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Abstract: In this paper, we reformulate a problem of pricing American put options to linear complementarity problem. The space and the time are discretised with the finite difference method in the Crank-Nickolson approach, which leads to present the put option price as a solution of the linear complementarity problem. For solving this problem and evaluating the put options we use a fast algorithm. We apply our study for an example on oil options.

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1 Introduction

ExxonMobil Corporation is an American oil and gas company, headed by Darren Woods and headquartered in Irving, a suburb of Dallas. It has 45 refineries in twenty-five countries which have a distillation capacity of 6.3 million barrels of oil per day. It also has 42,000 service stations in more than 100 countries under the Exxon, Esso and Mobil brands. ExxonMobil is also a major producer of petrochemical products.

Oil is not like any other material and its price would not be set by the market. Indeed, on the free market, price fluctuations are natural and depend on a large number of random economic, political or even geological events. These fluctuations constitute a permanent threat to oil producers because they place these operators at a significant risk of financial loss. So, how to manage this risk? This is the question we want to answer through this paper. For that we propose the use of the American options and we will show how this tool makes it possible to manage the problem effectively. An option is defined as a derivative product or a contract between two parties. It gives the buyer or the seller the right to buy or sell a call option and put option. There are several types of options: American, European, Asian and Bermudian options. The most usable options in the market are the European and the American options. The European options can only be exercised at the expiration date, while the American options can be exercised at any time until the expiration date. The first pricing option models are designed to evaluate European options. Black-Scholes model was the celebrated one for pricing European option with constant volatility. This model was published in 1973. The Black-Scholes formula makes it possible to calculate the theoretical value of a European option from a certain number of parameters. However, contrary to the previous model, the binomial model calculates the value of an option by decomposing the maturity T expressed in years of the option into n equal periods of maturity Δt. Firstly, it was proposed by Cox et al. (1979). By using this model we can simply calculate the price of an European option. In Black and Scholes (1973), Cox et al. (1979), Davis et al. (1993), Dupire (1994) and Wu (2004), the authors describe models for pricing European options. Improved models that have incorporated constant volatility with European options generally have larger pricing errors in comparison with models that used American options. Estimate the price of American options is one of the most difficult problems of options theory. The difficulty is that the American options have no explicit solution contrarily to the European

options. Hence the interest of thinking about methods of pricing American options. There are many methods for pricing American options. For example in Breen (1991), the author presented the accelerated binomial option pricing model; this model can be considered as a binomial approximation of the Geske-Johnson continuous time model for the value of American put option. In Brennan and Schwartz (1977), the authors developed an algorithm to evaluate the price of American put option when this put option has a limited life. The work of Bunch and Johnson (1992), shows the development of an expression for the American put price. In addition the authors calculated put prices using modified Geske-Johnson approach. In Oosterlee (2003), Oosterlee proposed a nonlinear multi-grid method for solving linear complementarity problems. The problem of evaluating American put options leads to the resolution of a 2D partial differential equation with a free boundary. Recently, in Grossinho et al. (2018), authors presented a generalisation of nonlinear Black-Scholes equation for pricing American call options, they transformed the free boundary problem for the Black-Scholes nonlinear equation to the Gamma variational inequality, and they used the projective successive over relaxation method for solving the Gamma variational inequality.

In this paper, we propose an application of the Black and Scholes; this application is linked to oil options. Investors or hedgers can use options in the oil market to obtain the right to buy or sell physical crude or futures contracts at a fixed price before the options expire. In contrast to futures contracts, options do not have to be exercised at maturity, which gives the contract holder more flexibility. Traders have the ability to collect premiums by selling oil options. If traders do not expect oil prices to change sharply in any direction (up or down), oil options offer them the opportunity to make a profit by selling out-of-the-money oil options. Uçal, İ. and Kahraman (2009) developed a new method to evaluate investments in reality, this model consists in evaluating fuzzy real options and it gives an application in oil investment valuations. We can cite as example of models evaluating oil options the works (Cortazar and Schwartz, 2002; Smith and McCardle, 1999). The model described in this work is the Black-Scholes model. This model assumes that the underlying asset does not pay any dividend forth duration of the option, but in our work we give interest to paying dividends by adding a dividend rate to the Black-Scholes equation, and we discuss the impact of the variation of dividend rate on the American put option price. The main purpose of this paper is to prove that the resolution of pricing American put option problem leads to resolution of a linear complementarity problem. For prove it we use the finite difference method to discretise the space and the time, this discretisation results a tridiagonal matrix. In addition, using the concept of P-matrix we obtain an important condition for the uniqueness of the solution. We propose for the resolution of the linear complementarity problem an efficient and fast algorithm that has a finite numbers of steps presented in Achik et al. (2020). In addition, we will use the results found by solving the linear complementarity problem to apply them to an example on oil options.

This paper is organised as follows. The American put option pricing model is described in Section 2. In Section 3, we present a discretisation of the space and the time, and then we obtain a linear complementarity problem. This last one is solved by a numerical algorithm. Numerical results are performed in Section 4. In the last section, we present an example of oil options. We close this paper with a conclusion.

2 Presentation of the model

The Black-Scholes model is a mathematic model which the price of underlying asset is a stochastic process in continuous time. This model developed in 1973 by Fischer Black

and Myron Scholes. In this model, we assumed that the dynamics of the underlying asset follows the following process $\frac{dS_t}{S_t} = (\mu - y)dt + \sigma dW_t$, where S represents the underlying asset, μ the expected instantaneous rate of return of the underlying asset, y the continuous dividend rate, σ^2 is the instantaneous variance of the return on the assets and W is a Wiener process. We also assume that there is no possibility of arbitrage.

The put option $V(S, t)$ must satisfy the following partial derivative equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - y)S\frac{\partial V}{\partial S} = rV \tag{1}$$

In the fact, we have $dS = (\mu - y)Sdt + \sigma SdW$, according to Ito's lemma we obtain $dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (\mu - y)S\frac{\partial V}{\partial S}\right)dt + \sigma S\frac{\partial V}{\partial S}dW$, we pose $\Pi = -V + \frac{\partial V}{\partial S}S$, so we have the following expressions $\Delta\Pi = -\Delta V + \frac{\partial V}{\partial S}\Delta S$, $\Delta S = (\mu - y)S\Delta t + \sigma S\Delta W$ and $\Delta V = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (\mu - y)S\frac{\partial V}{\partial S}\right)\Delta t + \sigma S\frac{\partial V}{\partial S}\Delta W$. Therefore we obtain

$$\Delta\Pi = -\left[\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (\mu - y)S\frac{\partial V}{\partial S}\right)\Delta t + \sigma S\frac{\partial V}{\partial S}\Delta W\right]$$
$$+ \frac{\partial V}{\partial S}\left[(\mu - y)S\Delta t + \sigma S\Delta W\right]$$

as a result we have $\Delta\Pi = -\frac{\partial V}{\partial t}\Delta t - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\Delta t$. Then, we have to use the concept of lack of arbitrage opportunity. If an investment strategy Π is invested entirely in risk-free assets, the return of this strategy will be $[(r - y)\Pi - yV]\Delta t$ for the period Δt, where r is the risk-free rate of interest because the right term of the previous equation doesn't contain any random variable, it cannot be greater or smaller than $[(r - y)\Pi - yV]\Delta t$, otherwise a possibility of risk-free profit would exist.

Therefore $[(r - y)\Pi - yV]\Delta t = \Delta\Pi$. By replacing the terms Π and $\Delta\Pi$ by their expressions, we obtain the following partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - y)S\frac{\partial V}{\partial S} = rV.$$

In the simple case of the put option, we can use the three Dirichlet conditions that are $V(0, t) = \max(E - 0, 0) = E$, $\lim_{S_t \to \infty} V(S_t, t) = 0$ and $V(S_T, T) = \max(E - S_T, 0)$, where E represents the exercise price of the option and T is its expiry date.

The evaluation of American options poses an additional difficulty, due to the freedom of exercise. Let us say that we will restrict to this paper in the case of the simple American put option, but according to the argument of the absence of arbitrage, the freedom of exercise makes the price of the put option $V(S, t)$, for example, can never be less than its intrinsic value, $V(S, t) \geq \max(E - S, 0)$, the freedom of exercise implies the presence of a free bound: a value $S_f(t)$ of the underlying, such that for all $S \in [0, S_f(t)]$, the rational holder must exercise the option. And for all $S \in [S_f(t), \infty]$, it is economically preferable not to exercise it. Note that the value of $S_f(t)$ is not only unknown a priori, but varies with time. Equation (1) governs the form of the function $V(S, t)$, as long as $S \in [S_f(t), \infty]$, so long as it is not exercised. On the other hand, only the following inequation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - y)S\frac{\partial V}{\partial S} - rV \leq 0 \text{ is validated on the region } [0, S_f(t)].$$

So according to the value of the underlying we have two different systems:

In the region $[0, S_f(t)]$

$$\begin{cases} V(S,t) - \max(E - S, 0) = 0 \dfrac{\partial V(S,t)}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V(S,t)}{\partial S^2} \\[2mm] + (r - y)S\dfrac{\partial V(S,t)}{\partial S} - rV(S,t) \leq 0 \end{cases}$$

and the other region $[S_f(t), \infty]$

$$\begin{cases} V(S,t) - \max(E - S, 0) \geq 0 \dfrac{\partial V(S,t)}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V(S,t)}{\partial S^2} \\[2mm] + (r - y)S\dfrac{\partial V(S,t)}{\partial S} - rV(S,t) = 0 \end{cases}$$

With appropriate boundary conditions, and if we knew $S_f(t)$, these systems could be solved. Since the bound $S_f(t)$ is unknown, a complementarity approach is used to evaluate the option without explicit reference to $S_f(t)$. The systems (2) and (3) are combined in the form $D.K = 0;\ -D \geq 0$ and $K \geq 0$, such that

$$D = \frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + (r - y)S\frac{\partial V(S,t)}{\partial S} - rV(S,t)$$

and $K = V(S,t) - \max(E - S, 0)$, with the following boundary conditions

$$V(0,t) = \max(E - 0, 0) = E,\ \lim_{S_t \to \infty} V(S_t, t) = 0$$

and $V(S_T, T) = \max(E - S_T, 0)$.

A transformation of the variable S_t, makes it possible to obtain a formulation of the evaluation problem that is useful from an algorithmic point of view. By applying the transformation $x_t = \ln S_t$ then we have the following partial differential equation

$$\frac{\partial P}{\partial t}(x,t) + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial x^2}(x,t) + (r - y - \frac{\sigma^2}{2})\frac{\partial P}{\partial x}(x,t) = rP(x,t) \tag{2}$$

where $P(x,t)$ an American put option.

In the fact, we already have $dS = (\mu - y)Sdt + \sigma SdW$. Let $x = \ln S$ then, $\frac{\partial x}{\partial S} = \frac{1}{S}$; $\frac{\partial^2 x}{\partial S^2} = -\frac{1}{S^2}$; $\frac{\partial x}{\partial t} = 0$, by applying the lemma of Ito we have

$$dx = \left(\frac{\partial x}{\partial S}(\mu - y)S + \frac{\partial x}{\partial t} + \frac{1}{2}\frac{\partial^2 x}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial x}{\partial S}\sigma SdW$$

so $dx = (\mu - y - \frac{\sigma^2}{2})dt + \sigma dW$, then the partial differential equation is as follows

$$\frac{\partial P}{\partial t}(x,t) + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial x^2}(x,t) + (r - y - \frac{\sigma^2}{2})\frac{\partial P}{\partial x}(x,t) = rP(x,t).$$

The value of the American put option $P(x_t, t)$ can be obtained by solving the following complementarity problem $F.G = 0$, $-F \geq 0$ and $G \geq 0$, such that

$$F = \frac{\partial P(x,t)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 P(x,t)}{\partial x^2} + (r - y)\frac{\partial P(x,t)}{\partial x} - rP(x,t)$$

and $G = P(x,t) - \max(E - e^x, 0)$ with the following boundary conditions $\lim_{x \to -\infty} P(x,t) = \lim_{x \to -\infty} \max(E - e^x, 0) = E$, $\lim_{x \to \infty} P(x,t) = 0$ and $P(x,T) = \max(E - e^x, 0)$.

3 Discretisation and localisation of pricing American option problem

The location of the space of the variables consists in limiting an originally infinite space. The dimension of the time is intrinsically bounded by the expiry date of the option: $t \in [0, T]$. The underlying dimension must be artificially bounded for the purposes of numerical resolution: $S \in [S^-, S^+]$. It is usually sufficient to fix the value of S^+ at two or three times the value of the exercise price. The treatment of S^- is discussed below.

We obtain a discretisation of the continuous space of the variables S and t by subdividing the time axis into M intervals of length $\Delta t = \frac{T}{M}$, and the axis of the underlying in N intervals of length $\Delta S = \frac{S^+ - S^-}{N}$. The dicretised space of the original variables (S, t) then corresponds to $(N + 1) \times (M + 1)$ points $(S^- + n\Delta S, m\Delta t); 0 \leq m \leq M; 0 \leq n \leq N$.

The dicretised space of the transformed variables (x, t) is represented by the $(N + 1) \times (M + 1)$ points $(x^- + n\Delta x, m\Delta t); 0 \leq m \leq M; 0 \leq n \leq N$, such that $x^- = \ln S^-$, and $\Delta x = \frac{x^+ - x^-}{N}$ with $x^+ = \ln S^+$. Note that the discretisation used is uniform in the x-dimension, and not induced by the discretisation of S. For numerical tests, we will set the values $\Delta S = S^-$ and $x^- = \ln \Delta S$; this assures us that the lower bound S^- converges to 0 with ΔS, without meeting the indeterminate form $\ln(0)$.

The value of the option at the grid points of the space (x, t) will be denoted $P_n^m = P(x^- + n\Delta x, m\Delta t), 0 \leq m \leq M, 0 \leq n \leq N$. The boundary conditions will be as follows $P_0^m = \max(E - e^x, 0) = E$, $P_N^m = \max(E - e^x, 0) = 0$ and $P_n^M = \max(E - e^{(x^- + n\Delta x)}, 0)$.

We will use the finite difference method in the Crank-Nicholson approach Umeorah and Mashele (2019). This approach makes it possible to replace the partial derivatives of the left part of (4) by approximations

$$\frac{P_n^{m+1} - P_n^m}{\Delta t} + \frac{1}{4}\sigma^2 \frac{P_{n+1}^{m+1} - 2P_n^{m+1} + P_{n-1}^{m+1}}{(\Delta x)^2} + \frac{1}{4}\sigma^2 \frac{P_{n+1}^m - 2P_n^m + P_{n-1}^m}{(\Delta x)^2}$$
$$+ (r - y - \frac{\sigma^2}{2})\frac{P_{n+1}^{m+1} - P_{n-1}^{m+1}}{2\Delta x} + (r - y - \frac{\sigma^2}{2})\frac{P_{n+1}^m - P_{n-1}^m}{2\Delta x} - \frac{1}{2}rP_n^{m+1} - \frac{1}{2}rP_n^m$$

This expression, once multiplied by Δt, will be represented matricially in the following form $Ap - Bq$, with $A_{ii} = -a$, $A_{i,i+1} = 1 - b$, $A_{i,i+2} = -c$ and $A_{ij} = 0$ for all $i > j$ or $i + 2 < j$; $p_i = P_i^m$, $q_i = P_i^{m+1}$, $B_{ii} = a$, $B_{i,i+1} = 1 + b$, $B_{i,i+2} = c$ and $B_{ij} = 0$ for all $i > j$ or $i + 2 < j$, such that $a = \frac{\sigma^2 \Delta t}{4(\Delta x)^2} - \left(r - y - \frac{\sigma^2}{2}\right)\frac{\Delta t}{2\Delta x}$, $b = \left(-\frac{\sigma^2}{2(\Delta x)^2} - \frac{r}{2}\right)\Delta t$ and $c = \frac{\sigma^2 \Delta t}{4(\Delta x)^2} + \left(r - y - \frac{\sigma^2}{2}\right)\frac{\Delta t}{2\Delta x}$. Since the values of P_0^m, P_0^{m+1}, P_N^m and P_N^{m+1}

are dictated by the boundary conditions, it is possible to decompose $(Ap - Bq)$ in the form $Mp^m + s^m - C^{m+1}$ such that M is a tridiagonal matrix such that $m_{i+1,i} = -a$, $m_{ii} = 1 - b$ and $m_{i,i+1} = -c$; $p^m = \left(P_1^m, ..., P_{N-2}^m, P_{N-1}^m\right)^T$; $s^m = (-a.E, 0, ..., 0)^T$ and $C^{m+1} = \Lambda P^{m+1} + Q^{m+1}$, where $P^{m+1} = \left(P_1^{m+1}, P_2^{m+1}, ..., P_N^{m+1}\right)^T$, $Q^{m+1} = \left(a.P_0^{m+1}, 0, ..., 0, c.P_N^{m+1}\right)^T$ and Λ is a tridiagonal matrix defined by $\Lambda_{i+1,i} = a$, $\Lambda_{i,i} = 1 + b$ and $\Lambda_{i,i+1} = c$.as a result we come across a linear complementarity problem such as $Mp^m - b^{m+1}$ such that $b^{m+1} = C^{m+1} - s^m$, where M is a matrix of size $(N-1) \times (N-1)$.

Once the space discretises, and once the partial derivative operator adapts to this space, it is possible to write a discrete version of the problem of continuous complementarity.

The matrix notation allows us to enlarge the problem to the entire space of the underlying $(\forall m)$ by rewriting the previous problem in the form of the linear complementarity problem (LCP). Find $p^m \in \mathbb{R}^{N-1}$ such as $(Mp^m - b^{m+1}).(p^m - p^M) = 0$, $p^m - p^M \geq 0$, and $Mp^m - b^{m+1} \geq 0$ where the vector p^M is the terminal vector of the exercise price characterised by the function $\max(E - e^x, 0)$.

There are M such problems, one for each time step $0 \leq m \leq M - 1$. Pricing the value of the option requires solving these M problems; although we must calculate the value of the option at all points of the discretisation of space (x, t).

To prove the existence and the uniqueness of the solution of the LPC, we will prove that the matrix M associated with this problem is a P-matrix i.e. all its principal minors are strictly positive. See Samelson et al. (1958), Cottle et al. (1992) and Murty (1972). In the fact, we have M is a tridiagonal matrix such that $m_{i,i-1} = -a$, $m_{ii} = 1 - b$ and $m_{i,i+1} = -c$.

We assume that $r > \frac{\sigma^2}{2}(1 + \frac{1}{\Delta x}) + y$. Since M is a tridiagonal matrix we have $\det M_{[N-1]} = (1 - b) \det M_{[N-2]} - ac \det M_{[N-3]}$. The determinant of matrix $M_{[N-2]}$ is as a function of $1 - b$ and $-ac$. Same principle for all the determinants of submatrices of order $n \leq N - 2$. However, all the main minors of the matrix M are written according to $1 - b$ and $-ac$ just prove that they are strictly positive. Regarding $1 - b$ as $b \leq 0$ then $1 - b > 0$. We need to prove that $-ac > 0$. In the fact, we have $r > \frac{\sigma^2}{2}(1 + \frac{1}{\Delta x}) + y$, then $r - \frac{\sigma^2}{2} - y > \frac{\sigma^2}{2\Delta x}$, as $r > \frac{\sigma^2}{2}(1 + \frac{1}{\Delta x}) + y$, as a result $r > \frac{\sigma^2}{2} + y$ so $r - \frac{\sigma^2}{2} - y > 0$, we obtain $(r - \frac{\sigma^2}{2} - y)^2 \frac{\Delta t^2}{4\Delta x^2} - \frac{\sigma^4 \Delta t^2}{16\Delta x^4} > 0$. Finally we have $-ac > 0$ which prove that M is a $P-$matrix.

4   Numerical results and discussion

The previous linear complementarity problem is rewritten using the following definitions $w^m = Mp^m - b^{m+1}$, $z^m = p^m - p^M$, $q^{m+1} = b^{m+1} - Mp^M$ to take the following form $w^m = Mz^m - q^{m+1}$, where $w_n^m \geq 0$, $z_n^m \geq 0$ and $w_n^m.z_n^m = 0$ for $n = 1, ..., N - 1$, and we solved this LCP with the algorithm defined in Achik et al. (2020).

In this section, we present the results obtained for the theoretical evaluation of some American options.

Tables 1–4 present the theoretical prices of American put options with well chosen parameters.

Table 1 Theoretical prices of American put options with $E = 50$ et $y = 0$

Parameters of option	Price S	Price of put option
$\sigma = 0.1, y = 0, r = 0.08, T = 3$ Months	40	13.58593
	45	7.58757
	50	4.82847
	55	2.92694
	60	0.73105
$\sigma = 0.1, y = 0, r = 0.08, T = 6$ Months	40	13.73088
	45	7.88133
	50	5.04984
	55	2.96403
	60	0.93261
$\sigma = 0.1, y = 0, r = 0.08, T = 9$ Months	40	13.78373
	45	8.07311
	50	5.19233
	55	3.28880
	60	1.16783
$\sigma = 0.1, y = 0, r = 0.08, T = 12$ Months	40	13.83644
	45	8.19331
	50	5.53081
	55	3.67775
	60	1.21886

The results obtained in this section considering the dividend payment with American put options. The following parameters were used for Tables 1–4: $S \max = 60$; $S \min = 40$; $\sigma = 0.1$ and $r = 0.09$. We see from Table 1 and Figure 1 that the put option value decreases from 13.83644 to 0.73105 when the value of the underlying asset increases. Regarding the value of put option as a function of time, we observe that the value increases but in a slow manner. In Table 2, we see that (Figure 2) the values of the options are increased by 14.78233 to 1.44579 compared to Table 1 and always the values reduced according to the underlying asset. Same notes for Table 3, see Figure 3, there is a decrease in values from 15.53908 to 2.52468 and increase over time. For Table 4 and Figure 4 we want to see the impact of dividend payment on the value of put options, we see that if the dividend rate increases the value of put option also increases. We will apply these results to the production of ExxonMobil, it has the capacity to produce 6.3 million barrels of oil per day. The price of a barrel is worth 45 \$. If there is an anticipation of a drop of barel prices it can protect its products by buying an American style put with the following characteristics (see Figure 5):

- Exercise price: 50\$

- Maturity: March

- Put price: 5 \$

- Quota: 567 000 000.

It pays the following amount: 5\$ × 567,000,000, i.e., 2,835,000,000\$.
    Before the deadline:
    It can at any time resell its put and close its position, with a gain or a loss depending on the evolution of the market and the level of the premium of the put that the buyer bought.

Table 2 Theoretical prices of American put options with $E = 50$ et $y = 0.01$

Parameters of option	Price S	Price of put option
$\sigma = 0.1$, $y = 0.01$, $r = 0.08$, $T = 3$ Months	40	14.37729
	45	8.60162
	50	5.80464
	55	4.21726
	60	1.44579
$\sigma = 0.1$, $y = 0.01$, $r = 0.08$, $T = 6$ Months	40	14.41384
	45	8.80728
	50	5.90012
	55	4.43180
	60	1.59519
$\sigma = 0.1$, $y = 0.01$, $r = 0.08$, $T = 9$ Months	40	14.44530
	45	8.83438
	50	6.14857
	55	4.64194
	60	1.73669
$\sigma = 0.1$, $y = 0.01$, $r = 0.08$, $T = 12$ Months	40	14.78233
	45	9.23091
	50	6.39398
	55	4.84752
	60	1.87087

Table 3 Theoretical prices of American put options with $E = 50$ et $y = 0.02$

Parameters of option	Price S	Price of put option
$\sigma = 0.1$, $y = 0.02$, $r = 0.08$, $T = 3$ Months	40	15.26232
	45	9.61940
	50	6.95023
	55	5.32344
	60	2.52468
$\sigma = 0.1$, $y = 0.02$, $r = 0.08$, $T = 6$ Months	40	15.37509
	45	9.75302
	50	7.16351
	55	5.53846
	60	2.56026
$\sigma = 0.1$, $y = 0.02$, $r = 0.08$, $T = 9$ Months	40	15.40961
	45	10.07622
	50	7.40842
	55	5.66554
	60	2.77848
$\sigma = 0.1$, $y = 0.02$, $r = 0.08$, $T = 12$ Months	40	15.53908
	45	10.31356
	50	7.74551
	55	5.88627
	60	2.98647

At maturity, two cases should be distinguished:

- *First case*: The barrel price went up. The barrel price rose above the exercise price of 50$. The put buyer will choose not to exercise the option. The put buyer will have

lost the premium amount which corresponds to the cost of insurance against a fall in the market, but the price of the barrel has increased.

- *Second case*: The price of the barrel fell. The barrel price is worth 40$. The option is valued, and its price at maturity is equal to the difference between the exercise price and the barrel price: 50$–40$ = 10 $, i.e., 5,670,000,000 $ in total for 567,000,000 barrel of oil. The put buyer can keep the prices of his barrels; he realises a profit of 2,835,000,000 $ (5,670,000,000 $ – 2,835,000,000$), ie the value of the option at maturity minus the total amount of the premium paid). In this case, the option acts as insurance which protects the value of the barrel in the event of a fall in the market. This is called a hedging transaction: by paying a premium the investor has in some way taken out insurance to protect him against market risk. Indeed, the profit made on the option makes it possible to partially offset the latent loss on the shares.
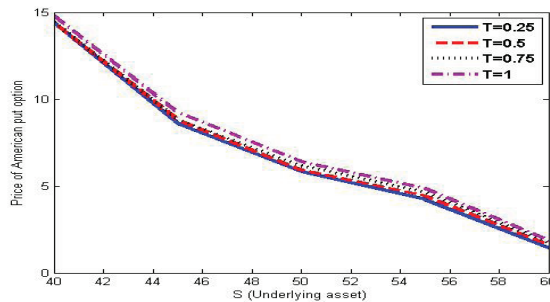
Table 4 American put options prices with different values of time with $E = 50$

| T	$y = 0$	$y = 0.01$	$y = 0.02$
0.25	4.82847	5.80464	6.95023
0.5	5.04984	5.90012	7.16351
0.75	5.19233	6.14857	7.40842
1	5.53081	6.39398	7.74551

Figure 1 American put options prices with different values of the underlying asset with $y = 0$ (see online version for colours)
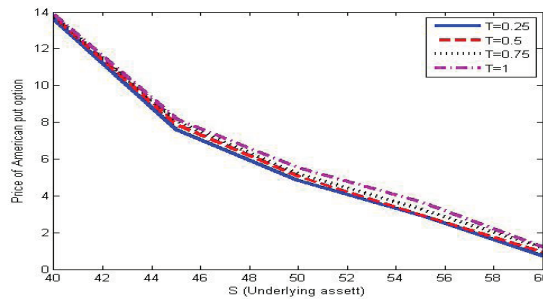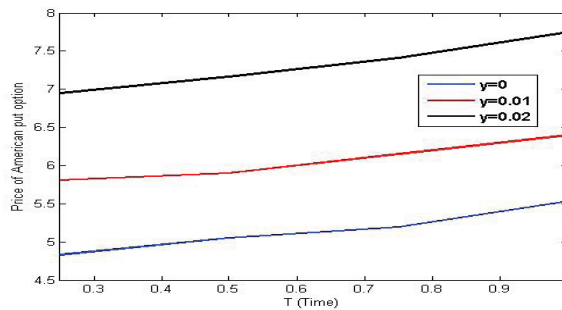

Figure 2 American put options prices with different values of the underlying asset with $y = 0.01$ (see online version for colours)
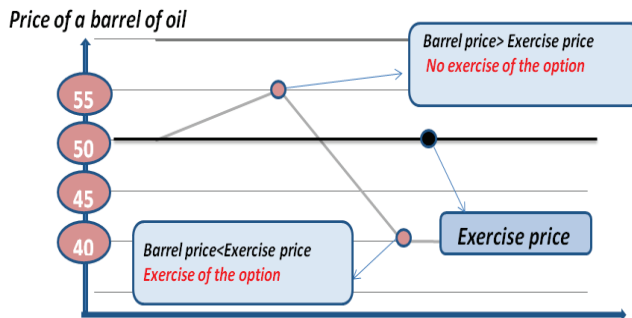
Figure 3 American put options prices with different values of the underlying asset with $y = 0.02$ (see online version for colours)
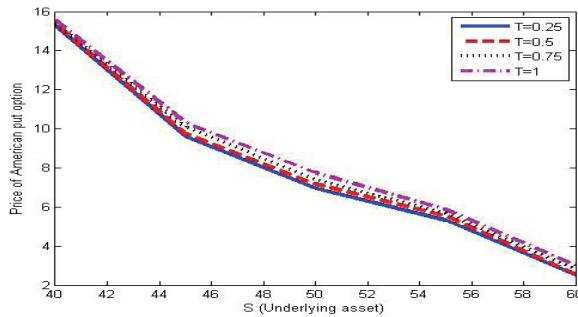
Figure 4 American put options prices with different dividend rates (see online version for colours)

Figure 5 Graphical representation of the option exercise case (see online version for colours)

5 Conclusion

In this paper, we are interested in the Black-Scholes model by adding the dividend rate, as we saw in the discussion that this rate makes a change on the put option price, more precisely, the values of the American put option increase when the dividend rate increase, and the values of the price reduced according to the underlying asset and increase according to the time. Regarding the oil options, it was noted that these options are very useful to help producers keep prices per barrel and to take advantage of the premium if there is a price fluctuation.

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