Minimising weighted completion time on a single machine under uncertain weights

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Abstract: This paper has investigated the single-machine scheduling problem regarding the minimisation of the total weighted completion time, with the known processing times, while weights are uncertain. Uncertainty in weights is modelled using a scenario set, which contains explicitly listed scenarios of weights (the discrete-scenario case) or the Cartesian product of the intervals that contain possible values of weights (the interval-data case). Two main criteria are investigated: minimising the maximum objective function (min-max version) and minimising the maximum regret (min-max regret version). The computational complexity of the min-max (regret) versions of the single-machine scheduling problem in the cases of the discrete scenario as well as interval data is discussed, respectively, and on this basis, the approximation of corresponding NP-hard problems is further analysed.

Keywords: single-machine scheduling; weighted completion time; min-max; min-max regret; complexity; approximation.

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1 Introduction

The uncertainty existing in the parameters of a model may result from different resources in different application areas, which can be either the unpredictable change in the future, fluctuations inherent in the system, importance of the jobs, or incomplete information. Uncertainty can be represented by a fuzzy set or expressed by the concept of discrete scenarios as well as interval data. The latter two ways of describing uncertainty have been reported by previous studies (Kouvelis and Yu, 1997). In the case of discrete scenarios, the scenario set is explicitly described, where a scenario describes a situation that could occur in the real world. In the case of interval data, every numerical parameter can take any number within the range between given lower and upper bounds.

Decision makers may lack confidence or experience regret when they use the results derived from estimated values for uncertain parameters. However, risk-averse decision makers prefer to hedge against the risk. The min-max (regret) criteria are proposed by Kouvelis and Yu (1997). These criteria enable the decision makers to make the decisions that behave reasonably upon uncertainty or imprecise input data (Mulvey et al., 1995; Yang and Yu, 2002; Mandal et al., 2009). The min-max criterion intends to get feasible schedules with the best performance in the worst case. The min-max regret criterion, tries to hedge against the worst-case performance, via a schedule minimising the maximum deviation between the resolved value and the optimal one for this scenario. Conservative decisions can be obtained through these two criteria, so as to hedge against uncertainties of the input data (Mastrolilli et al., 2013; Sun et al., 2014; Ramezani et al., 2015).

There are extensive studies on the complexity of the min-max (regret) versions during the last twenty years (Lebedev and Averbakh, 2006; Aissi et al., 2009; Lu et al., 2012; Tadayon and Smith, 2015; Choi and Chung, 2016; Drwal and Rischke, 2016; Bougeret et al., 2019). For a survey of min-max (regret) problems we refer the reader to Kasperski and Zieliński (2014). NP-hard problems can be solved by heuristic algorithm (Laha and Chakraborty, 2010), but the approximation of which is also a research hotspot. Up to now, there are some approximation results known for NP-hard combinatorial optimisation problems (Kasperski and Zieliński, 2006; Aissi et al., 2009, 2010; Conde, 2010, 2012; Poss, 2017; Chassein and Goerigk, 2018) and scheduling problems (Drwal and Rischke, 2016; Aissi et al., 2011; Choi and Chung, 2016; Kasperski and Zieliński, 2008, 2016, 2019; Feuerstein et al., 2014; Nip et al., 2015; Pereira, 2016; Bougeret et al., 2019; Detti et al., 2019; Györgyi and Kis, 2019). Aissi et al. (2009) obtained approximation solution of combinatorial optimisation problem by constructing a fictitious scenario. Kasperski and Zieliński (2006) proposed an approximation algorithm based on midpoint scenario for combinatorial optimisation problem in the interval-data case. Pereira (2016) investigated the min-max regret version of a single-machine scheduling problem by using interval processing times and total weighted completion time objective, in which the approximation ratio in the midpoint scenario was analysed. Detti et al. (2019) analysed the complexity and the approximation of single machine problem concerning the joint scheduling of multiple jobs and a maintenance activity.

In comparison to a large number of published papers related to uncertain processing times, a limited amount of literature has been devoted to the research of uncertain weights (Averbakh, 2001; Averbakh and Lebedev, 2005; Aloulou and Della Croce, 2008; Kasperski et al., 2013; Kasperski and Zieliński, 2016; Gilenson and Shabtay, 2019). Weight expresses the importance of one element relative to the other elements in the system set. The uncertainty of the weight parameter is essential in the research of the combinational optimisation problem. Averbakh (2001) presented a selecting problem with uncertain weights and demonstrated that the min-max regret version of the problem is NP-hard in the case of the discrete scenario but is polynomial-time solvable in the case of interval data. Subsequently, Averbakh and Lebedev (2005) presented a programming problem with uncertain function coefficients. The complexity of the min-max regret version of the problem is opposite to that of the problem in literature (Averbakh, 2001). Kasperski et al. (2013) studied a similar problem under the assumption of uncertain element costs. They showed that the min-max (regret) problems are approximable within given constant factor only when P = NP. In the scheduling problem, Kasperski and Zieliński (2016) proposed a general framework for solving single machine scheduling problems in which processing times, due dates, and weights are uncertain. Gilenson and Shabtay (2019) studied a multi-scenario scheduling problem on a single-machine and a two-machine flow-shop system, then they analysed the complexity of the problem where weights are scenario-dependent. Aloulou and Della Croce (2008) have proved that the min-max version of the single-machine scheduling problem with uncertain weights and total weighted completion time objective is NP-hard for the discrete-scenario case. In this paper, we will investigate the scheduling problem proposed by Aloulou and Della Croce (2008) further. We will discuss the complexity of the min-max version of the scheduling problem in the interval-data case and the complexity of the min-max regret version in the discrete-scenario case as well as the interval-data case, respectively. Then, we will present performance ratio of the approximation algorithm for the corresponding NP-hard problems.

The rest part of the paper is structured as follows. Some background concepts and definitions are elaborated in Section 2. The computational complexity analysis of the min-max (regret) versions corresponding to the single machine scheduling problem under discrete scenarios and interval data is given in Section 3. In Section 4, we present a performance ratio of polynomial-time approximation algorithm for the corresponding NP-hard problem. Then, two numerical examples are demonstrated in Section 5. Finally, some concluding remarks are provided in Section 6.

2 Problem establishment

A single machine is used to process n independent jobs $J_1, J_2, \ldots, J_n$, denoted by $J = \{J_1, J_2, \ldots, J_n\}$. Let p_i denote the processing time of job J_i. The assumption is made that the machine remains constantly available, while it can process only one job each time. Job splitting and pre-emption are prohibited. All the jobs can be processed at time zero. For simplicity, we denote every job J_i with its subscript i. Suppose that π is a feasible schedule representing the permutation of jobs in J. Let i_π denote the position occupied by job i and $C_i(\pi)$ be the corresponding completion time of job i in schedule π. $\prod$ is the set containing all feasible schedules. Weight w_i expresses the importance of job i relative to the other jobs in the job set. The uncertainty of the weights is described by a set. The set including all the possible scenarios can be described in two ways: discrete scenarios and interval data. In the case of the discrete scenario, the scenario set is explicitly described. Let S denote the set, and $|S| = k.$ For interval data, the weights can be any value from the corresponding intervals, $w_i \in \tilde{w}_i$. $\tilde{w}_i = [\underline{w}_i, \overline{w}_i]$ expresses a range of possible values of uncertain weight. Scenario set S is the Cartesian product of the corresponding interval $\tilde{w}_i$ for all $i \in J$. w_i^s is the weight of job i under scenario s. A vector $s = (w_1^s, w_2^s, \cdots, w_n^s)$ of n nonnegative real numbers will be called a scenario.

The considered single-machine scheduling problem can be presented as follows.

Problem **P** is defined as

$$\min_{\pi \in \Pi} \sum_{i \in J} w_i C_i(\pi), \tag{1}$$

where w_i be the uncertain weight.

The problem **P** aims to obtain an optimal schedule for which the total weighted completion time is minimised. We denote $F(\pi, s) = \sum_{i \in J} w_i^s C_i(\pi)$ the total weighted completion time of π under scenario s. Let $F^*(s) = \min_{\pi \in \Pi} F(\pi, s)$ be the optimal value of the problem **P** under a given scenario s.

The min-max version of problem **P** aims to obtain a feasible schedule with the best value in the worst case for all scenarios, as expressed in equation (2):

$$F_{robust} = \min_{\pi \in \Pi} \max_{s \in S} F(\pi, s). \tag{2}$$

In the case of the discrete scenario, the min-max version corresponding to problem **P** is represented as D-MIN-MAX **P**; in the interval-data case, it is represented as I-MIN-MAX **P**.

Given a feasible schedule $\pi \in \prod$, its absolute regret is defined as $R(\pi, s) = F(\pi, s) - F^*(s)$, for $s \in S$. The maximum regret of schedule π is denoted by $R(\pi) = \max_{s \in S} R(\pi, s) = \max_{s \in S} [F(\pi, s) - F^*(s)]$. A scenario s_π that maximises the absolute regret is called the worst-case scenario for π. The maximal regret criterion is a worst-case measure of performance, which represents the maximal possible deviation of a given schedule from the optimal value.

The min-max regret version of problem **P** aims to get a feasible schedule that minimises the maximum regret, which can be expressed as

$$R_{robust} = \min_{\pi \in \Pi} R(\pi) = \min_{\pi \in \Pi} \max_{s \in S} \left[F(\pi, s) - F^*(s) \right]. \tag{3}$$

In the case of the discrete scenario, the min-max regret version corresponding to problem **P** is represented as D-MIN-MAX REGRET **P**; in the interval-data case, it is represented as I-MIN-MAX REGRET **P**.

In order to define the approximation algorithm performance ratio of the robust scheduling problem, let $F(\pi) = \max_{s \in S} F(\pi, s)$, $R(\pi) = \max_{s \in S} R(\pi, s) = \max_{s \in S} \left[F(\pi, s) - F^*(s) \right]$ be the objective value of schedule *π*. We define the concept of the approximation algorithm performance ratio as follows.

Definition 1: The performance ratio of feasible schedule *π* is defined as

$$r(\pi) = \frac{F(\pi)}{\min_{\pi \in \Pi} F(\pi)} (\text{MIN-MAX } \mathbf{P}). \tag{4}$$

$$r(\pi) = \frac{R(\pi)}{\min_{\pi \in \Pi} R(\pi)} (\text{MIN-MAX REGRET } \mathbf{P}). \tag{5}$$

Definition 2: For a function *f*, an algorithm can be regarded as the *f*(*n*)-approximation algorithm, once it returns a solution *π* such that $r(\pi) \leq f(n)$, for any instance of size *n* of problem **P**.

3 Complexity

Theorem 1: D-MIN-MAX REGRET **P** is NP-hard, even when $|S| = 2$ and $p_i = 1$ for all $i \in J$.

Proof: D-MIN-MAX REGRET **P** can be expressed as

$$R_{robust} = \min_{\pi \in \Pi} \max_{s \in S} \left[\sum_{i \in J} w_i^s C_i(\pi) - F^*(s) \right], \tag{6}$$

where $|S| = k$.

D-MIN-MAX **P** has been studied by Aloulou and Della Croce (2008), which can be stated as

$$F_{robust} = \min_{\pi \in \Pi} \max_{s \in S} \sum_{i \in J} w_i^s C_i(\pi). \tag{7}$$

They have proved that the robust problem is NP-hard, even for $|S| = 2$ and $p_i = 1$ for all $i \in J$.

Clearly, the version

$$\min_{\pi \in \Pi} \max_{s \in S} \left[F(\pi, s) - C \right] = \min_{\pi \in \Pi} \max_{s \in S} \left[\sum_{i \in J} w_i^s C_i(\pi) - C \right], \tag{8}$$

where C is a constant, is also NP-hard, even for $|S| = 2$ and $p_i = 1$ for all $i \in J$.

The weighted shortest processing time (WSPT) rule can provide an optimal sequence for minimising the total weighted completion time on a single machine (Smith, 1956). According to the WSPT rule, $F^*(s) = \min_{\pi \in \Pi} F(\pi, s)$ can be easily solved, but the value of $F^*(s)$ depends on scenario s. In comparison to the constant C, the problem is more complicated. Based on the analysis above, we can ascertain that the D-MIN-MAX REGRET **P** is also NP-hard, even when $|S| = 2$ and $p_i = 1$ for all $i \in J$. □

Theorem 2: I-MIN-MAX **P** is polynomial-time solvable.

Proof: I-MIN-MAX **P** can be stated as equation (7), where S is the Cartesian product of the corresponding interval $\tilde{w}_i$ for all $i \in J$.

To solve I-MIN-MAX **P**, the optimisation problem $\min_{\pi \in \Pi} F(\pi, \bar{s})$ needs to be solved, where $\bar{s}$ represents the scenario which assigned each weight w_i is the same as the corresponding interval upper bound $\bar{w}_i$.

$$\begin{aligned} &\min_{\pi \in \Pi} \max_{s \in S} F(\pi, s) = \min_{\pi \in \Pi} \max_{s \in S} \sum_{i \in J} w_i^s C_i(\pi) = \min_{\pi \in \Pi} \sum_{i \in J} \bar{w}_i C_i(\pi) \\ &= \min_{\pi \in \Pi} \sum_{i \in J} w_i^{\bar{s}} C_i(\pi) = \min_{\pi \in \Pi} F(\pi, \bar{s}). \end{aligned} \tag{9}$$

According to the WSPT rule, the problem $\min_{\pi \in \Pi} F(\pi, \bar{s})$ can be easily solved in $O(n\log n)$ time. I-MIN-MAX **P** can be resolved directly, and the complexity has the same order with the problem $\min_{\pi \in \Pi} F(\pi, \bar{s})$ in $O(n\log n)$ time. □

Then we study the complexity of I-MIN-MAX REGRET **P** in the case that all the processing times are identical, i.e., $p_i = p$ for all $i \in J$. For simplicity, let problem $\left(1|p|\sum C_i\right)$ denote the single-machine scheduling problem to minimise the total flow time, where the processing time of each job is uncertain and can take any value on the corresponding interval of uncertainty. Let problem $\left(1|p_i{=}p, w|\sum w_i C_i\right)$ be the single-machine scheduling problem of minimisation the total weighted completion time, where the processing times are p, the weights of the jobs are uncertain and can be arbitrary values on the corresponding intervals of uncertainty.

Lemma 1: Problem $\left(1|p_i{=}p, w|\sum w_i C_i\right)$ is equivalent to the problem $\left(1|p|\sum C_i\right)$ for the interval data.

Proof: For the interval data, $\tilde{p}_i = [\underline{p}_i, \bar{p}_i]$ expresses the processing time interval of job i for the problem $\left(1|p|\sum C_i\right)$ and $\tilde{w}_i = [\underline{w}_i, \bar{w}_i]$ expresses the weight interval of job i for the problem $\left(1|p_i{=}p, w|\sum w_i C_i\right)$, where $\underline{w}_i = \underline{p}_i / p$, $\bar{w}_i = \bar{p}_i / p$ for all $i \in J$.

Given any scenario $p_i^s \in \tilde{p}_i$ and any sequence π for the problem $\left(1\middle|p\middle|\sum C_i\right)$, we generate a scenario $w_i^{s'} \in \tilde{w}_i$ with $w_i^{s'} = p_i^s / p$ and a sequence π' with $i_{\pi'} = n - i_\pi + 1$ for the problem $\left(1\middle|p_i{=}p, w\middle|\sum w_i C_i\right)$. Let F_1 and F_2 be the objective values of problem $\left(1\middle|p\middle|\sum C_i\right)$ and problem $\left(1\middle|p_i{=}p, w\middle|\sum w_i C_i\right)$, respectively. The following result is obtained:

$$F_2 = \sum_{i=1}^{n} w_i^{s'} \cdot i_{\pi'} \cdot p = \sum_{i=1}^{n} w_i^{s'} \cdot (n - i_\pi + 1) \cdot p = \sum_{i=1}^{n} (n - i_\pi + 1) \cdot p_i^s = F_1. \tag{10}$$

Hence, the sequence π' is the optimal schedule for the problem $\left(1\middle|p_i{=}p, w\middle|\sum w_i C_i\right)$, only if the sequence π is the optimal schedule for the problem $\left(1\middle|p\middle|\sum C_i\right)$. □

Theorem 3: The min-max regret (robust) version of problem $\left(1\middle|p\middle|\sum C_i\right)$ is NP-hard (Lebedev and Averbakh, 2006).

The following corollary shows the immediate consequence of Lemma 1 and Theorem 3.

Corollary 1: I-MIN-MAX REGRET **P** is NP-hard, even when $p_i = p$, for all $i \in J$.

4 Approximation

Since most of the min-max (regret) versions of single-machine scheduling problems are NP-hard, the study of approximation for such problems is particularly important.

4.1 D-MIN-MAX ***P***

Theorem 4: Solving the weighted completion time problem with each weight equal to its average overall scenarios is a *k*-approximation for D-MIN-MAX **P**, where *k* equals the number of scenarios.

Proof: D-MIN-MAX **P** can be stated as equation (7), where $|S| = k$.

Construct a single scenario scheduling problem for the minimisation of the total weighted completion time on a single machine, whose weight is $w_i^a = \sum_{s \in S} \frac{w_i^s}{k}$ for all $i \in J$, where a is an average scenario.

$$\begin{aligned} &\min_{\pi \in \Pi} F(\pi, a) = \min_{\pi \in \Pi} \sum_{i \in J} w_i^a C_i(\pi) = \min_{\pi \in \Pi} \sum_{i \in J} \sum_{s \in S} \frac{w_i^s}{k} C_i(\pi) \\ &= \min_{\pi \in \Pi} \frac{1}{k} \sum_{s \in S} \sum_{i \in J} w_i^s C_i(\pi) = \min_{\pi \in \Pi} \sum_{s \in S} \frac{1}{k} F(\pi, s). \end{aligned} \tag{11}$$

Let π^* be an optimal schedule of the single scenario scheduling problem.

Clearly, $U = \max_{s\in S} F(\pi^*, s)$ is an upper bound of D-MIN-MAX **P**, and $L = \sum_{s\in S} \frac{1}{k} F(\pi^*, s)$ is a lower bound of D-MIN-MAX **P**, since

$$\begin{aligned} L &= \sum_{s\in S} \frac{1}{k} F(\pi^*, s) = \min_{\pi\in\Pi} \frac{1}{k} \sum_{s\in S} F(\pi, s) \le \min_{\pi\in\Pi} \frac{1}{k} k \max_{s\in S} F(\pi, s) \\ &= \min_{\pi\in\Pi} \max_{s\in S} F(\pi, s). \end{aligned} \tag{12}$$

Finally, we have $U = \max_{s\in S} F(\pi^*, s) \le \sum_{s\in S} F(\pi^*, s) = kL.$ □

4.2 D-MIN-MAX REGRET *P*

Theorem 5: Solving the weighted completion time problem with each weight equal to its average overall scenarios is a *k*-approximation for D-MIN-MAX REGRET **P**, where *k* equals the number of scenarios.

Proof: D-MIN-MAX REGRET **P** can be expressed as equation (6), where $|S| = k$.

Construct a single scenario scheduling problem to minimise the total weighted completion time on a single machine, defined as

$$\min_{\pi\in\Pi} F(\pi, a) = \min_{\pi\in\Pi} \sum_{i\in J} w_i^a C_i(\pi), \tag{13}$$

where *a* is an average scenario, whose weight is $w_i^a = \sum_{s\in S} \frac{w_i^s}{k}$ for all $i \in J$.

$$\begin{aligned} \min_{\pi\in\Pi} F(\pi, a) &= \min_{\pi\in\Pi} \sum_{i\in J} w_i^a C_i(\pi) = \min_{\pi\in\Pi} \sum_{i\in J} \sum_{s\in S} \frac{w_i^s}{k} C_i(\pi) \\ &= \min_{\pi\in\Pi} \frac{1}{k} \sum_{s\in S} \sum_{i\in J} w_i^s C_i(\pi) = \min_{\pi\in\Pi} \sum_{s\in S} \frac{1}{k} F(\pi, \mathrm{s}). \end{aligned} \tag{14}$$

Let π^* be an optimal schedule of the single scenario scheduling problem.

Clearly, $U = \max_{s\in S} \left[F(\pi^*, s) - F^*(s) \right]$ is an upper bound of D-MIN-MAX REGRET **P**. However,

$$\begin{aligned} &\sum_{s\in S} \frac{1}{k} \left[F(\pi^*, s) - F^*(s) \right] = \min_{\pi\in\Pi} \frac{1}{k} \sum_{s\in S} \left[F(\pi, s) - F^*(s) \right] \\ &\le \min_{\pi\in\Pi} \frac{1}{k} k \max_{s\in S} \left[F(\pi, s) - F^*(s) \right] = \min_{\pi\in\Pi} \max_{s\in S} \left[F(\pi, s) - F^*(s) \right]. \end{aligned} \tag{15}$$

So $L = \sum_{s\in S} \frac{1}{k} \left[F(\pi^*, s) - F^*(s) \right]$ is a lower bound of D-MIN-MAX REGRET **P**.

Finally, we have $U = \max_{s\in S}\left[F(\pi^*, s) - F^*(s)\right] \le \sum_{s\in S}\left[F(\pi^*, s) - F^*(s)\right] = kL.$ □

4.3 I-MIN-MAX REGRET **P**

In this section, the performance ratio of polynomial-time approximation algorithm is derived for I-MIN-MAX REGRET **P** upon all processing times are identical, i.e., $p_i = p$ for all $i \in J$.

Theorem 6: Solving the weighted completion time problem with each weight equal to the average of the upper and lower bounds from the corresponding interval is a 2-approximation for the problem I-MIN-MAX REGRET **P**.

Proof: I-MIN-MAX REGRET **P** can be expressed as equation (6), where S is the Cartesian product of the interval $\tilde{w}_i$ for all $i \in J$.

Let m be the average scenario of the upper and lower bounds from the corresponding interval, i.e., for all $i \in J$, $w_i^m = \frac{1}{2}\left(\underline{w}_i + \overline{w}_i\right)$, and π^* be the optimised schedule in the scenario m. Consider any one feasible schedule $\pi \in \prod$, we have

$$\begin{aligned}
0 \le F(\pi, m) - F(\pi^*, m) &= \sum_{i\in J} p \cdot i_\pi \cdot w_i^m - \sum_{i\in J} p \cdot i_{\pi^*} \cdot w_i^m = \sum_{i\in J} p \cdot (i_\pi - i_{\pi^*}) \cdot w_i^m \\
&= \sum_{i\in J} p \cdot (i_\pi - i_{\pi^*}) \cdot \frac{\overline{w}_i + \underline{w}_i}{2} \\
&= \frac{1}{2}\left[\sum_{\{i:i_\pi > i_{\pi^*}\}} p \cdot (i_\pi - i_{\pi^*}) \cdot \overline{w}_i + \sum_{\{i:i_\pi > i_{\pi^*}\}} p \cdot (i_\pi - i_{\pi^*}) \cdot \underline{w}_i\right] \\
&\quad + \frac{1}{2}\left[\sum_{\{i:i_\pi < i_{\pi^*}\}} p \cdot (i_\pi - i_{\pi^*}) \cdot \overline{w}_i + \sum_{\{i:i_\pi < i_{\pi^*}\}} p \cdot (i_\pi - i_{\pi^*}) \cdot \underline{w}_i\right],
\end{aligned} \tag{16}$$

which is equivalent to the following inequality:

$$\begin{aligned}
&\sum_{\{i:i_\pi > i_{\pi^*}\}} p \cdot (i_\pi - i_{\pi^*}) \cdot \overline{w}_i + \sum_{\{i:i_\pi < i_{\pi^*}\}} p \cdot (i_\pi - i_{\pi^*}) \cdot \underline{w}_i \\
&\ge \sum_{\{i:i_{\pi^*} > i_\pi\}} p \cdot (i_{\pi^*} - i_\pi) \cdot \overline{w}_i + \sum_{\{i:i_{\pi^*} < i_\pi\}} p \cdot (i_{\pi^*} - i_\pi) \cdot \underline{w}_i.
\end{aligned} \tag{17}$$

Moreover,

$$\begin{aligned}
R(\pi) &= \max_{s\in S}\left[F(\pi, s) - F^*(s)\right] \ge \max_{s\in S}\left[F(\pi, s) - F(\pi^*, s)\right] \\
&= \sum_{\{i:i_\pi > i_{\pi^*}\}} p \cdot (i_\pi - i_{\pi^*}) \cdot \overline{w}_i + \sum_{\{i:i_\pi < i_{\pi^*}\}} p \cdot (i_\pi - i_{\pi^*}) \cdot \underline{w}_i.
\end{aligned} \tag{18}$$

Thus, we have

$$R(\pi) \geq \sum_{\{i:i_\pi < i_{\pi^*}\}} p \cdot (i_{\pi^*} - i_\pi) \cdot \bar{w}_i + \sum_{\{i:i_{\pi^*} < i_\pi\}} p \cdot (i_{\pi^*} - i_\pi) \cdot \underline{w}_i \ . \tag{19}$$

However,

$$F(\pi^*, s_{\pi^*}) - F(\pi, s_{\pi^*}) \leq \left[\sum_{\{i:i_{\pi^*} > i_\pi\}} p \cdot (i_{\pi^*} - i_\pi) \cdot \bar{w}_i + \sum_{\{i:i_{\pi^*} < i_\pi\}} p \cdot (i_{\pi^*} - i_\pi) \cdot \underline{w}_i \right], \tag{20}$$

where scenario s_{π^*} represents the worst-case scenario for schedule π^*. Clearly,

$$\begin{aligned} F(\pi^*, s_{\pi^*}) - F^*(s_{\pi^*}) &\leq F(\pi, s_{\pi^*}) - F^*(s_{\pi^*}) \\ &+ \left[\sum_{\{i:i_{\pi^*} > i_\pi\}} p \cdot (i_{\pi^*} - i_\pi) \cdot \bar{w}_i + \sum_{\{i:i_{\pi^*} < i_\pi\}} p \cdot (i_{\pi^*} - i_\pi) \cdot \underline{w}_i \right], \end{aligned} \tag{21}$$

Moreover,

$$R(\pi^*) = F(\pi^*, s_{\pi^*}) - F^*(s_{\pi^*}), R(\pi) \geq F(\pi, s_{\pi^*}) - F^*(s_{\pi^*}). \tag{22}$$

The following inequality can be derived,

$$\begin{aligned} R(\pi^*) &\leq R(\pi) + \left[\sum_{\{i:i_{\pi^*} > i_\pi\}} p \cdot (i_{\pi^*} - i_\pi) \bar{w}_i + \sum_{\{i:i_{\pi^*} < i_\pi\}} p \cdot (i_{\pi^*} - i_\pi) \underline{w}_i \right] \\ &\leq R(\pi) + R(\pi) = 2R(\pi). \end{aligned} \tag{23}$$

Therefore, for any feasible schedule π, $R(\pi^*) \leq 2R(\pi)$. □

5 Numerical example

Two single-machine scheduling problems are demonstrated to illustrate the above theorems, where weight uncertainty is modelled via discrete scenarios as well as interval data, respectively.

Example 1: In the five-job case, $n = 5$, $k = 2$; $p_1 = 2$, $p_2 = 3$, $p_3 = 13$, $p_4 = 10$, $p_5 = 7$, $w_1^1 = 3$, $w_2^1 = 4$, $w_3^1 = 25$, $w_4^1 = 9$, $w_5^1 = 15$, $w_1^2 = 10$, $w_2^2 = 14$, $w_3^2 = 15$, $w_4^2 = 12$, $w_5^2 = 5$. We seek to obtain the approximation algorithm performance ratio of the min-max (regret) versions.

There are 120 possible sequences for this instance. The values of $F(\pi, s)$ and $R(\pi)$ are listed in Table 1.

Consider the D-MIN-MAX **P**, according to Theorem 4, we calculate the average weights: $w_1^a = 6.5$, $w_2^a = 9$, $w_3^a = 20$, $w_4^a = 10.5$, $w_5^a = 10$. Then the optimal schedule for the single scenario problem is $\pi^* = \pi_2$. Clearly, $L = \sum_{s \in S} \frac{1}{2} F(\pi^*, s) = 973$ is a lower bound of the D-MIN-MAX **P**, and $U = \max_{s \in S}(\pi^*, s) = 1{,}041$ is an upper bound of the D-MIN-MAX **P**. Obviously, $U / L = 1.0699 < 2$. Table 1 shows that the optimal value for the D-MIN-MAX **P** is F_{robust} = 1,031, and the optimal schedule is π_{52}. The approximation algorithm performance ratio of the min-max version is 1.0097.

Consider the D-MIN-MAX REGRET **P**, $F^*(s_1)$ = 1,005 and $F^*(s_2)$ = 865 can be obtained from Table 1. The average scenario is $w_1^a = 6.5$, $w_2^a = 9$, $w_3^a = 20$, $w_4^a = 10.5$, $w_5^a = 10$. Then the optimal schedule for the average scenario problem is $\pi^* = \pi_2$. Clearly, $L = \sum_{s \in S} \frac{1}{k}\left[F(\pi^*, s) - F^*(s)\right] = 38$ is a lower bound of the D-MIN-MAX REGRET **P**, and $U = R(\pi^*) = 40$ is an upper bound of the D-MIN-MAX REGRET **P**. Hence, $U / L = 1.0526$. Actually, the optimal value for the D-MIN-MAX REGRET **P** is R_{robust} = 40, and the optimal schedule is π_2. The approximation algorithm performance ratio of the min-max regret version is 1.

From Example 1, we can see that the optimal solutions of the min-max (regret) versions in the case of the discrete scenario are more acceptable, due to the higher stability of the performances. Risk-averse decision makers would favour the min-max version ensuring at most 1,031 in all scenarios. If slight performance degradation is accepted in the worst-case scenario, the optimal schedule π_2 of the min-max regret version outperforms the optimal schedule π_{52} of the min-max version. Based on the performance ratio of the min-max (regret) versions, it is demonstrated that constructing a single scenario scheduling problem with each weight equal to its average overall scenarios is a very effective approximation method.

Table 1 $F(\pi, s)$ and $R(\pi)$ values

Schedule	$F(\pi, s_1)$	$F(\pi, s_2)$	$R(\pi)$
π_1 = (1, 2, 3, 4, 5)	1,078	871	73
π_2 = (1, 2, 3, 5, 4)	1,041	905	40
π_3 = (1, 2, 5, 3, 4)	1,086	945	81
π_4 = (1, 5, 2, 3, 4)	1,084	1,028	163
π_5 = (5, 1, 2, 3, 4)	1,085	1,088	223
π_6 = (1, 2, 4, 3, 5)	1,211	865	206
π_7 = (1, 2, 4, 5, 3)	1,256	905	251
π_8 = (1, 2, 5, 4, 3)	1,219	939	214
π_9 = (1, 5, 2, 4, 3)	1,217	1,022	212
π_{10} = (5, 1, 2, 4, 3)	1,218	1,082	217
π_{11} = (1, 3, 2, 4, 5)	1,055	1,008	143
π_{12} = (1, 3, 2, 5, 4)	1,018	1,042	177
π_{13} = (1, 3, 5, 2, 4)	1,016	1,125	260
π_{14} = (1, 5, 3, 2, 4)	1,061	1,165	300
π_{15} = (5, 1, 3, 2, 4)	1,062	1,225	360
π_{16} = (1, 3, 4, 2, 5)	1,068	1,112	247
π_{17} = (1, 3, 4, 5, 2)	1,066	1,195	330
π_{18} = (1, 3, 5, 4, 2)	1,029	1,229	364
π_{19} = (1, 5, 3, 4, 2)	1,074	1,269	404
π_{20} = (5, 1, 3, 4, 2)	1,075	1,329	464

Table 1 $F(\pi, s)$ and $R(\pi)$ values (continued)

Schedule	$F(\pi, s_1)$	$F(\pi, s_2)$	$R(\pi)$
π_{21} = (1, 4, 2, 3, 5)	1,224	969	219
π_{22} = (1, 4, 2, 5, 3)	1,269	1,009	264
π_{23} = (1, 4, 5, 2, 3)	1,267	1,092	262
π_{24} = (1, 5, 4, 2, 3)	1,230	1,126	261
π_{25} = (5, 1, 4, 2, 3)	1,231	1,186	321
π_{26} = (1, 4, 3, 2, 5)	1,201	1,106	241
π_{27} = (1, 4, 3, 5, 2)	1,199	1,189	324
π_{28} = (1, 4, 5, 3, 2)	1,244	1,229	364
π_{29} = (1, 5, 4, 3, 2)	1,207	1,263	398
π_{30} = (5, 1, 4, 3, 2)	1,208	1,323	458
π_{31} = (2, 1, 3, 4, 5)	1,079	873	74
π_{32} = (2, 1, 3, 5, 4)	1,042	907	42
π_{33} = (2, 1, 5, 3, 4)	1,087	947	82
π_{34} = (2, 5, 1, 3, 4)	1,088	1,007	142
π_{35} = (5, 2, 1, 3, 4)	1,086	1,090	225
π_{36} = (2, 1, 4, 3, 5)	1,212	867	207
π_{37} = (2, 1, 4, 5, 3)	1,257	907	252
π_{38} = (2, 1, 5, 4, 3)	1,220	941	215
π_{39} = (2, 5, 1, 4, 3)	1,221	1,001	216
π_{40} = (5, 2, 1, 4, 3)	1,219	1,084	219
π_{41} = (2, 4, 1, 3, 5)	1,224	943	219
π_{42} = (2, 4, 1, 5, 3)	1,269	983	264
π_{43} = (2, 4, 5, 1, 3)	1,270	1,043	265
π_{44} = (2, 5, 4, 1, 3)	1,233	1,077	228
π_{45} = (5, 2, 4, 1, 3)	1,231	1,160	295
π_{46} = (2, 4, 3, 1, 5)	1,213	1,043	208
π_{47} = (2, 4, 3, 5, 1)	1,214	1,103	238
π_{48} = (2, 4, 5, 3, 1)	1,259	1,143	278
π_{49} = (2, 5, 4, 3, 1)	1,222	1,177	312
π_{50} = (5, 2, 4, 3, 1)	1,220	1,260	395
π_{51} = (2, 3, 1, 4, 5)	1,068	973	108
π_{52} = (2, 3, 1, 5, 4)	1,031	1,007	142
π_{53} = (2, 3, 5, 1, 4)	1,032	1,067	202
π_{54} = (2, 5, 3, 1, 4)	1,077	1,107	242
π_{55} = (5, 2, 3, 1, 4)	1,075	1,190	325
π_{56} = (2, 3, 4, 1, 5)	1,080	1,049	184
π_{57} = (2, 3, 4, 5, 1)	1,081	1,109	244
π_{58} = (2, 3, 5, 4, 1)	1,044	1,143	278

Table 1 $F(\pi, s)$ and $R(\pi)$ values (continued)

Schedule	$F(\pi, s_1)$	$F(\pi, s_2)$	$R(\pi)$
π_{59} = (2, 5, 3, 4, 1)	1,089	1,183	318
π_{60} = (5, 2, 3, 4, 1)	1,087	1,266	401
π_{61} = (3, 1, 2, 4, 5)	1,044	1,108	243
π_{62} = (3, 1, 2, 5, 4)	1,007	1,142	277
π_{63} = (3, 1, 5, 2, 4)	1,005	1,225	360
π_{64} = (3, 5, 1, 2, 4)	1,006	1,285	420
π_{65} = (5, 3, 1, 2, 4)	1,051	1,325	460
π_{66} = (3, 1, 4, 2, 5)	1,057	1,212	347
π_{67} = (3, 1, 4, 5, 2)	1,055	1,295	430
π_{68} = (3, 1, 5, 4, 2)	1,018	1,329	464
π_{69} = (3, 5, 1, 4, 2)	1,019	1,389	524
π_{70} = (5, 3, 1, 4, 2)	1,064	1,429	564
π_{71} = (3, 2, 4, 1, 5)	1,057	1,186	321
π_{72} = (3, 2, 4, 5, 1)	1,058	1,246	381
π_{73} = (3, 2, 5, 4, 1)	1,021	1,280	415
π_{74} = (3, 5, 2, 4, 1)	1,019	1,363	498
π_{75} = (5, 3, 2, 4, 1)	1,064	1,403	538
π_{76} = (3, 2, 1, 4, 5)	1,045	1,110	245
π_{77} =(3, 2, 1, 5, 4)	1,008	1,144	279
π_{78} =(3, 2, 5, 1, 4)	1,009	1,204	339
π_{79} =(3, 5, 2, 1, 4)	1,007	1,287	422
π_{80} =(5, 3, 2, 1, 4)	1,052	1,327	462
π_{81} = (3, 4, 1, 2, 5)	1,069	1,288	423
π_{82} = (3, 4, 1, 5, 2)	1,067	1,371	506
π_{83} = (3, 4, 5, 1, 2)	1,068	1,431	566
π_{84} = (3, 5, 4, 1, 2)	1,031	1,465	600
π_{85} = (5, 3, 4, 1, 2)	1,076	1,505	640
π_{86} = (3, 4, 2, 1, 5)	1,070	1,290	425
π_{87} = (3, 4, 2, 5, 1)	1,071	1,350	485
π_{88} = (3, 4, 5, 2, 1)	1,069	1,433	568
π_{89} = (3, 5, 4, 2, 1)	1,032	1,467	602
π_{90} = (5, 3, 4, 2, 1)	1,077	1,507	642
π_{91} = (4, 1, 2, 3, 5)	1,236	1,045	231
π_{92} = (4, 1, 2, 5, 3)	1,281	1,085	276
π_{93} = (4, 1, 5, 2, 3)	1,279	1,168	303
π_{94} = (4, 5, 1, 2, 3)	1,280	1,228	363
π_{95} = (5, 4, 1, 2, 3)	1,243	1,262	397
π_{96} = (4, 1, 3, 2, 5)	1,213	1,182	317

Table 1 $F(\pi, s)$ and $R(\pi)$ values (continued)

Schedule	$F(\pi, s_1)$	$F(\pi, s_2)$	$R(\pi)$
π_{97} = (4, 1, 3, 5, 2)	1,211	1,265	400
π_{98} = (4, 1, 5, 3, 2)	1,256	1,305	440
π_{99} = (4, 5, 1, 3, 2)	1,257	1,365	500
π_{100} = (5, 4, 1, 3, 2)	1,220	1,399	534
π_{101} = (4, 2, 3, 1, 5)	1,226	1,147	282
π_{102} = (4, 2, 3, 5, 1)	1,227	1,207	342
π_{103} = (4, 2, 5, 3, 1)	1,272	1,247	382
π_{104} = (4, 5, 2, 3, 1)	1,270	1,330	465
π_{105} = (5, 4, 2, 3, 1)	1,233	1,364	499
π_{106} = (4, 2, 1, 3, 5)	1,237	1,047	232
π_{107} = (4, 2, 1, 5, 3)	1,282	1,087	277
π_{108} = (4, 2, 5, 1, 3)	1,283	1,147	282
π_{109} = (4, 5, 2, 1, 3)	1,281	1,230	365
π_{110} = (5, 4, 2, 1, 3)	1,244	1,264	399
π_{111} = (4, 3, 1, 2, 5)	1,202	1,282	417
π_{112} = (4, 3, 1, 5, 2)	1,200	1,365	500
π_{113} = (4, 3, 5, 1, 2)	1,201	1,425	560
π_{114} = (4, 5, 3, 1, 2)	1,246	1,465	600
π_{115} = (5, 4, 3, 1, 2)	1,209	1,499	634
π_{116} = (4, 3, 2, 1, 5)	1,203	1,284	419
π_{117} =(4, 3, 2, 5, 1)	1,204	1,344	479
π_{118} =(4, 3, 5, 2, 1)	1,202	1,427	562
π_{119} =(4, 5, 3, 2, 1)	1,247	1,467	602
π_{120} =(5, 4, 3, 2, 1)	1,210	1,501	636

Example 2: Consider a three-job instance where $n = 3$; $p_1 = p, p_2 = p, p_3 = p$; $\tilde{w}_1 = [3,16]$, $\tilde{w}_2 = [12,18]$, $\tilde{w}_3 = [12,30]$. We seek to obtain the approximation algorithm performance ratio of the min-max regret version.

There are six possible sequences for this instance π_1 = (3, 2, 1), π_2 = (3, 1, 2), π_3 = (1, 2, 3), π_4 = (1, 3, 2), π_5 = (2, 1, 3), π_6 = (2, 3, 1). The values $F(\pi, s)$ and $R(\pi)$ are listed in Table 2.

Consider the I-MIN-MAX REGRET **P**, the average scenario m is $w_1^m = 9.5$, $w_2^m = 15$, $w_3^m = 21$. Under the scenario m, $\pi^* = \pi_1$ is the optimal schedule. The maximal regret $R(\pi^*) = 10p$ can be obtained from Table 2. Clearly, for any feasible schedule π, $R(\pi^*) \leq 2R(\pi)$. Actually, the optimal schedule of the I-MIN-MAX REGRET is π_1 with a maximum regret $10p$. Obviously, the approximation algorithm performance ratio of the min-max regret version is 1.

From Example 2, it is demonstrated that the optimal value of the min-max regret version in the case of interval data depends on the values of the lower and upper bounds for the corresponding interval. In particular, with the increasing upper bound or decreasing lower bound, the maximum regret of each feasible schedule either remains stable or increases, which may influence the resulted optimal value. For example, we decrease $\underline{w}_3$ from 12 to 4, the updated min-max regret optimal schedule will change to π_6 with a maximum regret 22*p* and the performance ratio of the approximation algorithm is 1.182.

Table 2 $F(\pi, s)$ and $R(\pi)$ values

Schedule	*F(π, s) (scenario s)*	*R(π)*
π_1	$p \cdot (3w_1^s + 2w_2^s + w_3^s)$	10*p*
π_2	$p \cdot (2w_1^s + 3w_2^s + w_3^s)$	21*p*
π_3	$p \cdot (w_1^s + 2w_2^s + 3w_3^s)$	54*p*
π_4	$p \cdot (w_1^s + 3w_2^s + 2w_3^s)$	42*p*
π_5	$p \cdot (2w_1^s + w_2^s + 3w_3^s)$	45*p*
π_6	$p \cdot (3w_1^s + w_2^s + 2w_3^s)$	22*p*

6 Conclusions

In this paper, the single-machine scheduling problem to minimise the total weighted completion time with deterministic processing times and uncertain weights is investigated. We propose several computational complexity and approximation results on the min-max (regret) versions of the scheduling problem. As shown in Table 3, the min-max (regret) versions have the same complexity and approximation in the case of the discrete scenario. In the case of the interval data, the min-max version is a polynomial-time solvable problem; however, the min-max regret version is NP-hard. The min-max regret version has a 2-approximation algorithm for the interval-data case while the performance ratio of approximation algorithms is related to the number of scenarios in the case of the discrete scenario. However, finding an approximation algorithm of the min-max regret version for the interval-data case without restrictions on the processing times remains a challenging open question.

Table 3 Complexity and approximation of the min-max (regret) versions

Problem P		*Complexity*	*Approximation*
Min-max	D-MIN-MAX **P** $(\lvert S \rvert = k)$	NP-hard (Aloulou and Della Croce, 2008)	$\leq k$ (Theorem 4)
	I-MIN-MAX **P**	*O*(*n*log*n*) (Theorem 2)	
Min-max regret	D-MIN-MAX REGRET **P** $(\lvert S \rvert = k)$	NP-hard (Theorem 1)	$\leq k$ (Theorem 5)
	I-MIN-MAX REGRET **P** $(p_i = p)$	NP-hard (Corollary 1)	≤ 2 (Theorem6)

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