, 4.9)	(2.2, 2.3, 2.4; 2.1, 2.3, 2.6)	(2.5, 2.9, 3.3; 2.4, 2.9, 3.6)	(5.5, 5.7, 5.9; 5.3, 5.7, 6.1)
5	(5.9, 6.5, 7.1; 5.6, 6.5, 7.2)	(3.6, 4.1, 4.6; 3.5, 4.1, 4.7)	(4.4, 5.1, 5.8; 4.2, 5.1, 6.6)	(6.5, 7.4, 8.3; 5.6, 7.4, 9.2)

Table 2 CCs between EVs of IF inputs-outputs

	a_1	a_2	b_1	b_2
a_1	1	0.9877	0.9661	0.9175
a_2	0.9877	1	0.993	0.9451
b_1	0.9661	0.9933	1	0.9385
b_2	0.9157	0.9451	0.9385	1

Table 3 Ranking of five DMUs

DMUs	θ_k^*	*Rank*	ρ_k^{1*}	*Rank*	χ_k^{1*}	*Rank*
1	0.8574	3	0.855	2	0.9972	3
2	1	1	1	1	1	1
3	0.8599	2	0.853	3	0.99	2
4	1	1	1	1	1	1
5	1	1	1	1	1	1

5.2 Application to the health sector

An application is presented to illustrate and validate the PIOCCR and PIOSBM models and PIOME. In this section, DMUs are CHCs in Meerut District of Uttar Pradesh, India. The performance of each CHC is determined based on two IF inputs: number of doctors (say $\tilde{x}_{1j}^I$), number of pharmacists (say $\tilde{x}_{2j}^I$) and two IF outputs: number of inpatients (say $\tilde{y}_{1j}^I$), number of outpatients (say $\tilde{y}_{2j}^I$) for DMU_j, j = 1, 2, 3, ..., 12. The IF input-output data are given in Table 4.

The CCs between EVs of IF input and IF output data are determined using equation (4.1) and listed in Table 5. Since all the CCs in Table 5 are positive, the inclusion of the IF input and IF output data is justified and the PIOCCR and PIOSBM models are consistent.

The PIOCCR, PIOSBM efficiencies and PIOMEs of all hospitals are evaluated using Models 6, 7 and equation (3.27) respectively, which are shown in Table 6. Finally, the ranks of the hospitals using the ranking approach described in Subsection 4.1 are obtained (see Table 6). The hospitals are ranked as $H1 = H2 = H3 > H5 > H9 > H6 > H11 > H12 > H8 > H4 > H10 > H7$ based on PIOCCR model. The hospitals are ranked as $H1 = H2 = H3 > H5 > H9 > H12 > H8 > H11 > H6 > H10 > H4 > H7$ based on PIOSBM model. The hospitals are ranked as $H1 = H2 = H3 > H12 > H8 > H9 > H5 > H7 > H10 > H11 > H6 > H4$ based on PIOME. The hospitals H1, H2 and H3 are PIOM-efficient and have the most efficient combinations of IF input data.

In Table 6, the efficiency scores of hospitals in %age are as follows:

$H1$(100), $H2$(100), $H3$(100), $H12$(99), $H8$(98.3), $H9$(98), $H5$(93), $H7$(91), $H10$(90), $H11$(88), $H6$(75), $H4$(71). According to efficiency scores of hospitals, the least efficient hospital is H4. The inefficient hospitals (having efficiency scores <100%) are not utilising their inputs fully. The inefficiency scores in %age of DMUs are as follows:

$H1(0)$, $H2(0)$, $H3(0)$, $H12(1)$, $H8(1.7)$, $H9(2)$, $H5(7)$, $H7(9)$, $H10(10)$, $H11(12)$, $H6(25)$, $H4(29)$. Consequently, the inefficient hospitals should reduce their inputs by their respective inefficiency scores in %age to become efficient.

Table 4 IF input-output data of 12 hospitals

DMU	*IF inputs*		*IF outputs*	
	$\tilde{x}_{1j}^I$	$\tilde{x}_{2j}^I$	$\tilde{y}_{1j}^I$	$\tilde{y}_{2j}^I$
H1	(5, 10, 19; 3, 10, 28)	(3, 5, 8; 2, 5, 10)	(3,640, 3,650, 3,665; 3,635, 3,650, 3,695)	(134,130, 134,137, 134,140; 134,125, 134,137, 134,145)
H2	(4, 9, 17; 3, 9, 26)	(3, 5, 7; 2, 5, 9)	(4,150, 4,160, 4,175; 4,148, 4,160, 4,195)	(116,060, 116,062, 116,070; 116,055, 116,062, 116,075)
H3	(8, 11, 19; 5, 11, 29)	(2, 4, 5; 1, 4, 6)	(4,360, 4,370, 4,380; 4,357, 4,370, 4,398)	(94,060, 94,066, 94,070; 94,055, 94,066, 94,075)
H4	(3, 8, 15; 2, 8, 22)	(1, 1, 3; 1, 1, 5)	(485, 492, 500; 483, 492, 515)	(24,325, 24,329, 24,334; 24,322, 24,329, 24,338)
H5	(6, 10, 18; 5, 10, 26)	(3, 4, 6; 2, 4, 10)	(2.460, 2,464, 2,470; 2,458, 2,464, 2,475)	(99,745, 99,748, 99,750; 99,742, 99,748, 99,755)
H6	(7, 11, 21; 5, 11, 31)	(2, 3, 4; 1, 3, 5)	(1,360, 1,368, 1,378; 1,358, 1,368, 1,398)	(49,398, 49,401, 49,405; 49,395, 49,401, 49,409)
H7	(8, 10, 16; 5, 10, 25)	(1, 2, 6; 1, 2, 18)	(1,055, 1,062, 1,080; 1,050, 1,062, 1,083)	(37,769, 37,772, 37,776; 37,765, 37,772, 37,779)
H8	(7, 11, 18; 5, 11, 29)	(2, 4, 7; 1, 4, 19)	(1,295, 1,302, 1,310; 1,290, 1,302, 1,325)	(82,838, 82,841, 82,845; 82,835, 82,841, 82,849)
H9	(8, 12, 19; 5, 12, 28)	(2, 5, 7; 1, 5, 15)	(1,660, 1,671, 1,690; 1,657, 1,671, 16,105)	(100,590, 100,596, 100,600; 100,586, 100,596, 100,605)
H10	(9, 15, 19; 8, 15, 22)	(2, 4, 6; 1, 4, 18)	(1,010, 1,018, 1,035; 1,008, 1,018, 1,045)	(64,349, 64,351, 64,358; 64,345, 64,351, 64,360)
H11	(6, 11, 16; 5, 11, 18)	(3, 5, 8; 2, 5, 20)	(1,500, 1,504, 1,515; 1,495, 1,504, 1,535)	(80,050, 80,056, 80,060; 80,045, 80,056, 80,065)
H12	(4, 8, 11; 3, 8, 14)	(3, 4, 6; 1, 4, 7)	(1,960, 1,965, 1,972; 1,958, 1,965, 19,85)	(58,160, 58,167, 58,170; 58,157, 58,167, 58,174)

Source: Chief Medical Office, Head Office, Meerut, India

Table 5 CCs between EVs of IF inputs-outputs

	a_1	a_2	b_1	b_2
a_1	1	0.31	0.11	0.22
a_2	0.31	1	0.26	0.61
b_1	0.11	0.26	1	0.83
b_2	0.22	0.61	0.83	1

Table 6 Ranking of 12 hospitals

DMUs	θ_k^*	*Rank*	ρ_k^{1*}	*Rank*	χ_k^{1*}	*Rank*
H1	1	1	1	1	1	1
H2	1	1	1	1	1	1
H3	1	1	1	1	1	1
H4	0.55	10	0.39	11	0.71	12
H5	0.86	4	0.80	4	0.93	7
H6	0.66	6	0.49	9	0.75	11
H7	0.35	12	0.32	12	0.91	8
H8	0.59	9	0.58	7	0.983	5
H9	0.75	5	0.73	5	0.98	6
H10	0.48	11	0.43	10	0.90	9
H11	0.64	7	0.56	8	0.88	10
H12	0.62	8	0.61	6	0.99	4

6 Conclusions

In real life problems, the data are not always available in precise form. To deal with such data, we have considered them as TIFNs. So, in this paper, IOCCR model has been extended to IFIOCCR model, IOSBM to IFIOSBM model and IOME to IFIOME using expected value approach in which data are taken as TIFNs. IFIOCCR, IFIOSBM models and IFIOME are developed to measure the performance efficiencies of the DMUs. Further, a ranking approach has been developed and applied for comparing and ranking the DMUs in terms of PIOCCR, PIOSBM and PIOME, which presents not only a full ranking but also gives the information that to what degree PIOME is bigger/lesser than other. The proposed methods are applied to an illustrative example and a health sector (real life problem) to determine the performances of five DMUs and twelve CHCs (hospitals) respectively and they are ranked accordingly. The CCs between EVs of IF inputs and IF outputs are determined to justify the data and to validate the proposed models. The hospitals input-oriented mix inefficiency values in %age are: $H1(0)$, $H2(0)$, $H3(0)$, $H12(1)$, $H8(1.7)$, $H9(2)$, $H5(7)$, $H7(9)$, $H10(10)$, $H11(12)$, $H6(25)$, $H4(29)$. The input-oriented mix-inefficiency score in % of a hospital represents the degree by which all the inputs be reduced to become the hospital fully efficient. Hence, based on the models developed, the finding of this paper is that all the input-oriented mix-inefficient hospitals are suggested to reduce their inputs in order to become fully input-oriented mix-efficient.

In future, this paper can be extended to the fully IFIOME (FIFIOME) and the corresponding ranking approach.

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Compliance with ethical standards

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Ethical approval

This article does not contain any studies with human participants performed by any of the authors.

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