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Abstract: In this paper, we propose a continuous time fractional stochastic volatility model which extends the Barndorff-Nielsen and Shephard (2001) (BNS) model. Our model is the fractional BNS model, where we model the volatility as a fractional Lévy-driven Ornstein-Uhlenbeck process. We allow the memory parameter to be flexible so that our model can potentially produce short- or long-memory in volatility. We derive the analytical formula for option pricing using Fourier inversion technique. We numerically study the effect of memory parameter on the option prices and the calibration result indicates that the fractional model significantly improves the performance of the original BNS model.

Keywords: option pricing; stochastic volatility; Lévy-driven OU process; fractional Lévy; fractional calculus.


Biographical notes: Zhigang Tong is a doctoral student at Department of Mathematics and Statistics in University of Ottawa, Canada. He is interested in applying time series models to pricing financial derivatives. He is currently working on studying the probabilistic and statistical properties of fractional processes and their modifications, with particular emphasis on applications to those models in finance.

1 Introduction

There are many models for the uncertainty in future instantaneous volatility of stock returns. In the original Black and Scholes’s (1973) model, the volatility is modelled as a constant. As a result, when the implied volatility calculated from the empirical option data is plotted against option’s strike price and time to maturity, the resulting graph should be a flat surface. However, in practice, the implied volatility surface is not flat and tends to vary with the strike price and time to maturity. This disparity is now known as the volatility skew (see e.g., Rubinstein, 1994; Jackwerth and Rubinstein, 1996). To overcome this problem, a class of so-called stochastic volatility models is developed, where the volatility is a function of some stochastic process. The representative models
are Hull and White’s (1987) model, Heston’s (1993) model, Schöbel and Zhu’s (1999) model and BNS model (Barndorff-Nielsen and Shephard, 2001). The analytical formulas for European options are known for the latter three models.

All of these standard stochastic volatility models are able to reproduce some empirical stylised facts regarding derivative securities and implied volatilities, but they fail to capture the well-documented evidence of volatility persistence and particularly occurrence of fairly pronounced implied volatility skew effects even for rather long maturity options. In practice, the decrease of the skew amplitude when time to maturity increases turns out to be much slower than it goes according to the standard stochastic volatility model. It has been empirically observed that the autocorrelation function of the squared returns is usually characterised by its slow decay towards zero. This decay is not exponential, as in short memory processes, but at a hyperbolic rate.

One way to solve this problem is to model volatility as a long memory or spurious long memory process. In this direction, one can emphasise two streams of research: discrete time series approach and continuous time stochastic processes approach. In the time series framework most of models are the extensions of classical GARCH models. Baillie et al. (1996) and Bollerslev and Mikkelsen (1996) incorporate long memory fractional differencing into the GARCH model. The resulting model is called the fractionally integrated GARCH model or the FIGARCH model. There are many other nonlinear short memory models that exhibit spurious long memory in volatility. For example, the component GARCH models proposed by Engle and Lee (1999) constitute a convenient alternative method of incorporating long-memory-like features into a short-memory model. Christoffersen et al. (2008) find the component models significantly superior to the GARCH(1, 1) model in capturing European option prices.

From the option pricing perspective, a more suitable approach is to use continuous time stochastic processes to model prices of financial assets, while time series methodology can be employed in some discretisation schemes. Comte and Renault (1998) propose a continuous time fractional stochastic volatility (FSV) model. They assume that the stochastic volatility is driven by exponential of fractional Ornstein-Uhlenbeck (OU) process; that is the standard OU process where the Brownian motion is replaced by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Comte et al. (2012) consider a fractional affine stochastic volatility model, where the volatility process is driven by a fractional square root process. Due to the complex structures of the long memory stochastic processes, they cannot derive the analytical formulas for option pricing. Instead, they introduce some discretisation schemes and price options using Monte-Carlo simulations. Chronopoulou and Viens (2012a) study the stochastic volatility model of Comte and Renault (1998). Chronopoulou and Viens (2012b) also study two discrete time models: a discretisation of the continuous model of Comte and Renault (1998) via an Euler scheme and a discrete time model in which the return is a zero-mean i.i.d. sequence and the volatility is exponential of a fractional ARIMA process. In order to deal with the pricing problem, Chronopoulou and Viens (2012a, 2012b) construct a multinomial recombining tree using sampled values of the volatility.

In a recent work by Gatheral et al. (2014), they show that log-volatility behaves essentially as a fractional Brownian motion with Hurst exponent $H < \frac{1}{2}$, at any reasonable time scale. They adopt the FSV model of Comte and Renault (1998) and name
the new model rough FSV (RFSV) to underline that, in contrast to FSV, $H < \frac{1}{2}$. They demonstrate that RFSV model is consistent with financial time series data and enables them to obtain improved forecasts of realised volatility. They show that the FSV model will generate a term structure of volatility skew that is inconsistent with the observed one for very short expirations. They challenge the traditional view that volatility is a long memory process by arguing the classical estimation procedures will identify the spurious long memory of volatility. Bayer et al. (2016) show how the RFSV model can be used to price claims on both the underlying and the integrated volatility. Forde and Zhang (2015) study the asymptotics for the RFSV model.

In this paper, we extend the works of Barndorff-Nielsen and Shephard (2001) and propose a fractional BNS model, where we model the volatility as a fractional Lévy-driven OU process. The fractional Lévy-driven OU process is a generalisation of the fractional Brownian motion process and has been applied to the volatility modelling in Anh et al. (2002), Bender and Marquardt (2009), Bishwal (2011), Fink (2015) and Klüppelberg and Matsui (2015) and the credit risk modelling in Biagini et al. (2013) and Fink (2013). However, unlike their works where the memory parameter is restricted in $[0, \frac{1}{2})$, we allow memory parameter to take the value in $(-\frac{1}{2}, \frac{1}{2})$ so that our model can potentially introduce long memory or short memory into the volatility process. Moreover, unlike the RFSV model, we do not restrict the mean reversion parameter in the underlying OU process to be zero.

It is known that the models where the stock price process is replaced by a fractional process can imply the existence of arbitrage (see e.g., Rogers, 1997; Sottinen, 2001). However, the fractional volatility model is not affected by this consideration, because it is the volatility process that is driven by fractional noise. Since the underlying volatility cannot be traded, when we change the physical measure to the risk-neutral measure, we will implicitly include a parameter to measure the market price of risk. We provide a closed-form solution for option prices using Fourier inversion techniques. Recall that there is no option pricing formula given in the FSV model of Comte et al. (2012) and the RFSV model of Bayer et al. (2016), rather the authors rely on simulations.

We numerically study the effects of memory parameter on the option prices. We adopt a specific type of BNS model, namely gamma-OU model. We show that the fractional integration parameter has a sizable effect on the option prices across maturities and moneyness. We also calibrate the fractional model to a set of option prices and the result indicates that the volatility process does not display long memory but instead, it has a rougher path compared to the standard Lévy-driven case. Our calibration result also demonstrates the superior performance of the fractional model in pricing options.

The structure of the paper is as follows. In Section 2, we start with a brief overview of Lévy process and Lévy-driven OU process. Then we introduce the fractional calculus. With these preliminaries and tools, we review the BNS model in Section 3 and present the fractional BNS model in Section 4. The main result, the conditional characteristic function of log stock price, is included in Section 4. Simulation and numerical results based on a particular Lévy process, gamma-OU process, are given in Section 5. In Section 6, we calibrate the particular fractional model to a set of option data.
2 Preliminaries

2.1 Lévy processes

In this section Lévy processes are introduced together with several important definitions and properties. See Bertoin (1996) and Sato (1999) for a more exhaustive treatment on Lévy processes.

Let \((\Omega, \mathcal{F}, P, \mathcal{F}_t)\) be a filtered probability space which satisfies the usual conditions.

Definition 2.1: a càdlàg (sample paths that are almost surely continuous from the right and have limits from the left), adapted, real valued stochastic process \(L = \{L(t)\}_{t \geq 0}\) with \(L(0) = 0\) a.s. is called a Lévy process if the following conditions are satisfied:

- \(L\) has independent increments, i.e., \(L(t) - L(s)\) is independent of \(\mathcal{F}_s\) for any \(0 \leq s < t\)
- \(L\) has stationary increments, i.e., for any \(s, t \geq 0\) the distribution \(L(t + s) - L(t)\) does not depend on \(t\)
- \(L\) is stochastically continuous, i.e., for every \(t \geq 0\) and \(\epsilon > 0\):

\[
\lim_{\epsilon \downarrow 0} P\left( \left| L(t) - L(s) \right| > \epsilon \right) = 0.
\]

A Lévy process \(L(t)\) is infinitely divisible, which indicates that the characteristic function of marginal random variable \(L(t)\) can be expressed as follows:

\[
\Phi_{L(t)}(u) = E\left[ \exp \left( i u L(t) \right) \right] = \exp \left( i \psi_{L(t)}(u) \right),
\]

where \(\psi_{L(t)}(u)\) is the characteristic exponent of the Lévy process at unit time.

The Lévy-Khintchine formula (see Sato, 1999) gives the expression for characteristic exponent \(\psi_{L(t)}(u)\) as follows:

\[
\psi_{L(t)}(u) = i u - \frac{1}{2} \sigma^2 u^2 + \int_{-\infty}^{\infty} \left( \exp(\text{i}ux) - 1 - \text{i}ux1_{|x|\leq1} \right) \nu(dx),
\]

where \(\gamma \in \mathbb{R}, \sigma \in \mathbb{R}, v\) is a positive measure satisfying:

\[
\int_{-\infty}^{\infty} \min\left(1, |x|^2 \right) \nu(dx) < \infty.
\]

The measure \(\nu\) is called the Lévy measure of the distribution \(L\). Throughout the whole paper we will always assume that \(\nu\) additionally satisfies:

\[
\int_{-\infty}^{\infty} |x|^2 \nu(dx) < \infty.
\]

We only work with the case that \(L(t)\) is a subordinator, i.e., a process with no Brownian component, non-negative drift and only positive increments.

We consider the stochastic integral with respect to Lévy process. The following theorem of Rajput and Rosinski (1989) will be used in the paper:
Theorem 1: For $f \in L^2[0, T]$ the integral $\int_0^T f(t) dL(t)$ exists as an $L^2(\Omega)$-limit of approximating step functions. Moreover, we have for $u \in \mathbb{R}$:

$$E\left[ \exp\left( iu \int_0^T f(t) dL(t) \right) \right] = \exp\left( \int_0^T \psi_{L(t)}(uf(t)) \right).$$

2.2 Lévy-driven OU processes

The general Lévy-driven OU process $Y = \{Y(t)\}_{t \geq 0}$ is defined as the solution to the stochastic differential equation of the form

$$dY(t) = -\lambda Y(t) + dL(\lambda t), \quad Y(0) > 0,$$

where the subordinator $L$ is also referred to as the background driving Lévy process (BDLP) of $Y$.

It is easy to verify that a (strong) solution $Y = \{Y(t)\}_{t \geq 0}$ to (2) is given by

$$Y(t) = \exp(-\lambda t)Y(0) + \int_0^t \exp(-\lambda(t-s)) dL(\lambda s), \quad t \geq 0.$$ 

Up to indistinguishability, this solution is unique (Sato, 1999). In the case that $Y(t)$ is a stationary OU process, let $\psi_{L(1)}(u)$ be the characteristic exponent of $L(1)$ and $\psi_{Y(1)}(u)$ be the characteristic exponent of $Y(1)$, then they are related through the formula (see e.g., Barndorff-Nielsen and Shephard, 2001):

$$\psi_{L(1)}(u) = u \frac{d}{du} \psi_{Y(1)}(u).$$

2.3 Fractional integration and fractional derivative

In this section, we provide definitions of fractional integration and fractional derivative, which we will require in the following discussions. Following Samko et al. (1993), we have:

Definition 2.2: Let $f \in L^1[a, b]$ and $d > 0$, the Riemann-Liouville left-sided fractional integrals on $(a, b)$ of order $d$ are defined by

$$(I_{a+}^d f)(s) = \frac{1}{\Gamma(d)} \int_a^s f(u)(s-u)^{d-1} du, \quad a < s < b.$$ 

Definition 2.3: Let $f \in L^1(\mathbb{R})$ and $d > 0$, the Riemann-Liouville fractional integrals on $\mathbb{R}$ is defined as

$$(I_{-\infty}^d f)(s) = \frac{1}{\Gamma(d)} \int_{-\infty}^s f(u)(s-u)^{d-1} du.$$ 

Definition 2.4: Let $0 < d < 1$, the Riemann-Liouville fractional derivatives on interval $(a, b)$ can be defined as
\[ (D_a^d, f)(u) = \frac{1}{\Gamma(1-d)} \frac{d}{du} \int_u^\infty f(s)(u-s)^{-d} \, ds, \quad a < s < b. \]

**Definition 2.5:** Let \( 0 < d < 1 \), the Riemann-Liouville fractional derivatives on \( \mathbb{R} \) can be defined as

\[ (D_a^d, f)(u) = \frac{1}{\Gamma(1-d)} \frac{d}{du} \int_u^\infty f(s)(u-s)^{-d} \, ds. \]

Let \( 0 < d < 1 \). From Samko et al. (1993), we know that the fractional derivative \( D_a^d \) and fractional integration \( I_a^d \) have the following properties:

- for any \( f \in \mathcal{L}^1[a, b] \), we have
  \[ D_a^d I_a^d f = f \]
- for any \( h \) such that \( h = I_a^d f \), we have
  \[ I_a^d D_a^d h = h. \]

In this paper, we will use the unified notation both for fractional integral and fractional derivative by \( I_a^d = D_a^{-d} \) for \( d < 0 \). We will make use of the following result (see Samko et al., 1993):

**Theorem 2:** The relation

\[ I_a^d I_a^d f = I_a^{d+2d} f \]

is valid in any of the following cases:

1. \( d_2 > 0, d_1 + d_2 > 0, f(x) \in \mathcal{L}^1[a, b] \)
2. \( d_2 < 0, d_1 > 0, f(x) \in I_{a_1}^{-d_2} (\mathcal{L}) \)
3. \( d_1 < 0, d_1 + d_2 < 0, f(x) \in I_{a_1}^{d_1-d_2} (\mathcal{L}) \)

where \( I_{a_1}^d (\mathcal{L}) \) denotes the space of functions \( f(x) \) such that \( f = I_a^d h, h \in \mathcal{L}^1[a, b] \).

Few functions have a fractional integral expressible in terms of elementary functions. Exceptions include:

- for \( f(x) = c \), \( (I_0^d, f)(t) = c \frac{t^d}{\Gamma(d+1)} \)
- for \( f(x) = x^a \), \( (I_0^d, f)(t) = \frac{\Gamma(a+1)}{\Gamma(a+d+1)} t^{a-d} \) for \( a > -1 \)
- for \( f(x) = \exp(ax) \), \( (I_0^d, f)(t) = E_i(d, a) \frac{a^{-d} \exp(at)(d, at)}{\Gamma(d)} \), where \( E_i(d, a) \) is a Miller-Ross function and \( \gamma(d, at) \) is a lower incomplete gamma function.
3 The BNS model

The Barndorff-Nielsen-Shephard (BNS) model was introduced in Barndorff-Nielsen and Shephard (2001) and it is a specific class of stochastic volatility models in which the squared volatility process is a Lévy-driven OU process. In the model the price of the stock jumps downwards when an up-jump in volatility takes place. Between the jumps, the stock price moves continuously and the volatility decays also continuously. BNS model implies volatility clusters as well as heavy tails. The closed-form solution for the characteristic function of the asset price is available for some special cases of BNS models (see e.g., Nicolato and Venardos, 2003).

Let \((\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in [0, T]})\) be a filtered probability space and \(T\) is a fixed horizon date. In Barndorff-Nielsen et al. (2002), the squared volatility process under an equivalent martingale measure \(Q\) is defined as

\[
\frac{d\sigma^2(t)}{\lambda} = -\lambda \sigma^2(t) dt + dL(\lambda t),
\]

where \(L = \{L(t)\}_{t \in [0, T]}\) is a subordinator. Letting

\[
\hat{\theta} = \sup \{\theta \in \mathbb{R} : \log E[\exp(\theta L(1))] < \infty\},
\]

we assume that

\[
\hat{\theta} > 0 \quad \text{and} \quad \lim_{\theta \to \hat{\theta}} \log E[\exp(\theta L(1))] = \infty.
\]

The log stock price process under \(Q\) follows the dynamics

\[
dX(t) = d \log(S(t)) = \left[r - q - \frac{1}{2} \sigma^2(t) - \lambda \psi_{L(t)}(-i \rho)\right] dt + \sigma(t) dB(t) + \rho dL(\lambda t),
\]

where \(r\) is the risk-free interest rate and \(q\) is the continuous dividend. \(B(t)\) is a Brownian motion and independent of the BDLP \(L(t)\) for \(t \in [0, T]\). The parameter \(\rho\) is introduced to capture the co-movement of the volatility and stock price processes.

The filtration \((\mathcal{F}_t)_{t \in [0, T]}\) is assumed to be generated by \(B(t)\) and \(L(\lambda t)\) for \(t \in [0, T]\). The conditional characteristic function of the log price can be written as

\[
\phi_{X(t+\tau)}(u) = E^Q \left[ \exp(iuX(t+\tau)) \bigg| \mathcal{F}_t \right] = \exp \left[ iuX(t) + \left( r - q - \lambda \psi_{L(t)}(-i \rho)\right) \tau - \frac{1}{2 \lambda} (u^2 + iu) \left( 1 - \exp(-\lambda \tau) \right) \sigma^2(t) \right]
\]

\[
\times \exp \left[ \lambda \int_0^\tau \psi_{L(s)} \left( \rho u - \frac{1}{2} u(1 - i u) \left( 1 - \exp(-\lambda (\tau - s)) \right) \right) ds \right].
\]

Denote the time \(t\) of a European call option with a strike price \(K\) and time of maturity \(t + \tau\) on an underlying asset \(S(t)\) by \(C(t, K, \tau, S(t))\). We can write the price of call option as
Option pricing in stochastic volatility models driven by

\begin{align*}
C(t, K, \tau, S(t)) &= S(t) \exp(-qt)Q_t \left( X(t + \tau) > \log(K) \right| \mathcal{F}_t \\
&\quad - \exp(-r\tau)KQ_t \left( X(t + \tau) > \log(K) \right| \mathcal{F}_t, 
\end{align*}

(7)

where \( Q_t \), \( j = 1, 2 \) are two probabilities that can be obtained from the characteristic function of log price:

\begin{align*}
Q_1 \left( X(t + \tau) > \log(K) \right| \mathcal{F}_t) &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{\exp(-iu \log(K) - (r-q)\tau - X(t))\phi_{x(t+\tau)}(u-i)}{iu} \right) \, du,
\end{align*}

(8)

and

\begin{align*}
Q_2 \left( X(t + \tau) > \log(K) \right| \mathcal{F}_t) &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{\exp(-iu \log(K))\phi_{x(t+\tau)}(u)}{iu} \right) \, du.
\end{align*}

(9)

4 The fractional BNS model

Let \((\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in [0, T]})\) be a filtered probability space, where the filtration \((\mathcal{F}_t)_{t \in [0, T]}\) is the same as that in the BNS model. Under the equivalent martingale measure \( Q \), we define the fractional BNS model by

\begin{equation}
\begin{aligned}
dX(t) &= \left( r - q - \frac{1}{2} \sigma^2(t) - \lambda \psi(\lambda t) - ip \right) dt + \sigma(t) dB(t) + \rho dL(\lambda t),
\end{aligned}
\end{equation}

(10)

where the volatility \( \sigma^2(t) \) follows the following dynamics:

\begin{align*}
\sigma^2(t) &= Y^d(t), \\
Y^d(t) &= (I_{d^2}^t, Y(t)), \\
dY(t) &= -Y(t) dt + dL(\lambda t), \ Y(0) > 0.
\end{align*}

(11-13)

We assume \(-\frac{1}{2} < d < \frac{1}{2}\) and \( d \) is connected to Hurst exponent through \( H = d + \frac{1}{2} \). We also assume that \( Y \in \mathcal{L}^d[0, T] \) so that \( Y(t) \) exists for \(-\frac{1}{2} < d < \frac{1}{2}\). It is clear that \( Y(t) \) is a Lévy-driven OU process whereas \( \sigma^2(t) \) is fractional integration of \( Y(t) \). From Basse and Pedersen (2009), we know that \( \sigma^2(t) \) is a semimartingale when \( d \in \left( 0, \frac{1}{2} \right) \) and

\[ \int_0^1 |Y|^{(1-d)} \nu(dy) < \infty, \] where \( \nu(dy) \) is the Lévy measure of the distribution \( Y \).

Furthermore, \( \sigma^2(t) \) is not a semimartingale when \( d \in \left( -\frac{1}{2}, 0 \right] \). We note that the model as formulated above is a generalisation of BNS model. By imposing the constraint that \( d = 0 \), we will obtain the original BNS model.
Let $\tilde{Y}^d(t)$ be the fractional integration of centred $Y(t)$, that is

$$\tilde{Y}^d(t) = \left( I^d_t \tilde{Y} \right)(t),$$

where $\tilde{Y}(t) = Y(t) - E(Y(t))$.

The process (14) is asymptotically equivalent to the (weakly) stationary process

$$\tilde{Y}^d(t) = \left( I^d_t \tilde{Y} \right)(t).$$

$\tilde{Y}^d(t)$ can be considered as an OU process driven by a fractional Lévy process. Fractional Lévy process can be constructed as a convolution of a classical Lévy process and a Volterra-type kernel. Two most prominent examples of such kernels are the Mandelbrot-van-Ness kernel and Molchan-Golosov kernel. The former one allows for stationary increments and is considered by Marquardt (2006), while the latter one is considered in Bender and Marquardt (2009) and Fink (2013, 2015). In this paper, we adopt the Mandelbrot-van-Ness kernel for the fractional Lévy process.

We have the following result:

**Lemma 3:** for $h > 0$,

$$\gamma^{Y^d}(h) = \frac{\text{Cov}\left( \tilde{Y}^d(t+h), \tilde{Y}^d(t) \right)}{\text{Var}\left( \tilde{Y}^d(t) \right)} \sim Ch^{2d-1} \text{ if } d \neq 0,$$

and

$$\gamma^{Y^d}(h) = \exp(-\lambda h) \text{ if } d = 0.$$

Marquardt (2006) proves this result when $d \in \left(0, \frac{1}{2}\right)$. We can follow the similar idea and also Pipiras and Taqqu (2000) to prove the result for the case $d \in \left(\frac{1}{2}, 0\right)$. Depending on the value of $d$, we will have three cases for the process $\tilde{Y}^d$ :

- when $d = 0$, $\tilde{Y}^d$ has exponentially decaying correlation
- when $-\frac{1}{2} < d < 0$, $\tilde{Y}^d$ is a short-memory process since $\sum_h |\gamma^{Y^d}(h)| < \infty$
- when $0 < d < \frac{1}{2}$, $\tilde{Y}^d$ is a long-memory process since $\sum_h |\gamma^{Y^d}(h)| = \infty$.

In the BNS model, the volatility process $\sigma^2(t)$ can only have exponentially decaying correlation, whereas in the fractional BNS model, it is more flexible and can be a short- or long-memory process depending on $d$. $d < 0$ implies a negative correlation between the increments of the fractional Lévy process, whereas $d > 0$ implies a positive correlation.
To obtain the analytical formula for option pricing, we need to find the conditional characteristic function $\phi_{X(t+\tau)}(u)$ of $X(t+\tau)$ in the fractional BNS model. This will be achieved by representing the process $\sigma^2(t)$ in (11) in terms of the Lévy-driven OU process via fractional integration. We can then compute $\phi_{X(t+\tau)}(u)$ from the following theorem:

**Theorem 4:** the conditional characteristic function $\phi_{X(t+\tau)}(u)$ of $X(t+\tau)$ under the probability measure $Q$ for the fractional BNS model is given by

$$
\phi_{X(t+\tau)}(u) = \exp \left[ iu(t-q-\lambda\psi_{\tau}(X_{t-\lambda})) \right. \\
+ \lambda \int_0^\tau \psi_{\tau}(\xi) \left( -\frac{1}{2} u(z_{t+\tau}) \phi_{t-z_{t+\tau}}(d+1,-\frac{\lambda}{2}) + \rho dL(z) \right) ds \\
- \frac{1}{2} u(i+u) \left( \frac{1}{\Gamma(d+1)} \int_0^\tau \left[ (t+\tau-s)^d - (t-s)^d \right] Y(s) ds \right. \\
+ iuX(t) - \frac{1}{2u} (i+u) \phi_{t-z_{t+\tau}}(d+1,-\frac{\lambda}{2}) Y(t) \right].
$$

**Proof:** from Theorem 2 we can decompose the integrated volatility as

$$
\int_t^{t+\tau} \sigma^2(s) ds = \int_0^{t+\tau} Y^d(s) ds - \int_0^\tau Y^d(s) ds = Y^{d+1}(t+\tau) - Y^{d+1}(t)
$$

$$
= \int_0^{t+\tau} \frac{(t+\tau-s)^d}{\Gamma(d+1)} Y(s) ds - \int_0^\tau \frac{(t-s)^d}{\Gamma(d+1)} Y(s) ds
$$

$$
= \frac{1}{\Gamma(d+1)} \int_0^{t+\tau} \left[(t+\tau-s)^d - (t-s)^d\right] Y(s) ds \\
+ \frac{1}{\Gamma(d+1)} \int_0^{t+\tau} (t+\tau-s)^d Y(s) ds.
$$

We know

$$
\phi_{X(t+\tau)}(u) = \mathbb{E}^Q \left[ \exp(iuX(t+\tau)) | \mathcal{F}_t \right]
$$

$$
= \mathbb{E}^Q \left[ \exp \left( iuX(t) + iu \int_t^{t+\tau} \left( -q - \frac{1}{2} \sigma^2(s) - \lambda\psi_{\tau}(\xi) \right) ds \\
+ iu \int_t^{t+\tau} \sigma(s) dB(S) + iu \int_t^{t+\tau} \rho dL(\lambda S) \right) | \mathcal{F}_t \right]
$$

$$
= \exp \left( iuX(t) + iu(t-q-\lambda\psi_{\tau}(\xi)) \right)
$$

$$
= \mathbb{E}^Q \left[ \exp \left( -\frac{1}{2} iu \int_t^{t+\tau} \sigma^2(s) ds \\
+ \int_t^{t+\tau} \sigma(s) dB(S) + iu \int_t^{t+\tau} \rho dL(\lambda S) \right) | \mathcal{F}_t \right].
$$
Noting that
\[
E^Q \left[ \exp \left( \frac{-u \sigma^2}{2} s + u \int_{t}^{t+\tau} \sigma(s) dB(s) + \int_{t}^{t+\tau} \rho dL(s) \right) \right] = E^Q \left[ \exp \left( \frac{-u \sigma^2}{2} s + 2u \int_{t}^{t+\tau} \sigma(s) dB(s) \right) +iu \int_{t}^{t+\tau} \rho dL(s) \right] \mathcal{F}_t \right] \]  
= E^Q \left[ \exp \left( \frac{-u \sigma^2}{2} s - \frac{1}{2} u^2 \int_{t}^{t+\tau} \sigma(s) dB(s) \right) +iu \int_{t}^{t+\tau} \rho dL(s) \right] \mathcal{F}_t \right] \]  
= E^Q \left[ \exp \left( \frac{-u \sigma^2}{2} s - \frac{1}{2} u^2 \int_{t}^{t+\tau} \sigma(s) dB(s) + iu \int_{t}^{t+\tau} \rho dL(s) \right) \right] \mathcal{F}_t \right] \]  
\tag{18}

Using (16), we can compute the expectation in the last line of (18) as
\[
E^Q \left[ \exp \left( \frac{-u \sigma^2}{2} s - \frac{1}{2} u^2 \int_{t}^{t+\tau} \sigma(s) dB(s) \right) \right] \mathcal{F}_t \right] = \exp \left( \frac{-u \sigma^2}{2} s - \frac{1}{2} u^2 \int_{t}^{t+\tau} \sigma(s) dB(s) \right) \mathcal{F}_t \right] \]  
= \exp \left( \frac{-u \sigma^2}{2} s - \frac{1}{2} u^2 \int_{t}^{t+\tau} \sigma(s) dB(s) + iu \int_{t}^{t+\tau} \rho dL(s) \right) \mathcal{F}_t \right] \]  
\tag{19}

Using
\[
Y(t + \tau) = \exp(-\lambda \tau)Y(t) + \int_{t}^{t+\tau} \exp(-\lambda(t + \tau - s)) dL(s),
\]
we have
\[
E^Q \left[ \exp \left( \frac{-u \sigma^2}{2} s - \frac{1}{2} u^2 \int_{t}^{t+\tau} \sigma(s) dB(s) + iu \int_{t}^{t+\tau} \rho dL(s) \right) \right] \mathcal{F}_t \right] = E^Q \left[ \exp \left( \frac{-u \sigma^2}{2} s - \frac{1}{2} u^2 \int_{t}^{t+\tau} \sigma(s) dB(s) \right) \right] \mathcal{F}_t \right] \]  
= E^Q \left[ \exp \left( \frac{-u \sigma^2}{2} s - \frac{1}{2} u^2 \int_{t}^{t+\tau} \sigma(s) dB(s) + iu \int_{t}^{t+\tau} \rho dL(s) \right) \right] \mathcal{F}_t \right] \]  
= \exp \left( \frac{-u \sigma^2}{2} s - \frac{1}{2} u^2 \int_{t}^{t+\tau} \sigma(s) dB(s) \right) \mathcal{F}_t \right] \]  
\times E^Q \left[ \exp \left( \frac{-u \sigma^2}{2} s - \frac{1}{2} u^2 \int_{t}^{t+\tau} \sigma(s) dB(s) \right) \right] \mathcal{F}_t \right] \]  
\tag{20}
By using (1), the last expectation in (20) becomes

\[
E^Q \left[ \exp \left( \frac{1}{2} u(i + u) \frac{1}{\Gamma(d + 1)} \int_t^{t + \tau} (t + \tau - s)^d \left[ \exp(-\lambda(s-w))dL(\lambda w)ds \right. \right. \right.

\left. + iup \int_t^{t + \tau} dL(\lambda s) \right] \mathcal{F}_t \bigg]\bigg]

\[
= E^Q \left[ \exp \left( \frac{1}{2} u(i + u) \frac{1}{\Gamma(d + 1)} \int_t^{t + \tau} \left( \int_t^{t + \tau} (t + \tau - s)^d \exp(-\lambda(s-w))dL(\lambda w) \right) \bigg] \mathcal{F}_t \bigg]

\[
= \exp \left[ \lambda \int_t^{t + \tau} \psi_{L(t)} \left( -\frac{1}{2} u(1-iu) \frac{1}{\Gamma(d + 1)} \right) \right.

\left. \int_t^{t + \tau} (t + \tau - s)^d \exp(-\lambda(s-w))ds + up \bigg] dw \right].
\]

Finally, combining (17)–(21), we obtain (15).

Note that when \( d \in (0, \frac{1}{2}) \), the path of the underlying volatility process \( \sigma^2 \) will be smoother than the case when \( d = 0 \). When \( d \in \left( \frac{1}{2}, 0 \right) \), the path of process \( \sigma^2 \) can be much rougher than that from the BNS model. However, what matters for option pricing is the integrated volatility \( \int_t^{t + \tau} \sigma^2(s)ds \). From (16), we see \( \int_t^{t + \tau} \sigma^2(s)ds = Y^{d+1}(t + \tau) - Y^{d+1}(t) \).

Since \( \frac{1}{2} < d + 1 < \frac{3}{2} \), the integrated volatility exists as long as \( Y \in L^2[0, T] \). We also note that comparing the characteristic functions of BNS model (6) and the fractional BNS model (15), we can see that the fractional model is inherently different from the BNS model in that it is non-Markovian. As a result, the characteristic function will be dependent on the history of the volatility through the term

\[
\frac{1}{2} u(i + u) \frac{1}{\Gamma(d + 1)} \int_0^{t + \tau} \left[ (t + \tau - s)^d - (t - s)^d \right] Y(s) ds.
\]

When \( d = 0 \), this term will disappear, then the characteristic function will be only dependent on the current volatility.

For \( t = 0 \), the conditional characteristic function \( \phi_{X(t \mid t)}(u) \) of \( X(t + \tau) \) for the fractional BNS model becomes

\[
\phi_{X(t \mid t)}(u) = \exp \left[ iu \tau \left( r - q - \lambda \psi_{L(t)}(-ip) \right) \right.

\left. + \lambda \int_0^\tau \psi_{L(t)} \left( -\frac{1}{2} u(1-iu)E_{c,-w}(d+1,-\lambda) + up \bigg] dw \right. \right.

\left. + iuX(0) - \frac{1}{2} u(i + u)E_{c,0}(d+1,-\lambda)Y(0) \bigg].
\]

We will use this to estimate the parameters for the fractional BNS model from the observed option prices.
5 Numerical example

In this section, we numerically study an option pricing model driven by a specific type of fractional Lévy process. We choose fractional gamma-OU process. More specifically, we assume $Y$ in (13) has marginal gamma($a$, $b$) law. The characteristic function is given by

$$\phi_{Y(t)}(u) = \left(1 - \frac{iu}{b}\right)^{-a} .$$

Then from (3), we know the characteristic exponent of corresponding BDLP $L$ is given by

$$\psi_{L(t)}(u) = \frac{iau}{b - iu} .$$

From this we can easily derive (see Schoutens, 2003) that $L(t)$ is a compound Poisson process, i.e.,

$$L(t) = \sum_{n=1}^{N} Z_n ,$$

where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity parameter $a$, i.e., $E[N(t)] = at$ and $\{Z_n\}_{n=N}$ is an independent and identically distributed sequence; each $Z_n$ follows a gamma(1, $b$) law.

To simulate $Y$ at the time points $t = n\Delta t$, $n = 0, 1, 2, \ldots$, first simulate at the same time points a Poisson process $N$ with intensity $a\lambda$, then

$$Y_{n\Delta t} = (1 - \Delta t) Y_{(n-1)\Delta t} + \sum_{n=N_{(n-1)\Delta t}}^{N_{n\Delta t}} Z_n .$$

We can simulate $\sigma^2$ based on the Grünwald-Letnikov’s method (Oldham and Spanier, 1974) to approximate fractional integreation/derivative:

$$\sigma_{n\Delta t}^2 = (\Delta t)^{-d} \sum_{j=0}^{N_{n\Delta t}} c_j^{(q)} Y_{(n-j)\Delta t} ,$$

where

$$c_0^{(q)} = 1, \quad c_j^{(q)} = \left(1 - \frac{1+q}{j}\right) c_{j+1}^{(q)} .$$

Figure 1 shows a single simulated path of $\sigma^2(t)$ for $d = -0.2$, $d = 0$ and $d = 0.2$ respectively. For the underlying gamma-OU process $Y$, we have used the parameters $\lambda = 10$, $a = 10$, $b = 100$ and $Y(0) = 0.08$ (same parameters are used in the simulation study in Schoutens, 2003). We find that the integration parameter $d$ influences the smoothness of the volatility process. The greater $d$ is, the smoother the path of $\sigma^2$ is.
Figure 1  Simulation of fractional gamma-OU process for different integration parameters

Notes: \( \sigma^2(t) = Y(t), \quad Y(t) = \int_0^t \frac{(t-s)^d}{\Gamma(1+d)} dY(s) \) and \( dY(t) = -\lambda Y(t)dt + dL(t) \). \( \lambda = 10, a = 10, b = 100 \) and \( Y(0) = 0.08 \).

We also investigate the influence of parameter \( d \) on the option prices. In the fractional gamma-OU model, the mean of volatility process \( \sigma^2(t) \) will depend on both \( d \) and \( t \):

\[
E(\sigma^2(t)) = \frac{a}{b} \frac{t^d}{\Gamma(d+1)},
\]

whereas in the gamma-OU model, the mean will be \( \frac{a}{b} \). Hence, we expect that the option prices from the fractional model and gamma-OU model will be different for different \( d \) and maturities. We compare the option prices from the fractional gamma-OU model with \( d = -0.2 \) and \( d = 0.2 \) with the gamma-OU model. In Figure 2, we plot the price differences between the fractional models and gamma-OU model across moneyness and maturity. It seems that for short-term options, the fractional model with \( d = -0.2 \) produces higher option prices compared to the standard model across the moneyness, whereas the model with \( d = 0.2 \) has lower prices. When maturities increase, the option prices from the model with \( d = -0.2 \) decrease relative to the standard model while the relative prices from the model with \( d = 0.2 \) increase. For long-term options, the fractional model with \( d = -0.2 \) produces lower option prices compared to the standard model across
the moneyness, whereas the model with \( d = 0.2 \) has higher prices. From this figure, we can clearly see that the memory parameter \( d \) has different impacts on the option prices for different maturities and strike prices.

**Figure 2** Option prices from the fractional gamma-OU model with different integration parameters minus that from the BNS model, (a) \( T - t = 0.5 \) (b) \( T - t = 1.0 \) (c) \( T - t = 1.5 \) (d) \( T - t = 2.0 \)

Notes: \( dx(t) = \left( r - q - \frac{1}{2} \sigma^2(t) - \lambda Y(t)(-i\rho) \right) dt + \sigma(t) dB(t) + \rho dL(\lambda t), \quad \sigma^2(t) = Y(t), \)

\[ Y(t) = \frac{1}{\Gamma(1 + d)} \int_0^t (t - s)^d dY(s) \quad \text{and} \quad dY(t) = -\lambda Y(t) dt + dL(\lambda t). \]

\( r = 2\% \), \( q = 0 \), \( \rho = -0.5 \), \( \lambda = 10 \), \( a = 10 \), \( b = 100 \), \( \lambda(0) = \log(1,000) \) and \( Y(0) = 0.08 \).
6 Calibration example

In this section, we calibrate the option prices resulting from the fractional gamma-OU model to a set of observed option prices. To compare the different models and obtain the parameters for the models, we minimise the average relative percentage error (ARPE) between the model prices and market prices, that is

\[
\min_{\theta} \frac{1}{n} \sum_{i=1}^{n} \frac{C_{i,\text{model}}(0; \theta) - C_{i,\text{market}}(0)}{C_{i,\text{market}}(0)},
\]

where \(C_{i,\text{market}}(0)\) and \(C_{i,\text{model}}(0; \theta)\) are the market and model prices at time 0 respectively. \(C_{i,\text{model}}(0; \theta)\) can be computed from (7), (8), (9) and (22).

For comparative purposes, we also compute some other measures of fit such as the root mean square root error (RMSE), the average absolute error as a percentage of the mean price (APE) and the average absolute error (AAE):

\[
\text{RMSE} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left( C_{i,\text{model}}(0; \theta) - C_{i,\text{market}}(0) \right)^2},
\]

\[
\text{APE} = \frac{\sum_{i=1}^{n} \left| C_{i,\text{model}}(0; \theta) - C_{i,\text{market}}(0) \right|}{\sum_{i=1}^{n} C_{i,\text{market}}(0)},
\]

\[
\text{AAE} = \frac{1}{n} \sum_{i=1}^{n} \left| C_{i,\text{model}}(0; \theta) - C_{i,\text{market}}(0) \right|.
\]

For calibration, we use the data set in Schoutens (2003). The data set consists of 77 call option prices on the S&P 500 index at the close of the market on 18 April 2002. On that day, the S&P 500 index closed at 1,124.47. We also have values of \(r = 1.9\%\) and \(q = 1.2\%\) per year. We note that the objective function in (23) is neither convex nor of any particular structure. It may have more than one global minimum and it is not possible to tell whether a unique minimum can be reached by a search algorithm. Our calibration is to do grid search on the long memory parameter \(d\). For a fixed \(d\), we minimise the objective function using the global optimisation algorithm. Finally, we choose values of \(d\) and other parameters that correspond to the smallest objective function. To approximate the Fourier transform that is needed to calculate the characteristic functions, we numerically evaluate the integrals using adaptive Simpson’s rule.

Table 1 provides the parameter estimates for both fractional gamma-OU and gamma-OU models. We find the memory parameter \(d\) is −0.0919, which indicates that the volatility process is a short memory process. Furthermore, we notice that when we include memory parameter into the gamma-OU model, the mean reversion parameter \(\lambda\) becomes much smaller, dropping from 0.8935 to 0.1069. It is due to the fact that when \(d < 0\), the fractional process itself can produce mean reverting effect.
Table 1  Parameter estimation

<table>
<thead>
<tr>
<th>Model</th>
<th>$\rho$</th>
<th>$\lambda$</th>
<th>$a$</th>
<th>$B$</th>
<th>$\sigma^2(0)$</th>
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<tbody>
<tr>
<td>Gamma-OU</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fractional gamma-OU</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td>$\lambda$</td>
<td>$a$</td>
<td>$b$</td>
<td>$Y(0)$</td>
<td>$d$</td>
</tr>
<tr>
<td>------------------</td>
<td>--------</td>
<td>------</td>
<td>------</td>
<td>--------</td>
<td>------</td>
</tr>
<tr>
<td>$-1.9309$</td>
<td>$0.8935$</td>
<td>$0.8600$</td>
<td>$16.7073$</td>
<td>$0.0167$</td>
<td></td>
</tr>
<tr>
<td>$-3.9241$</td>
<td>$0.1069$</td>
<td>$5.6695$</td>
<td>$26.2400$</td>
<td>$0.0128$</td>
<td>$-0.0919$</td>
</tr>
</tbody>
</table>

Table 2  APE, AAE, RMSE and ARPE for BNS and fractional BNS models

<table>
<thead>
<tr>
<th>Model</th>
<th>APE (%)</th>
<th>AAE</th>
<th>RMSE</th>
<th>ARPE (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gamma-OU</td>
<td>1.92</td>
<td>1.1859</td>
<td>2.0138</td>
<td>1.81</td>
</tr>
<tr>
<td>Fractional gamma-OU</td>
<td>1.77</td>
<td>1.0970</td>
<td>1.8129</td>
<td>1.58</td>
</tr>
</tbody>
</table>

Figure 3  Gamma-OU and fractional gamma-OU calibration of S&P 500 options, (a) gamma-OU (b) fractional gamma-OU

Notes: $r = 1.9\%$, $q = 1.2\%$, $X(0) = \log(1124.47)$. The other parameters are obtained from Table 1.
Table 2 compares the fractional model with the gamma-OU model in terms of different measures of fit. It is clear that the fractional model significantly improves the performance of the standard model. Figure 3 plots the option prices versus the strike prices for both the market prices and the model prices. Again, we can see that the fractional model yields an improvement on the standard model. We want to stress that this is a first look at the empirical performance of the fractional BNS model, and a more thorough study is beyond the scope of the paper.

7 Conclusions

In this paper we have developed a method for pricing options under the FSV model. Although there has been ample research on option pricing with short memory stochastic volatility, there is much less research on option pricing under a FSV framework. We have proposed the fractional BNS model which is an extension of the popular BNS model. We propose to use Fourier inversion techniques to obtain the closed-form solutions for option pricing.

We numerically study the effects of memory parameter on the option prices. We also estimate the parameters of the proposed model from the observed option data. We show that the volatility process is a short memory process instead of long memory process. Furthermore, the fractional model significantly improves the performance of the standard BNS model for option pricing.

Our goal in this paper is to study option pricing with the fractional Levy-driven stochastic volatility models and we only provide a first empirical application. In the future, we would like to evaluate the performance of our models by a comprehensive empirical study. In addition, we also want to study the asymptotic behaviour the fractional BNS models.

References


