Optimal trading strategy under linear-percentage temporary impact price dynamics with conditional value-at-risk as timing risk measure

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Abstract: This paper attempts to solve constrained optimal trading problem of minimising expected execution cost subject to non-negativity constraints for risk neutral as well as risk averse investors. The optimal trading problems are modelled under linear-percentage temporary impact price dynamics (LPT price model) of underlying asset’s execution price dynamics. The algorithm to solve optimal trading problem with non-negativity constraints under LPT price model by static approximation method (SA method) is detailed. Furthermore, a comprehensive discussion of applicability of other existing approaches in literature to solve proposed problems and, their limitations are presented to justify use of SA method for the same. In case of a risk averse investor, the conditional value-at-risk (CVaR) is taken as a measure for the timing risk. The extensive numerical illustrations on simulated data as well as on the real market data depict the practical significance of the proposed optimal trading problems.

Keywords: optimal trading; conditional value-at-risk; CVaR; dynamic programming problem; quadratic programming problem; risk averse; price impact.


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1 Introduction

The investment performance of a trading depends considerably on the way it has been acquired. Trading all the required number of shares of an asset at once, moves asset’s price up or down and thus, an investor lands up trading at different price (called execution price) than the price which he initially observed in the market. The prevailing price of an asset without any order of the trade is called as no-impact price. The effect in price due to immediate execution demand of a trade is called temporary price impact which affects the execution price only at the moment of trading. Whereas, permanent change in price due to supply-demand imbalance, arrival of some information about the asset, etc., is termed as permanent price impact (Loeb, 1983; Bertsimas and Lo, 1998; Barclay and Warner, 1993; Perold, 1998; Almgren and Chriss, 2000). Thus, execution price dynamics of an asset comprises of no-impact price path and the price impact factor. Many possible characterisations of execution price dynamics, based on the definition of no-impact price path and that of price impact function, have been proposed in literature (Bertsimas and Lo, 1998; Almgren and Chriss, 2000; Huberman and Stanzl, 2005; Gatheral and Schied, 2011; Moazeni et al., 2013; Khemchandani et al., 2016).

To reduce effect of the price impact and thus minimise execution cost, an investor divides trade into small packages to be traded in some fixed number of time periods rather than performing one single trade at once. The optimal trading strategy tells what number of shares investor should trade in different time periods so that the expected execution cost is minimum. Bertsimas and Lo (1998) are the first to give mathematical formulation of the optimal trading problem. Later on Almgren and Chriss (2000), Huberman and Stanzl (2005), Gatheral and Schied (2011), Moazeni (2011), Khemchandani et al. (2013), Moazeni et al. (2013), and Khemchandani and Chandra (2014) are few among the several academicians and market practitioners who studied the optimal trading problem.

Bertsimas and Lo (1998) have discussed three approaches, based on dynamic programming method (DP method), approximate dynamic programming method (ADP method) and discretisation method, to obtain solution of optimal trading problem of minimising expected execution cost. They introduced LPT price model under which the approach of DP method is elaborated. Whereas ADP and discretisation methods are briefly presented for general formulation of execution price path. In a trading strategy of buying some fixed number of shares of an asset, it is difficult to associate a physical meaning to a negative trade in any time period (Bertsimas et al., 1999). In view of this, in this paper, an attempt to present approaches based on ADP and discretisation methods to solve optimal trading problem of minimising expected execution cost subject to non-negativity constraints is made. The optimal trading problem is formulated under LPT price model. Furthermore, the algorithm based on static approximation (SA) method, which was initially suggested by Bertsimas et al. (1999), is detailed to solve proposed problem. The efficacy of SA method over other discussed methods is elaborated.

Apart from inclusion of non-negativity constraints, another important aspect of optimal trading problem is to include a measure for timing risk in its modelling. The timing risk is the risk which arises due to unfavourable price movement of stock in the trading time horizon. Thus, a risk averse investor finds it plausible to minimise a measure for the timing risk besides minimising expected execution cost. The literature which studies optimal trading problem with inclusion of a measure for timing risk includes Huberman and Stanzl (2005), Moazeni et al. (2013), Khemchandani et al. (2013), etc. In
this paper, conditional value-at-risk (CVaR) of execution cost under LPT price model is proposed as a measure for timing risk. The algorithm to solve proposed optimal trading problem for a risk averse investor by SA method is discussed. Furthermore, numerical illustrations are provided to assess the practical significance of proposed optimal trading problems and the dependence of their optimal trading strategies on parameters of LPT price model.

The rest of the paper is organised as follows. The formulation of optimal trading problem for a risk neutral investor under LPT price model, and different approaches to solve this problem and their comparisons are presented in Section 2. The formulation of optimal trading problem for a risk averse investor with CVaR of execution cost as timing risk measure, and algorithm of finding its solution by SA method are presented in Section 3. Section 4 gives numerical illustrations and Section 5 concludes the paper.

2 Optimal trading problem under LPT price model for a risk neutral investor

Consider an investor who intends to buy some fixed number of shares, say \( \bar{S} \), of an asset within a specified time interval. To reduce the price impact and thus minimise cost of execution, the investor would choose to divide \( \bar{S} \) into smaller parts to be traded at different time points in the given time interval. Let the time interval be divided into \( T \) time periods of equal length. Consider \( S_t \) and \( P_t \) as the number of shares purchased and the execution price of a share in \( t \)th time period, respectively. We assume that trading takes place at the end of a time period.

Mathematically, the problem of finding a trading strategy \((S_1, S_2, \ldots, S_T)\), which gives minimum expected execution cost to buy \( \bar{S} \) shares in \( T \) number of time periods, is as follows:

\[
\begin{align*}
\min_{\{S_t\}_{t=1}^T} & \quad E_t \left( \sum_{t=1}^T P_t S_t \right) \\
\text{subject to} & \quad \sum_{t=1}^T S_t = \bar{S}.
\end{align*}
\]

Throughout this paper, symbol \( E_t \) represents the conditional expectation given the no-impact price of a share, say \( \bar{P}_t \), realised in \((t-1)\)th time period, and the value of information factor, say \( X_t \), corresponding to \( t \)th period (Bertsimas and Lo, 1998). Thus, in (P1), the expectation \( E_t \) is considered in the sense that the present no-impact price \( \bar{P}_0 \) and the information factor \( X_1 \) for the first time period are known.

The complete formulation of (P1) requires specification of execution price dynamics \( P_t \). In this paper, we consider LPT price model introduced by Bertsimas and Lo (1998) for the optimal trading problem formulation. In LPT price model, the no-impact price \((\bar{P})\) and information factor \((X)\) are respectively modelled as geometric Brownian motion and the first order auto-regressive process (AR(1)), as follows:
\[ \tilde{P}_t = \tilde{P}_{t-1} e^{X_t}, \quad (1) \]

\[ X_t = \rho X_{t-1} + \eta_t; \quad \rho \in (-1, 1). \quad (2) \]

In (1), \( Z_t; t = 1, 2, \ldots, T, \) represent independent identically distributed normal random variables with mean \( \mu_Z \) and variance \( \sigma_Z^2 \). The variables \( \eta_t; t = 1, 2, \ldots, T, \) in (2), correspond to white noise factor with mean zero and variance \( \sigma_\eta^2 \). Let \( q = E_i(E^{Z_t}) = e^{\mu_Z + \frac{\sigma_Z^2}{2}}, \ t = 1, 2, \ldots, T. \)

In LPT price model, the price impact, as a percentage of no-impact price, is considered as a linear function of trade size and information factor. Let \( \Delta_t \) denote the price impact of \( t \)th time period. Then we have

\[ \Delta_t = \tilde{P}_t (\theta S_t + \gamma X_t); \quad \theta > 0, \quad (3) \]

where \( \theta \) and \( \gamma \) measures the degree of impact of trade size and information factor on execution price. For a buying trade \( \theta \) is positive.

The execution price \( (P_t) \) under LPT price model is sum of no-impact price \( \tilde{P}_t \) and the price impact factor \( \Delta_t \) as follows:

\[ P_t = \tilde{P}_t + \Delta_t = \tilde{P}_t (1 + \theta S_t + \gamma X_t) \quad (4) \]

Using (1) and (2), LPT price model (4) can be rewritten as follows:

\[ P_t = \tilde{P}_t e^{X_t} (1 + \theta S_t + \gamma X_t), \quad (5) \]

\[ X_t = \rho X_{t-1} + \eta_t; \quad \rho \in (-1, 1). \quad (6) \]

The problem (P1) under LPT price model, (5) and (6), is the basic formulation of optimal trading problem which allows negative trades in intermediate time periods. However, in practice, a negative trade in any time period in the optimal trading strategy of purchasing some fixed number of shares has no physical meaning. Thus, it is required to consider the following formulation of optimal trading problem subject to non-negativity constraints.

\[ \begin{align*}
\text{(P2)} \\
\min & \ E_i \left( \sum_{t=1}^{T} P_t S_t \right) \\
\text{subject to} & \\
\sum_{t=1}^{T} S_t = \mathbb{S}, \\
S_t & \geq 0; \quad t = 1, 2, \ldots, T.
\end{align*} \]

where \( P_t \) is driven by LPT price model, (5) and (6).
The variables $\bar{P}_{t-1}$ and $X_t$ in LPT price model are external state variables which change due to their own natural dynamics given by (1) and (2), respectively. Because of the presence of external variables in execution price dynamics $P_t$, the optimal trading strategies cannot be determined in advance of trading. One requires to revise these strategies in each time period according to the new realisation of external variables. Apart from need of consideration of new realisation of external variables in each time period, another important fact is the incorporation of non-negativity constraints in the problem formulation. Thus, a method to solve an optimal trading problem should take into consideration both of these aspects. The three approaches of Bertsimas and Lo (1998), which are based on DP, ADP and discretisation methods, take into account the new realisations of external variable in each time period. However, these approaches are discussed for optimal trading problem formulation (P1) which does not impose non-negativity constraints on trading strategies. By approach of DP method, Bertsimas and Lo (1998) have obtained closed form solution of (P1) under LPT price model. Furthermore, they have briefly mentioned approaches based on ADP and Discretisation methods to solve (P1) under general formulation of execution price path. In the discussion which follows, we briefly present DP method approach of Bertsimas and Lo (1998) and the computational difficulty of this approach to deal with non-negativity constraints. Moreover, in the successive subsections, we make an attempt to give approaches based on ADP and discretisation methods for solving (P2), which impose non-negativity constraints on trading strategies under LPT price model.

Despite presenting the possibility of solving (P2) under LPT price model by ADP and discretisation methods which thus overcome the computational difficulty associated with DP method to do the same, we will also mention some limitations of these methods. In view of limitations associated with approaches based on ADP and discretisation methods, and computational difficulty of DP method approach to incorporate non-negativity constraints, we resort to use SA method. We detail the algorithm to solve (P2) under LPT price model by SA method and justify its use over other methods.

### 2.1.1 DP method

Bertsimas and Lo (1998) have used DP method, introduced by Bellman (1957), to give the closed form solution of (P1) under LPT price model. The state variables for $t^{th}$ time period are no-impact price $\bar{P}_{t-1}$ realised in the last trading period, information factor $X_t$ of the current period, and $W_t$ as the number of shares that remain to be traded after time period $t - 1$. The control variable for $t^{th}$ time period is the number of shares $S_t$ to be traded in that period. The optimisation problem (P1) to be solved, via DP method, can be re-formulated as follows:
(DP problem)

$$
\begin{align*}
\min_{S_t} E_t \left( \sum_{i=1}^{T} P_i S_i \right)
\end{align*}
$$

subject to

$$
W_t = S_t, W_t = W_{t-1} - S_{t-1}; t = 2, 3, \ldots, T, W_{t-1} = 0.
$$

Approach based on DP method uses the fact that at any time period $t$, the part of optimal trading strategy $\{S_t^*, S_{t+1}^*, \ldots, S_T^*\}$ which corresponds to time periods $t$ to $T$ must be optimal for the remaining trading problem. That is, $\{S_t^*, S_{t+1}^*, \ldots, S_T^*\}$ is the optimal solution of minimising expected execution cost $E_t \left( \sum_{u=t}^{T} P_u S_u \right)$ in $t$th time period.

Let $V_t = (\tilde{P}_{t-1}, X_t, W_t) \left[ E_t \left( \sum_{u=t}^{T} P_u S_u \right) \right] \forall t = 1, 2, \ldots, T$ be the optimal execution cost-to-go which is incurred by trading into periods $t$ to $T$. For the last trading period $T$, optimal strategy is to execute all the remaining shares $W_T$ to finish the trade. Thus, $S_T$ equals to $W_T$, and optimal execution cost for the last trading period, $V_T$, is given as follows:

$$
V_T (\tilde{P}_{T-1}, X_T, W_T) = \min_{S_T} E_T \left( P_T S_T \right) = \min_{S_T} q\tilde{P}_{T-1} (1 + \theta S_T + \gamma X_T) S_T = q\tilde{P}_{T-1} (1 + \theta W_T + \gamma X_T) W_T.
$$

The Bellman equation, which relates $V_t$ to $V_{t+1}$, is given as follows:

$$
V_t (\tilde{P}_{t-1}, X_t, W_t) = \min_{S_t} E_t \left[ P_t S_t + V_{t+1} (\tilde{P}_{t}, X_{t+1}, W_{t+1}) \right] = \min_{S_t} E_t \left[ \tilde{P}_{t-1} e^{\rho t} (1 + \theta S_t + \gamma X_t) S_t + V_{t+1} (\tilde{P}_{t-1} e^{\rho t}, \rho X_t + \eta_{t+1}, W_t - S_t) \right].
$$

Starting from the last period $T$ and moving backwards recursively, Bertsimas and Lo (1998) have derived the optimal trading strategy of DP problem as a function of the state variables. The optimal cost of execution is given by $V_1 (\tilde{P}_0, X_1, S)$. We refer readers to see Bertsimas and Lo (1998) for details.

Furthermore, Bertsimas and Lo (1998) have mentioned some computational difficulties in applying their approach to optimal trading problem subject to non-negativity constraints. Without any constraints, cost-to-go $V_t$ in any time period $t$ is a quadratic function of $S_t$, and thus, it is easy to get closed form solution of (8). Whereas, if non-negativity constraints are imposed, $V_t$ becomes a piece-wise quadratic function of $S_t$ with $3^t$ pieces. In that case, in $t$th time period one needs to obtain optimal solution for $3^t$ quadratic functions. Therefore, even for a small number of time periods, like $T = 20$, the number of computations of $V_t$ required for the first time period becomes.
3^{20-1} = 1,162,261,467. Thus, the calculation corresponding to large T is not feasible to do in practice.

Next, we present approaches based on ADP and discretisation methods to solve problem (P2) under LPT price model, which has non-negativity constraints in its formulation.

2.1.2 ADP method

Similar to DP, ADP method is also based on first finding an optimal decision for the last time period and further stepping backward in time to find optimal trading strategy. The major difference between DP and ADP method is that in ADP method optimal objective function in each time period is approximated by another function (Bertsekas, 2009).

In this section, we present an approach based on ADP method to solve (P2) under LPT price model where we approximate optimal cost-to-go $V_t$, which becomes piecewise quadratic function due to non-negativity constraints, by a function which is quadratic in whole space. For solving (P1) under LPT price model, the approach of DP method yields $V_t$ as $qP_{t-1}$ times multiple of a quadratic function in $X_t$ and $W_t$ (see Bertsimas and Lo, 1998).

In view of this under the proposed approach based on ADP method, we approximate $V_t$ in each time period with $\hat{V}_t$ which is assumed as $qP_{t-1}$ times multiple of a quadratic function in $X_t$ and $W_t$ (say $Y_t$). The approximation is performed such that mean square error of approximation is minimum. That is

$$V_t(\tilde{P}_{t-1}, X_t, W_t) \approx \hat{V}_t(\tilde{P}_{t-1}, X_t, W_t)$$

where $\hat{V}_t(\tilde{P}_{t-1}, X_t, W_t) = qP_{t-1}Y_t(H_t)$ with $H_t = [X_t, W_t]'$ and $Y_t(H_t) = H_t'A_tH_t + B_t'H_t + C_t$ such that mean square error of approximation is minimum.

Throughout the paper, matrices and vectors are represented by bold symbols.

For the last trading period $T$,

$$V_T(\tilde{P}_{T-1}, X_T, W_T) = \min_{S_T} E_T(P, S_T)$$

where $A_T = \begin{bmatrix} 0 & 1 \\ \gamma & \theta \end{bmatrix}$, $B_T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $C_T = 0$. From (9), $V_T(\tilde{P}_{T-1}, X_T, W_T)$ is already in the desired form. That is $V_T = \hat{V}_T$ and error of approximation is zero for the last trading period.

Furthermore, let there exist $A_{t+1}, B_{t+1}$ and $C_{t+1}$ such that

$$V_{t+1}(\tilde{P}_t, X_{t+1}, W_{t+1}) \approx \hat{V}_{t+1}(\tilde{P}_t, X_{t+1}, W_{t+1})$$

where
\[ V_i\left(\tilde{P}_{i}, X_i, W_i\right) = q\tilde{P}Y_{i}\left(H_{i}\right) \]

with \( H_{i} = \left[X_{i}, W_{i}\right]' \) and \( Y_{i}\left(H_{i}\right) = H_{i}'A_{i}H_{i} + B_{i}'H_{i} + C_{i} \) such that mean square error of approximation is minimum.

Using quadratic approximation of \( \tilde{V}_{i} \), next we need to find \( V_{i}\left(\tilde{P}_{i}, X_i, W_i\right) \) and its approximation \( \tilde{V}_{i} \).

\[ V_{i}\left(\tilde{P}_{i}, X_i, W_i\right) = \min_{0 \leq S_i \leq W_i} E_i\left[P_{i}S_i + \tilde{V}_{i}\left(\tilde{P}_{i}, X_i, W_i\right)\right] \]

\[ = \min_{0 \leq S_i \leq W_i} E_i\left[\tilde{P}_{i}\left(1 + \theta S_i + \gamma X_i\right)S_i + q\tilde{P}Y_{i}\left(H_{i}\right)\right] \]

\[ = E_i\left(\tilde{P}_{i}\right)\left[\min_{0 \leq S_i \leq W_i} E_i\left[(1 + \theta S_i + \gamma X_i)S_i + qY_{i}\left(H_{i}\right)\right]\right] \]

\[ = q\tilde{P}_{i}\left[\min_{0 \leq S_i \leq W_i} E_i\left[(1 + \theta S_i + \gamma X_i)S_i + q\left(H_{i}'A_{i}H_{i} + B_{i}'H_{i} + C_{i}\right)\right]\right]. \]  

Consider the following:

\[ H_{i} = \left[\begin{array}{c} X_{i} \\ Y_{i} \end{array}\right] \]

\[ = \left[\begin{array}{c} \rho X_i + \eta_i \\ W_i - S_i \end{array}\right] = DH_{i} + e_2S_i + e_1 \eta_{i+1} \]  

where \( D = \left[\begin{array}{cc} \rho & 0 \\ 0 & 1 \end{array}\right], e_2 = \left[\begin{array}{c} 0 \\ 1 \end{array}\right] \) and \( e_1 = \left[\begin{array}{c} 1 \\ 0 \end{array}\right] \).

Moreover, it can be checked that

\[ 1 + \theta S_i + \gamma X_i = 1 + \theta S_i + \gamma \left[H_{i}'H_{i}\right]. \]  

Using (11) and (12) in (10), we get

\[ V_{i}\left(\tilde{P}_{i}, X_i, W_i\right) = q\tilde{P}_{i}\left[\min_{0 \leq S_i \leq W_i} \left\{ \left(\theta + qe_{i}'A_{i}e_{2}\right)S_i^2 \right. \right. \]

\[ \left. + \left(1 + \gamma e_{i}'H_{i} - qH_{i}'D'A_{i}e_{2} - qe_{i}'A_{i}DH_{i} - qB_{i}'e_{2}\right)S_i \right. \]

\[ \left. + \left(qH_{i}'D'A_{i}DH_{i} + q\gamma^2 e_{i}'A_{i}e_{1} + qB_{i}'DH_{i} + qC_{i}\right)\right]\right]. \]  

The unconstrained minimisation in (13) without constraint \( 0 \leq S_i \leq W_i \) would yield following point of minima.

\[ \hat{S}_i = -\frac{1 + \gamma e_{i}'H_{i} - qH_{i}'D'A_{i}e_{2} - qe_{i}'A_{i}DH_{i} - qB_{i}'e_{2}}{2\left(\theta + qe_{i}'A_{i}e_{2}\right)} \]

\[ = \frac{H_{i}'M_i + N_i}{L_i} \]

where \( M_i = qD'(A_{i} + A_{i}')e_{2} - \gamma e_{i}, N_i = qB_{i}'e_{2} - 1 \) and \( L_i = 2\left(\theta + qe_{i}'A_{i}e_{2}\right) \).

Based on the value of \( \hat{S}_i \), the point of minima \( \hat{S}'_i \) for the constrained minimisation of (13) is given as follows (Bertsimas and Lo, 1998):
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\[
S_t^* = \begin{cases} 
0 & \text{if } \frac{H'M_t + N_t}{L_t} < 0, \\
\frac{H'M_t + N_t}{L_t} & \text{if } 0 \leq \frac{H'M_t + N_t}{L_t} \leq W_t, \\
W_t & \text{if } \frac{H'M_t + N_t}{L_t} > W_t.
\end{cases}
\]  

For the three cases of (14), corresponding optimal cost-to-go \( V_t(\tilde{P}_{t-1}, X_t, W_t) \) from (13) are given as follows. (In calculation, relation \( W_t = e_t^2H_t \) is used.)

\[
V_t(\tilde{P}_{t-1}, X_t, W_t) = \begin{cases} 
q \tilde{P}_{t-1} \left( H'M_t^2 + (B_t^2)H_t + C_t \right) & \text{if } \frac{H'M_t + N_t}{L_t} < 0, \\
q \tilde{P}_{t-1} \left( H'M_t^2 + (B_t^2)H_t + C_t \right) & \text{if } 0 \leq \frac{H'M_t + N_t}{L_t} \leq W_t, \\
q \tilde{P}_{t-1} \left( H'M_t^2 + (B_t^2)H_t + C_t \right) & \text{if } \frac{H'M_t + N_t}{L_t} > W_t.
\end{cases}
\]  

where

\[
A_t^1 = qD'A_{t-1}D, \quad B_t^1 = qB'_{t-1}D, \quad C_t^1 = q\sigma^2_t e_t'A_{t-1}e_t + qC_{t-1},
\]

\[
A_t^2 = qD'A_{t-1}D - \frac{1}{2L_t} (M_tM_t^2), \quad B_t^2 = qB'_{t-1}D - \frac{N_tM_t}{L_t}, \quad C_t^2 = q\sigma^2_t e_t'A_{t-1}e_t + qC_{t-1} - \frac{N_t^2}{2L_t},
\]

\[
A_t^3 = qD'A_{t-1}D - M_t e_t^2 + \frac{1}{2} L_t e_t^2 e_t^2, \quad B_t^3 = qB'_{t-1}D - N_t e_t^2, \quad C_t^3 = q\sigma^2_t e_t'A_{t-1}e_t + qC_{t-1}.
\]

Based on \( V_t(\tilde{P}_{t-1}, X_t, W_t) \), given by (15), an approximating function \( \tilde{V}_t(\tilde{P}_{t-1}, X_t, W_t) = q\tilde{P}_{t-1}Y_t(H_t) \) where \( Y_t(H_t) = H'M_t^2 + (B_t^2)H_t + C_t \) for some \( A_t, B_t, \) and \( C_t \) such that the mean square error of approximation is minimum, can be obtained.

Thus, starting from the last time period and then moving backward in time while approximating cost-to-go \( V_t \) by \( \tilde{V}_t \) in each time period, and obtaining optimal trade \( S_t^* \), the optimal trading strategy can be obtained. Although the discussed approach based on ADP method enables one to impose non-negativity constraints on feasible trading strategy, a limitation of this method is that the approximation error may be high during volatile market when the chances of negative trades are more.

2.1.3 Discretisation method

The approach based on discretisation method to solve (P2) under LPT price model depends upon the assumption that the external variable \( \tilde{P}_t \) and \( X_t \) can be discretised. Let \( \tilde{p}_l, \ l = 1, 2, \ldots, L \) be the \( L \) possible values assumed by \( \tilde{P}_t \) in \( t \)th time period with the condition \( \tilde{p}_{l(t+1)} = q\tilde{p}_l \). Similarly, let \( x_{r(t+1)} = \rho x_{r(t)} \). Thus, there are \( L \) and \( R \) possible values of \( \tilde{P}_t \) and \( X_t \), respectively in
each time period. Further, we assume that $S_t$, $t = 1, 2, \ldots, T$ are whole numbers which in turn implies that $W_t$, $t = 1, 2, \ldots, T$ are whole numbers. Thus, we have $S + 1$ possible values of $W_t$ and $S_t$ for all the trading periods except for the first period in which $W_1$ is equal to $S$. That is

$$W_t = S, W_t \in \{0, 1, 2, \ldots, S\} \forall t = 2, 3, \ldots, T, S_t \in \{0, 1, 2, \ldots, W_t\} \forall t = 1, 2, \ldots, T.$$  

Similar to DP and ADP methods, approach based on discretisation method also starts by evaluating optimal cost-to-go $V_T$ for the last trading period in which optimal trading strategy is to execute all the remaining shares to finish that trade ($S_T = W_T$). Using the $LR(S+1)$ possible values of $V_{t+1}(\tilde{P}_t, X_{t+1}, W_{t+1})$ evaluated corresponding to $L, R$ and $S + 1$ possible values of $\tilde{P}_t$, $X_{t+1}$ and $W_{t+1}$, respectively, the optimal value of cost-to-go $V_t$ corresponding to preceding time period is obtained. This algorithm can be summarised in following steps:

**Step 1** For $t = T$,

$$V_T(\tilde{P}_{T-1}, X_T, W_T) = \min_{S_T} \{ \tilde{P}_T S_T \}$$

$$= \min_{S_T} \{ \tilde{P}_T (1 + \theta S_T + \gamma X_T) S_T \}$$

$$= \tilde{P}_T (1 + \theta W_T + \gamma X_T) W_T$$ (16)

We have $LR(S+1)$ possible values of $V_T$ as $V_T(\tilde{P}_{T-1}, X_T, f)$

$$=[q_{\tilde{P}_{T-1}}(1 + \theta f + \gamma X)]$$

for all $l = 1, 2, \ldots, L; r = 1, 2, \ldots, R$ and

$$j = 0, 1, 2, \ldots, S.$$

**Step 2** For $t = T - 1$: (−1):2,

$$V_t(\tilde{P}_{t-2}, X_{t-1}, W_t) = \min_{S \in \{0, 1, 2, \ldots, W_t\}} \left[ P_t S_t + V_{t+1}(\tilde{P}_{t-1}, X_{t+1}, W_{t+1}) \right]$$

$$= \min_{S \in \{0, 1, 2, \ldots, W_t\}} \left[ \tilde{P}_t (1 + \theta S_t + \gamma X_t) S_t + V_{t+1}(\tilde{P}_{t-1}, X_{t+1}, W_t - S_t) \right]$$

Thus, for all $l = 1, 2, \ldots, L; r = 1, 2, \ldots, R$ and $j = 0, 1, \ldots, S$ we get

$$V_t(\tilde{P}_{t-1}, X_{t-1}, f) = \min_{S \in \{0, 1, 2, \ldots, W_t\}} \left[ \tilde{P}_t S_t + V_{t+1}(1 + \theta S_t + \gamma X_t) S_t + V_{t+1}(\tilde{P}_{t-1}, X_{t+1}, W_t - S_t) \right].$$ (17)

For each combination of triplet $(\tilde{P}_{t-1}, X_{t-1}, f)$, let $S_t(\tilde{P}_{t-1}, X_{t-1}, f)$ be the point of minima of (17).

**Step 3** For $t = 1,$

$$V_1(\tilde{P}_0, X_1, W_1) = \min_{S \in \{0, 1, 2, \ldots, W_1\}} \left[ \tilde{P}_0 S_1 + V_2(\tilde{P}_1, X_2, W_2) \right]$$

$$= \min_{S \in \{0, 1, 2, \ldots, W_1\}} \left[ \tilde{P}_1 (1 + \theta S_1 + \gamma X_1) S_1 + V_2(\tilde{P}_1, X_2, W_1 - S_1) \right]$$
For the first time period, $\tilde{P}_0$ and $X_1$ are known, and $W_1$ equals to $S$. Thus, we have

$$V_1(\tilde{P}_0, X_1, \bar{S}) = \min_{S_i \in \{0, 1, 2, ..., S\}} \left[ q\tilde{P}_0 (1 + \delta S_i + \gamma X_1) S_i + V_2(q\tilde{P}_0, \rho X_1, \bar{S} - S_i) \right]. \quad (18)$$

Let $S'_1(\tilde{P}_0, X_1, \bar{S})$ be the value at which minimisation of (18) is achieved.

Consider realisation of vectors $\tilde{P}_{t-1}$ and $X_t$ for $t = 2, 3, ..., T$ as $\tilde{p}_{t-1} \tilde{p}_{t-2}, ..., \tilde{p}_{t-1} \tilde{r}_{t}$ and $x_{t}, x_{t}, ..., x_{t}, \rho x_{t}$, respectively. Then, optimal trading strategy of (P2) under LPT price model is given as follows:

$$S'_1(\tilde{P}_0, X_1, \bar{S}), S'_2(\tilde{p}_{t-1} x_{t}, \bar{S} - S'_1), S'_3(\tilde{p}_{t-2} x_{t}, \bar{S} - S'_2), ..., S'_T(\tilde{p}_{t-1} x_{t}, \bar{S} - \sum_{t=1}^{T} S'_t).$$

Moreover, the optimal execution cost is given by $V_1(\tilde{P}_0, X_1, \bar{S})$.

Despite the fact that discretisation approach enables one to impose non-negativity constraints in the optimal trading problem easily, there are some limitations associated with the approach. For example, this method is more suitable whenever trading horizon is small enough so that the number of possible values of external variables $\tilde{P}_{t-1}$ and $X_t$ ($L$ and $R$, respectively) are small and method is feasible computationally. Moreover, the amount of error incurred due to assumption of using expected dynamics of $\tilde{P}_{t-1}$ and $(\tilde{p}_t = q\tilde{p}_{t-1})$ and $x_t = \rho x_{t-1}$, respectively) may be questionable whenever the market is volatile, and due to which variables $\tilde{P}_{t-1}$ and $X_t$ are highly random.

Having discussed approaches based on ADP and discretisation methods to solve (P2) under LPT price model followed by mentioning their limitations, we now detail the algorithm to solve the problem via SA method. The comparison of SA method with other discussed methods is also presented.

2.1.4 SA method

Unlike DP, ADP and discretisation methods, the SA method starts from the first time period and moves forward in time while solving optimal trading problems corresponding to each time period. By optimal trading problem corresponding to $i^{th}$ time period, we mean the problem of finding the optimal trading strategy corresponding to time periods $t$ to $T$, say $\{S_t, S_{t+1}, ..., S_T\}$, for trading the shares, say $\hat{S}$, which are left after the trading of first $t - 1$ time periods are accomplished. However, from the strategy $\{S_t, S_{t+1}, ..., S_T\}$ only the trade $S_t$ is executed in $i^{th}$ time period and in the next time period $t + 1$ the optimal trading problem is formulated again based on the new realisation of external variables $\tilde{P}_t$ and $X_{t+1}$.

Mathematically, the optimal trading problem corresponding to $i^{th}$ time period is as follows:
In the following proposition, we evaluate the expected execution cost of trading in time periods \( t \) to \( T \), \( E_t \left( \sum_{u=t}^{T} P_u S_u \right) \), under the LPT price model, (5) and (6).

**Proposition 1:** Under the LPT price model (5) and (6), we have

\[
E_t \left( \sum_{u=t}^{T} P_u S_u \right) = R^t S_t + S^t Q^t S_t
\]

where

\[
R_t = \begin{pmatrix}
q_{t+1} (1 + X_t) \\
q^2 \hat{P}_{t-1} (1 + \gamma p X_t) \\
\vdots \\
q^{T-t+1} \hat{P}_{t-1} (1 + \gamma p^{T-t} X_t)
\end{pmatrix}
\quad \text{and} \quad
S_t = \begin{pmatrix}
S_t \\
S_{t+1} \\
\vdots \\
S_{T-1+t}
\end{pmatrix}
\]

\[
Q_t = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}_{(T-t+1) \times (T-t+1)}
\]

**Proof:** First, we prove the following:

\[
E_t \left( P_u \right) = \hat{P}_{t-1} q^{u-t-1} \left( 1 + \theta S_u + \gamma p^{u-t} X_t \right); \quad \forall u = t, t+1, ..., T.
\]

From (5) and (6), we have
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\[ E_t(P_u) = E_t\left[ \hat{P}_u e^{P_u} (1 + \theta S_u + \gamma X_u) \right] \]

\[ = E_t\left[ \hat{P}_u e^{P_u} e^{\eta u} e^{X_u} \{ 1 + \theta S_u + \gamma (\rho X_{u-1} + \eta_u) \} \right] \]

\[ = E_t\left[ \hat{P}_u e^{P_u} e^{\eta u} e^{X_u} \{ 1 + \theta S_u + \gamma (\rho X_{u-2} + \eta_{u-1} + \eta_u) \} \right] \]

\[ = E_t\left[ \hat{P}_u e^{P_u} e^{\eta u} e^{X_u} \{ 1 + \theta S_u + \gamma (\rho^2 X_{u-2} + \rho \eta_{u-1} + \eta_u) \} \right] \]

\[ = (\text{Continuing in this way}) \]

\[ = E_t\left[ \hat{P}_u e^{P_u} e^{\eta u} e^{X_u} \{ 1 + \theta S_u + \gamma (\rho \eta_{u-1} + \eta_u) \} \right] \]

\[ = \hat{P}_u E_t (e^{P_u} \ldots E_t (e^{X_u}) \times \left[ 1 + \theta S_u + \gamma \left( \rho \eta_{u-1} + \eta_u \right) \right] \]

\[ = \hat{P}_u E_t (e^{P_u} \ldots E_t (e^{X_u}) \times \left[ 1 + \theta S_u + \gamma \left( \rho \eta_{u-1} + \eta_u \right) \right] \]

\[ = \hat{P}_u E_t (e^{P_u} \ldots E_t (e^{X_u}) \times \left[ 1 + \theta S_u + \gamma \left( \rho \eta_{u-1} + \eta_u \right) \right] \]

\[ \cdot \text{ For the } t^{th} \text{ time period, variables } \hat{P}_u \text{ and } X_u \text{ are known. Also } \{ Z_i : i = 1, 2, \ldots, T \} \text{ and } \{ \eta_i : i = 1, 2, \ldots, T \} \text{ are two independent sets of i.i.d. random variables.} \]

\[ = \hat{P}_u \gamma^{u-1} (1 + \theta S_u + \gamma \rho \eta^u X^u) \]

\[ \cdot \text{ } E_t (Z_t) = q = e^{\rho^u \sigma^2} \text{ and } E_t (\eta_t) = 0. \]

which proves (20).

From (20), we get

\[
E_t \left( \sum_{u=t}^{T} P_u S_u \right) = \sum_{u=t}^{T} E_t \left( P_u \right) S_u
\]

\[ = \sum_{u=t}^{T} \hat{P}_u e^{P_u} \gamma^{u-1} (1 + \theta S_u + \gamma \rho \eta^u X^u) S_u \]

\[ = \sum_{u=t}^{T} \hat{P}_u e^{P_u} \gamma^{u-1} (1 + \theta S_u + \gamma \rho \eta^u X^u) S_u + \sum_{u=t}^{T} \hat{P}_u e^{P_u} \gamma^{u-1} \theta S_u \]

\[ = R'S_t + S'_Q S_t \]

\[ \text{Remark 1: The square matrix } Q_t \text{ of (19), being a diagonal matrix with positive entries, is a positive definite matrix. Thus, the expected cost } R'S_t = S'_Q S_t \text{ is a convex quadratic function of } S_t. \text{ In view of this, the optimal trading problem (P3) is a convex quadratic programming problem (QP(t)) of minimising a convex quadratic function subject to linear constraints. } \]

\[ \text{(QP(t)) } \]

\[ \min_{S_t} \quad R'S_t + S'_Q S_t \]

subject to

\[ \sum_{u=t}^{T} S_u = \hat{S}, \]

\[ S_u \geq 0; \quad u = t, t + 1, \ldots, T. \]
Remark 2: The optimal trading problem (P2) for a risk-neutral investor can be rewritten as the following quadratic programming problem (QP problem).

\[
\text{(QP problem)}
\]

\[
\min_{\mathbf{S}_t} \mathbf{R}_t^\prime \mathbf{S}_t + \mathbf{S}_t^\prime \mathbf{Q}_t \mathbf{S}_t \\
\text{subject to} \sum_{t=1}^{T} S_t = \mathbf{S}, \quad S_t \geq 0; \quad t = 1, 2, ..., T,
\]

where \( S_1, \mathbf{R}_1 \) and \( \mathbf{Q}_1 \) are given by (19) for \( t = 1 \).

The algorithm to solve QP problem under LPT price model by SA method is as follows:

**Step 1** For \( t = 1 \),

Formulate QP(1) with decision variable vector \( \mathbf{S}_1 \), known external state variables \( \mathbf{P}_1, X_1 \) and \( \hat{\mathbf{S}}(= \mathbf{S}) \) as the total number of shares to be traded at the beginning.

Solve the problem to obtain the optimal trading strategy as \( \{S_{11}, S_{12}, ..., S_{1T}\} \).

Execute \( S_{11} \) in the first time period.

**Step 2** For \( t = 2: (+1): T - 1 \),

Formulate problem QP(\( t \)) with decision variable vector \( \mathbf{S}_t \), new realisation of external variables \( \mathbf{P}_{t-1}, X_t \) and updated \( \hat{\mathbf{S}} = \mathbf{S} - \sum_{u=1}^{T-1} S_{u}^{(u)} \), which are number of shares to be executed after the trading in time periods 1 to \( t - 1 \) is accomplished.

Solve the problem to obtain the optimal solution of QP(\( t \)) as \( \{S_{t1}, S_{t2}, ..., S_{tT}\} \).

Execute \( S_{t1} \) in the \( t \)th time period.

**Step 3** For \( t = T \),

Execute all the remaining shares, \( \hat{\mathbf{S}} = \mathbf{S} - \sum_{u=1}^{T-1} S_{u}^{(u)} \), in the last time period.

That is optimal solution for QP(\( T \)) is \( S_{T1}^{(T)} = \hat{\mathbf{S}} \).

With Steps 1 to 3, we solve a sequence of \( T \) convex quadratic programming problems QP(\( t \); \( t = 1, 2, ..., T \)) to obtain optimal trading strategy of QP problem as \( \{S_{11}, S_{12}, ..., S_{1T}\} \).

There are several compelling reasons for using SA method over other methods discussed so far. The sequence of problems QP(\( t \)); \( t = 1, 2, ..., T \), can be formulated by incorporating many other desired constraints like sector-balance constraints, turnover...
constraints, etc., apart from imposing essential non-negativity constraints. Thus, SA method gives possibility of imposing any desired constraints in the optimal trading problem definition which is not the case with other methods. Furthermore, due to the convex quadratic programming formulation of QP(t); \( t = 1, 2, \ldots, T \) this method is easier to apply from the computational point of view. In addition to these, unlike ADP and Approximation methods, there is no approximation error incurred in applying SA method to solve optimal trading problem.

3 Optimal trading problem under LPT price model for a risk averse investor

In this section, we present the formulation of optimal trading problem for a risk averse investor, who intends to minimise expected execution cost and a measure for the timing risk. We consider CVaR of execution cost under LPT price model as a measure for the timing risk.

Let \( C_s = \sum_{t=1}^{T} P_t S_t \) be the total cost of execution corresponding to trading strategy \( \mathcal{S} = (S_1, S_2, \ldots, S_T) \) and the random vector of execution price \( \mathcal{P} = (P_1, P_2, \ldots, P_T) \).

For a specified probability level \( \beta (0 < \beta < 1) \), the value-at-risk (VaR) of the execution cost \( C_s (VaR_\beta (C_s)) \) is the minimum \( \alpha \) such that, with probability \( \beta \) the execution cost will always be less than or equals to \( \alpha \). The CVaR of \( C_s (CVaR_\beta (C_s)) \) is the conditional expectation of execution cost above the amount \( VaR_\beta (C_s) \).

By Rockafellar and Uryasev (2000), minimising \( CVaR_\beta (C_s) \) with respect to all feasible trading strategies \( \mathcal{S} \), is same as minimising following function \( f_\beta (\alpha, \mathcal{S}) \) with respect to all real \( \alpha \) and feasible trading strategies \( \mathcal{S} \).

\[
f_\beta (\alpha, \mathcal{S}) = \alpha + \frac{1}{1-\beta} E[(C_s - \alpha)^+] = \alpha + \frac{1}{1-\beta} E \left[ \sum_{t=1}^{T} P_t S_t - \alpha \right]^+
\]

where \( \alpha^+ = \max \{ \alpha, 0 \} \).

Thus, we have

\[
\min_{\mathcal{S}} CVaR_\beta (C_s) = \min_{(\alpha, \mathcal{S})} f_\beta (\alpha, \mathcal{S}).
\]

For a risk averse investor, we propose the bi-criteria optimisation problem of minimising two objective functions, expected execution cost \( E(C_s) \) and \( f_\beta (\alpha, \mathcal{S}) \). Moreover, non-negativity constraints are also considered in the problem formulation. As it is generally not possible to minimise both the objective functions together, we introduce a trade-off parameter \( \lambda \) in the objective function to convert the bi-objective problem into the one with single objective of minimising a convex combination of \( E(C_s) \) and \( f_\beta (\alpha, \mathcal{S}) \) (Cai et al., 2000). Thus, for a risk averse investor following optimal trading problem is proposed.
(P4) \[
\min_{(\alpha, S)} \lambda E_t(C_S) + (1 - \lambda) f_{\beta}(\alpha, S)
\]
subject to
\[
\sum_{t=1}^{T} S_t = \overline{S},
\]
\[
S_t \geq 0; \quad t = 1, 2, ..., T.
\]

where parameter \( \lambda \) \((0 \leq \lambda \leq 1)\) is a measure for the risk aversion. For a risk neutral investor \( \lambda = 1 \), whereas for a risk averse investor \( \lambda < 1 \).

Since it is difficult to get an analytical expression for the CVaR of execution cost under LPT price model, we use Monte Carlo simulation to discretise it. By (4), in any time period \( t \), the execution price \( (P_t) \) is function of no-impact price \( (\tilde{P}_t) \) and the information factor \( (X_t) \), which evolve according to their own dynamics given by (1) and (2), respectively. Let \( q_1 \) and \( q_2 \) be the number of scenarios generated, by Monte Carlo simulation, for \( \tilde{P}_t \) and \( X_t \), respectively. Further let \( P^{(i)}_t \) be the execution price in \( i^{th} \) time period corresponding to \( \tilde{P}^{(i)}_t \) \((i = 1, 2, ..., q_1)\) and \( X^{(j)}_t \) \((j = 1, 2, ..., q_2)\). Thus, there are \( q_1q_2 \) number of realisations of \( P^{(i)}_t \) in each time period \( t \), which in turn give realisation of execution cost as \( C^{(i)}_S \).

The approximation of \( f_{\beta}(\alpha, S) \), obtained by generating samples \( C^{(i)}_S \) of execution cost, is given as follows:
\[
\hat{f}_{\beta}(\alpha, S) = \alpha + \frac{1}{1 - \beta} \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \left( C^{(i)}_S - \alpha \right)^+ \\
= \alpha + \frac{1}{1 - \beta} \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \left( \sum_{t=1}^{T} P^{(i)}_t S_t - \alpha \right)^+ \\
= \alpha + \frac{1}{1 - \beta} \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \left[ \sum_{t=1}^{T} P^{(i)}_t (1 + \theta S_t + \gamma X^{(j)}_t S_t - \alpha) \right].
\]

With the approximate value of function \( \hat{f}_{\beta}(\alpha, S) \), given by (21), and quadratic formulation of \( E_t(C_S) \), given by (19) for \( \lambda = 1 \), the problem (P4) can be reformulated as follows. (Here, we make a note that \( S = S_t \)).
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(P5)

\[
\min_{(\alpha, S)} \lambda \left( R_i S + S^T Q_i S \right) + (1 - \lambda) \left[ \alpha + \frac{1}{1 - \beta} \times \frac{1}{q_i q_j} \sum_{i=1}^{q_i} \sum_{j=1}^{q_j} \left( \sum_{t=1}^{T} \tilde{P}_{ij} \left( 1 + \theta S_t + \gamma X_t^{(j)} \right) S_t - \alpha \right) \right] \\
\text{subject to} \\
\sum_{i=1}^{q_i} S_i = \bar{S}, \\
S_i \geq 0; \quad t = 1, 2, ..., T.
\]

For \( \lambda = 0 \), (P5) reduces to single objective optimisation problem of minimising \( CVaR_\beta (C_S) \) only. Let \( (\alpha^*, S^*) \) be the optimal solution of the problem (P5) with \( \lambda = 0 \). Then by Rockafellar and Uryasev (2000), we have

\[
VaR_\beta (C_{S^*}) = \alpha^* \quad \text{and} \quad CVaR_\beta (C_{S^*}) = f_\beta (\alpha^*, S^*).
\]  

To simplify the formulation of (P5), let

\[
U_{ij} = \left( \sum_{t=1}^{T} \tilde{P}_{ij} \left( 1 + \theta S_t + \gamma X_t^{(j)} \right) S_t - \alpha \right) \quad \forall i = 1, 2, ..., q_i; \quad j = 1, 2, ..., q_j.
\]  

By definition \((x)^+ = \max\{x, 0\}\), we get

\[
U_{ij} \geq \sum_{t=1}^{T} \tilde{P}_{ij} \left( 1 + \theta S_t + \gamma X_t^{(j)} \right) S_t - \alpha, \quad \text{and} \quad U_{ij} \geq 0 \quad \forall i = 1, 2, ..., q_i; \quad j = 1, 2, ..., q_j.
\]  

Now in view of (23) and (24), (P5) can be reformulated as follows:

(QCQP problem)

\[
\min_{(\alpha, S)} \lambda \left( R_i S + S^T Q_i S \right) + (1 - \lambda) \left[ \alpha + \frac{1}{1 - \beta} \times \frac{1}{q_i q_j} \sum_{i=1}^{q_i} \sum_{j=1}^{q_j} U_{ij} \right] \\
\text{subject to} \\
U_{ij} \geq \sum_{t=1}^{T} \tilde{P}_{ij} \left( 1 + \theta S_t + \gamma X_t^{(j)} \right) S_t - \alpha; \quad i = 1, 2, ..., q_i; \quad j = 1, 2, ..., q_j, \\
U_{ij} \geq 0; \quad i = 1, 2, ..., q_i; \quad j = 1, 2, ..., q_j, \\
\sum_{i=1}^{q_i} S_i = \bar{S}, \\
S_i \geq 0; \quad t = 1, 2, ..., T.
\]
The inequalities (24), which appear as constraints in QCQP problem above, are quadratic in $S$. Further, the objective function of QCQP problem is also a quadratic function. Thus, for a risk averse investor, optimal trading problem formulation results in a quadratically constrained quadratic programming problem and hence the name QCQP problem. The QCQP class of problems are well studied in the areas like conic optimisation, semidefinite programming, and, therefore have all nice properties associated with the class.

### 3.1 Solution of QCQP problem: SA method

In this section, an approach to solve QCQP problem by SA method is presented. Apart from enabling to include desired constraints in the optimal trading problem, the SA method also allows inclusion of some risk measure for the timing risk in the objective function of the problem. Consider the following optimal trading problem with CVaR risk measure corresponding to $t^{th}$ time period, after the execution of first $t-1$ time periods after which we are left with $\hat{S}$ shares to be traded.

$$
QCQP(t) \quad \min \left( R_tS_t + S_t^Q, S_t \right) + (1-\lambda) \left[ \alpha + \frac{1}{1-\beta} \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} U_{ij} \right]
$$

subject to

$$U_{ij} \geq \sum_{u=t}^{q} P_{ij} \left( 1 + \delta S_{u} + \gamma X_{u}^{(j)} \right) - \alpha ; \quad i = 1, 2, ..., q_1; \quad j = 1, 2, ..., q_2;
$$

$$U_{ij} \geq 0; \quad i = 1, 2, ..., q_1; \quad j = 1, 2, ..., q_2;
$$

$$\sum_{u=t}^{q} S_{u} = \hat{S}
$$

$$S_{u} \geq 0; \quad u = t, t+1, ..., T,
$$

where $R_t$, $S_t$ and $Q_t$ are given by (19).

The algorithm to solve QCQP problem by SA method is similar to one discussed for QP problem before with the difference that here we solve sequence of problems $QCQP(t); \quad t = 1, 2, ..., T$ instead of quadratic programming problems. First QCQP(1) with decision variables $S_1$, external state variables $P_0$, $X_1$, and $\hat{S}$ equals to $\bar{S}$, is solved to obtain optimal trading strategy as $\{S_{(1)}^{(1)}, S_{(1)}^{(2)}, ..., S_{(1)}^{(T)}\}$. In the first time period $S_{(1)}^{(1)}$ is executed. Further, QCQP(2) is formulated with decision variables as $S_2$, external variables as $P_1$, $X_2$, and $\hat{S}$ equals to $\bar{S} - S_{(1)}^{(1)}$. From the optimal trading strategy of QCQP(2), say $\{S_{(2)}^{(1)}, S_{(2)}^{(2)}, ..., S_{(2)}^{(T)}\}$, $S_{(2)}^{(2)}$ is executed in the second time period, and next QCQP(3) is formulated. This procedure continues till we reach to last trading period where the optimal strategy is to execute all the remaining shares in that period. That is, $S_{(T)}^{(T)} = \hat{S} = \bar{S} = \sum_{u=t}^{q} S_{(u)}^{(u)}$. In this way, the optimal trading strategy of (QCQP problem) is obtained as $\{S_{(1)}^{(1)}, S_{(2)}^{(2)}, ..., S_{(T)}^{(T)}\}$. 
4 Numerical illustrations

To illustrate the practical significance of the proposed QP and QCQP problems for risk neutral and risk averse investors, respectively, we conduct numerical analysis. With the analysis presented, the dependence of optimal trading strategies of these problems on the risk aversion parameter and parameters of LPT price model is also studied. The numerical analysis is conducted on simulated as well as on the real market data drawn from NSE India (National Stock Exchange of India Ltd.) database. To solve QP and QCQP problems by SA method, the sequences of optimisation problems QP(\(t\)) and QCQP(\(t\)); \(t = 1, 2, \ldots, T\), are solved by MOSEK solver available within MATLAB software.

4.1 Simulated data

The numerical analysis is conducted on simulated data with following parameter values.

\[
T = 20, \bar{S} = 10,000, \hat{P}_0 = 550, X_0 = 0, \sigma_z = \frac{0.02}{\sqrt{13}}, \rho = 0.5 \quad \sigma_\eta = \sqrt{1 - \rho^2}, \beta = 0.95,
\]

\[
\mu_x = 0, -0.0005, 0.0005; \quad \theta = 5 \times 10^{-6}, 10 \times 10^{-6}, 20 \times 10^{-6}.
\]

The optimal trading problem of an investor who wishes to buy 10,000 shares of a stock within 20 trading time periods is considered (\(\bar{S} = 10,000, T = 20\)). The values \(\mu_x = 0, -0.0005\) and 0.0005 give \(q = 1, q < 1\) and \(q > 1\), respectively; which correspond respectively to stable, bearish and bullish market scenarios. The parameter \(\theta\) gives the impact of investor’s own trade on the execution price. The values \(5 \times 10^{-6}, 10 \times 10^{-6}\) and \(20 \times 10^{-6}\) of \(\theta\) correspond to 5%, 10% and 20% price impact respectively, in the stable market for the trade of 10,000 shares. The considered value of \(\sigma_\eta\) gives a unit variance for \(X_t\), whereas the value of \(\sigma_z\) implies a 2% daily standard deviation in the log return of no-impact price of stock (Bertsimas and Lo, 1998). The random variables \(\hat{P}_t\) and \(X_t\) are simulated by Monte Carlo simulation technique with the considered parameters. For a risk averse investor, 95% of probability level in CVaR is considered.

Figure 1 compares the optimal trading strategy of QCQP problem for different combinations of parameters involved in its formulation. The effect of dynamics of information factor \(X_t\) on optimal trading strategies and the sensitivity of these strategies towards risk aversion parameter \(\lambda\) and the parameter \(\gamma\) are analysed. The strategies are obtained for all the three market scenarios-stable, bearish and bullish.

It is evident from Figure 1 that optimal trading strategies exhibit reversal nature with information dynamics. Because of the temporary price impact in LPT price model, higher value realisation of information factor, and thus higher execution price in any time period results in lower trade in that period. Furthermore, for a particular \(\lambda\) and market scenario, the optimal trading strategy is more information driven for higher \(\gamma\) value. Here, trading strategy, being information dynamics driven means its movement is according to the movement of price dynamics but in opposite sense. With increase in \(\gamma\), the effect of information factor in execution price increases and hence it is efficient to trade in more reversal nature according to the information dynamics. This effect of \(\gamma\) is more prominent.
for large $\lambda$ values. The larger values of $\lambda$ give less risk aversion and, therefore trading behaviour depends mostly upon parameters of LPT price model like $\gamma$.

**Figure 1** Comparison of optimal trading strategies of QCQP problem obtained for different levels of risk aversion parameter $\lambda$ and parameter $\gamma$ with fixed $\theta = 5 \times 10^{-6}$

From the formulation of QCQP problem, it is apparent that lower $\lambda$ values correspond to higher risk aversion factor. Thus, for low value of $\lambda$ ($\lambda = 0.0001$), optimal trading strategies are aggressive in all the market scenarios. This is due to the fact that longer trade duration results in higher timing risk and to mitigate that, major portion of trading is completed by 10th time period in all the market scenarios. For the moderate value of $\lambda$ ($\lambda = 0.7$), the optimal trading strategies become less aggressive. In bearish market scenario, the trading is similar to naive strategy (equal division of total trade in all the time periods). This is because of the two conflicting facts-timing risk and bearish market behaviour. Due to timing risk, strategy tends to be aggressive to finish trade early. Whereas bearish market scenario expects decline in price of the stock in future, forcing optimal trading strategy to be passive with lower to higher number of trades as we progress in trading. These two facts together result in a naive like strategy. For bullish market, there is expectation of rise in price in future. This along with moderate value of $\lambda$ results in an aggressive strategy. However, strategy is less aggressive as compared to that of the case when $\lambda = 0.0001$.

The high value of $\lambda$ ($\lambda = 0.999$) results in negligible risk aversion effect and optimal trading strategy depends highly on market scenario. The optimal trading strategies for stable, bearish and bullish market are naive like, upward tending and downward tending, respectively. In bullish market, price of the stock is expected to rise in future and thus optimal strategy is to trade aggressively to finish trading early. Whereas in bearish market, the investor trades slowly while waiting for more favourable prices in future. For stable market, trading is similar to naive strategy.
Figure 2  Behaviour of optimal trading strategy of QP problem for different values of $\theta$ and fixed $\gamma = 0.001$ in the stable market ($q = 1$)

Figure 2 gives the effect of parameter $\theta$ on optimal trading strategies of QP problem in stable market scenario. For lower values of $\theta$, the effect of information factor in execution price is more and thus the optimal trading strategy is more information driven. Whereas higher values of $\theta$ lowers the impact of information variable in execution price, which results in optimal trading strategy tends more and more to behave like naive strategy.

Figure 3  Comparison of DP strategy (optimal trading strategy of DP problem), unconstrained QP strategy (optimal trading strategy of QP problem without non-negativity constraints) and constrained QP strategy (optimal trading strategy of QP problem with non-negativity constraints) obtained for set of parameters $\theta = 5 \times 10^{-6}$ and $\gamma = 0.01$, in the stable market ($q = 1$)
Figure 3 compares the optimal trading strategy of DP problem obtained via DP method, with the strategies of QP problem with and without non-negativity constraints which are obtained via SA method. The DP optimal trading strategy and unconstrained QP strategy are almost same. Thus, for the same problem formulation (DP problem and QP problem without non-negativity constraints are same), DP method and SA method result in almost same optimal trading strategies. Furthermore, the constrained QP strategy is similar to DP and unconstrained QP optimal trading strategies except for the time periods where latter two have negative trades, but the former has zero shares to be purchased.

4.2 Real market data

We conducted numerical analysis on the real market data drawn from NSE India database. The data considered consists of Nifty index’s and Reliance Industries Ltd. stock’s daily closing prices of all trading days in year 2014. We assume that the execution price of Reliance Industries Ltd. stock can be modelled with LPT price model. In general, return on indices is a common component in the prices of most equities (Bertsimas and Lo, 1998). As Reliance Industries Ltd. stock is one of the 50 stock of Nifty index, the return on the latter can be taken as the information factor in the LPT price dynamics of stock. Further, we assume that Reliance Industries Ltd. stock’s closing prices follow geometric Brownian motion. With these assumptions, we fit the calculated returns data of index to the dynamics of information factor \( \dot{X}_k = \rho X_{k-1} + \eta_k \), and closing prices data of the considered stock as geometric Brownian motion \( \dot{P}_k = \dot{P}_{k-1} e^{\mu Z_k} \). These need the estimation of the values of the parameters \( \rho, \sigma_\eta, \mu_Z \) and \( \sigma_Z \), which is accomplished using the least squares estimation method (Bertsimas et al., 1999).

The \( \eta_k \) is error term in the dynamics of \( X_k \). To minimise the sum of the square of errors for historical data values, we consider

\[
\frac{\partial}{\partial \rho} \sum_{k=1}^{M} (X_k - \rho X_{k-1})^2 = 2 \sum_{k=1}^{M} (X_k - \rho X_{k-1})(-X_{k-1}),
\]

where \( M \) is the number of historical data points. Equating above derivative to zero, we obtain following estimate for \( \rho \):

\[
\hat{\rho} = \frac{\sum_{k=1}^{M} X_k X_{k-1}}{\sum_{k=1}^{M} X_{k-1}^2}.
\]

Under the information factor dynamics, given by (2), \( X_k - \rho X_{k-1} \) follows normal distribution \( N(0, \sigma_\eta^2) \). An estimate of \( \sigma_\eta^2 \) is the sample variance of considered sample data, which is given as follows.

\[
\hat{\sigma}_\eta^2 = \frac{1}{M - 1} \sum_{k=1}^{M} \left[ (X_k - \hat{\rho} X_{k-1}) - (\bar{X} - \hat{\rho} \bar{X}) \right]^2
\]

where \( \bar{X} = \frac{1}{M} \sum_{k=1}^{M} X_k \).

For parameter estimation of geometric Brownian motion, consider the following relation of log-prices.
\[
\ln \tilde{P}_t = \ln \tilde{P}_{t-1} + Z_t.
\]

From the data of \(\tilde{P}_t\), we calculate corresponding values of \(Z_t\) \((Z_t = \ln \tilde{P}_t - \ln \tilde{P}_{t-1})\). The parameters \(\mu_Z\) and \(\sigma_Z^2\) are estimated by sample mean and sample variance, respectively.

\[
\hat{\mu}_Z = \frac{1}{M} \sum_{k=1}^{M} Z_k, \quad \sigma_Z^2 = \frac{1}{M-1} \sum_{k=1}^{M} (Z_k - \bar{Z})^2
\]

where \(\bar{Z} = \frac{1}{M} \sum_{k=1}^{M} Z_k\).

For considered historical data, the following parameter values are estimated.

\[
\hat{\rho} = 0.1682, \quad \hat{\mu}_Z = -2.9046 \times 10^{-4}, \quad \hat{\sigma}_Z^2 = 2.1545 \times 10^{-4}, \quad \hat{\rho}_Z = 6.2498 \times 10^{-5}, \quad q = 1.001.
\]

We take \(\tilde{P}_0 = Rs.895.2\), (year 2013’s last trading day price of a share of Reliance Industries Ltd. stock). Further, we consider \(T = 20, \quad \delta = 10,000, \quad \gamma = 0.1\) and \(\theta = 5 \times 10^{-6}\).

With estimated parameters and considered historical data, we obtain and analyse the optimal trading strategy corresponding to purchase of shares of Reliance Industries Ltd. stock for risk neutral and risk averse investors.

**Figure 4** Optimal trading strategy of QCQP problem for Reliance Industries Ltd. stock with risk aversion parameter \(\lambda = 0.999\)

Figure 4 and Figure 5 compare the optimal trading strategies for purchasing of 10,000 shares of the Reliance Industries Ltd. stock in 20 trading days. The strategies are obtained corresponding to the QCQP problem for different levels of risk aversion parameter. The average return of Nifty index is also plotted in both the figures. Since return on Nifty index is assumed as the information component in the price of Reliance Industries Ltd. stock, the optimal trading strategies of Figure 4 and Figure 5 show the reversal behaviour with respect to it, which is the characteristic of temporary price impact. It is evident from Figure 4 that due to less risk aversion \((\lambda = 0.999)\) the trading is more information driven. Whereas in Figure 5 due to small \(\lambda\) value \((\lambda = 0.0001)\), which
corresponds to high risk aversion factor, the strategy is aggressive to reduce the timing risk. Moreover, the bearish market scenario ($q > 1$) also leads to early finishing of trading.

**Figure 5** Optimal trading strategy of QCQP problem for Reliance Industries Ltd. stock for risk aversion parameter $\lambda = 0.0001$

<table>
<thead>
<tr>
<th>Shares purchased</th>
<th>Time</th>
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5 Conclusions and future work

In this paper, we attempted to solve optimal trading problems for risk neutral and risk averse investors with inclusion of non-negativity constraints in the problem formulation. Different approaches, based on ADP method, discretisation method and SA method, are discussed to solve optimal trading problem of minimising expected execution cost subject to non-negativity constraints (QP problem) under the LPT price model. Further, several compelling reasons are mentioned to use SA method over other discussed methods to solve proposed problems. For a risk averse investor, the formulation of optimal trading problem of minimising a convex combination of expected execution cost and CVaR of execution cost under LPT price model is presented. This problem formulation results in a quadratically constrained quadratic programming problem (QCQP problem) which is solved by SA method. The practical significance of proposed optimal trading problems is amply supported by the extensive numerical illustrations conducted on simulated as well as on the real market data. The dependence of optimal trading strategies of QP and QCQP problems upon the parameters of LPT price model is assessed. Moreover, the behaviour of these strategies in different market scenarios, namely stable, bearish, and bullish, is discussed.

We mention some directions of further extension of the work of this paper. In view of Bertsimas et al. (1999), this paper can be further extended for the trading of portfolio of assets with inclusion of constraints like turnover constraints, dollar-balance constraints, etc., in the problem formulation. Furthermore, on similar lines to work of Moazeni et al. (2013), one can further extend the presented work by including jump processes in execution price path to model the price impacts of other traders’ buy and sell orders.
Optimal trading strategy under linear-percentage temporary impact price

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