Compressive sensing multi-target diffusive source localisation using sparse recovery algorithms in sensor networks

Yong Zhang, Zhi Yan, Qi Chen, Teng Fei and Liyi Zhang*

Information Engineering College, Tianjin University of Commerce, Tianjin, 300134, China
Email: zhangyong@tjcu.edu.cn
Email: yanzhi1974@sina.com
Email: chq687002@163.com
Email: feiteng@tjcu.edu.cn
Email: zhangliyi@tjcu.edu.cn
*Corresponding author

Abstract: According to the multi-target diffusive source localisation in sensor networks, a compressive sensing sparse recovery algorithm was proposed for the mismatching problem of the target sources sparsity and the high-dimensional redundant sampling signals. Firstly, the compressive sensing system model and the related terms were given and explained. Then, the joint optimal estimation of the sparse diffusive source vector and the diffusion distribution state were realised with the variational Bayesian expectation maximisation algorithm (VB-EM). In which, the dynamic compressive sensing dictionary model of the real target source sparse representation was designed and adjusted with the grid division parameters optimisation for the dictionary mismatch problem solving. Finally, the simulation results show that the proposed compressive sensing method with VB-EM algorithm could effectively achieve the diffusive source parameters estimation and its diffusion distribution state prediction. Compared with the traditional compressive sensing sparse recovery algorithms, it could obtain higher robustness performance for the rapid and accurate localisation in complex environment.

Keywords: sensor network; compressive sensing; variational Bayesian expectation maximisation; diffusive source localisation.


Biographical notes: Yong Zhang is currently an Associate Professor of Information Engineering College of Tianjin University of Commerce. He received his PhD in Tianjin University, China, in 2012, MS in Yanshan University, China, in 2005 and BS in Liaoning Shihua University, China, in 2000. His current research interests include wireless sensor networks, gas source detection and localisation, intelligent detection and information processing theory.

Zhi Yan is a Lecturer of Information Engineering College of Tianjin University of Commerce. His research interests are intelligent detection and automatic control theory.

Qi Chen is an Associate Professor of Information Engineering College of Tianjin University of Commerce. She received her PhD in Tianjin University, China, in 2008. Her research interests are intelligent detection and information processing theory.

Teng Fei is a Senior Experimentalist of Information Engineering College of Tianjin University of Commerce. She received her PhD in Tianjin University, China, in 2016. Her research interests are signal detection and information processing.

Liyi Zhang is currently a Professor of Information Engineering College of Tianjin University of Commerce. He received his PhD in Beijing Institute of Technology, China, in 2003. His research interests are in the fields of signal detection and processing, and medical image processing.
1 Introduction

With the innovation of wireless communication network and embedded technology, wireless sensor networks has been rapidly developed and widely used in the field of multi-target source localisation, such as sensor network node localisation (Cheng et al., 2012; Huang et al., 2015; Kotwal et al., 2019; Moghaddamee et al., 2020), sound source localisation (Alexandridis and Mouchtaris, 2017; Lee, 2019), water pollution source localisation (Luo et al., 2014), gas diffusive source localisation (Zhao and Nehorai, 2007) and so on. It is the key point to determine the number and location of target sources accurately and quickly, which usually need a higher sampling rate of sensor node and process massive redundant signals in a shorter time period. In sensor networks, the energy, computing and communication capabilities and the signals sampling rate of sensor node was limited, it usually could not complete the procession and transmission of a large number of redundant high-dimensional signals directly and often needed to be implemented in a fusion center with the unlimited energy and relatively computing power.

In the real multi-target source localisation application, the number of targets is usually limited in the monitoring field with a certain sparse property. So, there is a great conflict between the sparsity of the target sources and the high-dimensional redundant sampling signals or data to achieve a fast and accurate performance, which has been a hot and significant research issue in the signal processing field. Recently, the compressive sensing (CS) theory provides a new perspective to effectively solve this mismatch or conflict for the sparse target sources localisation problem in sensor network (Xu et al., 2015).

With dividing the monitoring field into amounts of grids, the sparsity of grids within target sources provides a possibility for applying the compressive sensing theory for the multi-target source localisation problem. In which, the original signals could be undersampled by a sparse compressed sensing observation matrix with low sampling rate, and then the original sparse signals vector of target sources could be reconstructed with a high probability by different sparse recovery algorithms in the fusion center. It could not only effectively solve the limited resources problem of sensor nodes, but also achieve a better performance and higher accuracy than the traditional localisation methods. Therefore, it is widely used in the research of different source localisation in sensor networks.

Cevher firstly proposed a compressive sensing multi-target source localisation framework in sensor networks (Cevher et al., 2008, 2009). In which, the sparse target source vector was reconstructed with few amounts of observations based on a suitable compressive sensing matrix. But each sensor node needs to maintain a sparse representation dictionary and the cost of the sparse recovery algorithm was large. Feng proposed an underdetermined equation method for the compressive sensing target source localisation problem, and analysed the performance with the Basis pursuit (BP) and basis pursuit denoising (BPDN) sparse recovery algorithms (Feng et al., 2009). The main problem of this method was that the number of target sources should be known at first. Zhang designed a greedy matching pursuit (GMP) algorithm for the multiple sources localisation problem, in which, the number of multi-target sources and target position vector estimation could be realised simultaneously without the target sparsity (Zhang et al., 2011). In addition, the restricted isometry property (RIP) of the compressive sensing matrix (Candes and Wakin, 2008) was proved and the simulation result showed that the proposed method could improve the sparse recovery accuracy of the multi-target source vector. He improved the GMP algorithm with a greedy matching residual optimisation method and implement the K-sparse multi-target source vector reconstruction with a higher accuracy (He et al., 2012). Wang proposed an orthogonal matching pursuit (OMP) algorithm for the multi-target vector recovery, in which, the position vector and the number of the multi-target sources with an unknown sparsity (Ju et al., 2017).

The above compressive sensing multi-target source localisation methods were usually with fixed grids dividing method, in which the monitoring area was first divided into many fixed grids, and assume that the targets were localised in few fixed grids with unchanged positions. So, the compressive sensing matrix or dictionary is usually static and the source localisation problem which could be seen as a sparse vector recovery problem, it was based on a static dictionary with the traditional GMP sparse recovery algorithms, and in which the sparsity of the targets was usually required as a priori condition. However, in the real application, the environment and the multi-target sources are usually dynamic changed. There will be a dictionary mismatch problem between the static dictionary and the real sparse representation dictionary, which will seriously affect the localisation performance. In addition, the performance also depends on the size or density of the division grids. With increasing the grids number or division density, the complexity of the sparse recovery algorithm or the compressive sensing matrix would also be higher.

In order to solve this problem, You proposed an adaptive sparse grids division method using Bayesian learning algorithm for the multi-target source localisation (You et al., 2018). By adaptively adjusting the grid division size, the algorithm could get a higher positioning accuracy than the grid fixed division method. Yan proposed a two-step target localisation algorithm: First of all, the detection area was divided into some grids with large size and at the same time, the optimal estimation method is used for the target source prediction. Then, the initial candidate target grids were divided again and the target vector was reconstructed within a small sub-grid (Yan et al., 2017). Lin improved the target vector recovery accuracy with two kinds of grids dividing methods, which was implemented with a grids division optimisation method considering the structure of the sensor network (Lin et al., 2017).

In the multi-target source localisation application in sensor networks, the system model and the target sources parameters were usually unknown, which should be
reconstructed and estimated at the same time. It was usually defined as a joint optimal estimation problem and could be solved with the EM and variational Bayesian (VB) inference algorithms (Tzikas et al., 2008). The EM method was a special form of the variational Bayesian inference algorithm, which was usually applied for the lower dimension of target parameters, and the variational Bayesian inference method was suitable for dealing with the complex state system estimation problems with higher dimension target parameters (Blei et al., 2017). Yu proposed a variational expectation maximisation algorithm to achieve dictionary adaptation and target parameters estimation according to the dictionary mismatch problem caused by the dynamic changes of environment in the compressive sensing target source recovery problem (Yu et al., 2016). Sun proposed an optimal estimation method for the multi-target source location problem, in which, the estimation of the target sources parameters and diffusion distribution state reconstruction were based on the variational EM algorithm (Sun et al., 2017). But most of the compressive sensing sparse recovery algorithms could not effectively solve the problem of data redundancy and computational complexity, because the structure information of sensor networks and prior information of system model are ignored or not fully utilised.

In this paper, a compressive sensing diffusive target source localisation method was proposed based on the variational Bayesian expectation maximisation (VB-EM) sparse recovery algorithm, which was aimed at solving the joint optimisation of target source parameters estimation and the diffusion distribution state reconstruction in the dynamic environment. The structure of this paper was as follows: firstly, the compressive sensing system framework was given. Then, the optimal estimation of target source parameters and diffusion distribution reconstruction were implemented based on the VB-EM algorithm, in which, the state prior probability function was based on a Gaussian mixture model. At the same time, a dynamic dictionary model of the real target source sparse representation was designed with the grid division parameters, which was continuously optimised according to the joint optimal estimation results to solve the problem of dictionary mismatch. Finally, the simulation was given for the proposed algorithm.

2 Compressive sensing system model of multi-target localisation

It was assumed that there are \( K \) targets were randomly deployed in sensor networks with \( M \) sensor nodes. The monitoring field was divided into \( N \) grids, and the target position could be described with the grid points. Therefore, the problem of target source location could be indirectly transformed into a sparse estimation problem of its grid, and the state of target diffusion distribution should be also reconstructed at the same time. According to the compressive sensing theory, if \( K < M \ll N \), the \( K \) target sources vector could be seen with \( K \) sparsity according to the \( N \) grids monitoring area. It is a sparse recovery problem of the \( K \) sparsity targets source vector in the \( N \) dimensional space based on the \( M \) dimensional noise-containing sensor nodes observations.

\[
Y = \Phi\Psi\mathbf{x} + W = AX + W \tag{1}
\]

where \( A = \Phi\Psi \) was the compressive sensing matrix, \( W \) was the measurement Gaussian noise vector. The corresponding terms in equation (1) were described as follows:

(1) Target source sparse vector \( \mathbf{X} \in \mathbb{R}^N \)

\[X = (x_1, x_2, \ldots, x_N)^T \text{ was a } K \text{ sparsity vector with } N \times 1 \text{ dimension, if there was a target in the } n \text{ th grid, the element } x_n = 1; \text{ Otherwise } x_n = 0. \text{ So, } K \text{ targets in the monitoring area, there would be only } K \text{ elements were } 1 \text{ and the rest elements were } 0 \text{ in the } \mathbf{X} \text{ vector.}

(2) Sparse basis matrix \( \Psi \in \mathbb{R}^{N \times N} \)

Defined \( \Psi = (\psi_1, \psi_2, \ldots, \psi_N) \) as a \( N \times N \) sparse matrix, each column \( \psi_j = (\psi_{ij})_N \), \( i, j = 1, 2, \ldots, N \) of the matrix \( \Psi \) represented the diffusive substance concentration signals vector of the target at the \( j \) th grid measured by all the \( N \) grids. Suppose the signals of the \( j \) th target measured by the sensor node at the \( i \) th grid conform to the following diffusion model:

\[
c_i(r_i, q_j, k_i) = \frac{q_j}{4\pi k_i |r_i - r_j|} \text{erfc}\left(\frac{r_i - r_j}{2\sqrt{k_i(t - t_0)}}\right) \tag{2}
\]

where \( c_i(r_i, q_j, k_i) \) was the diffusive substance concentration measured by the sensor node with known position \( r_i \), \( q_j \) was the release rate, \( k_i \) was the environmental diffusion coefficient, \( d_{ij} = |r_i - r_j| \) was the Euclidean distance between sensor node grid \( r_i \) and target source grid \( r_j \), \( t_0 \) was the release start time, \( t \) was the release duration, and \( \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2} \text{d}y \) was the error compensation function. Obviously, the sparse basis matrix consist with \( \psi_{ij} = c_{ij} \) was different from the traditional sparse basis matrix in the OMP algorithms.

(3) Measurement matrix \( \Phi \in \mathbb{R}^{M \times N} \)

The measurement matrix \( \Phi \) represented the \( M \) \( (M \ll N) \) sensor nodes deployment or node scheduling activation scheme with known position grids. Generally, the measurement matrix could be defined as:

\[
\Phi = I_{M \times N} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}_{M \times N} \tag{3}
\]

(4) Compressive sensing matrix \( A \in \mathbb{R}^{M \times N} \)

As compressive sensing matrix, \( A = \Phi\Psi \) was usually defined as an over-complete dictionary, it could be described as:
Compressive sensing multi-target diffusive source localisation using sparse recovery algorithms

$$\mathbf{A}(\boldsymbol{\Theta}) = \begin{pmatrix}
  a_{i1}(\boldsymbol{\Theta}) & a_{i2}(\boldsymbol{\Theta}) & \cdots & a_{iN}(\boldsymbol{\Theta}) \\
  a_{21}(\boldsymbol{\Theta}) & a_{22}(\boldsymbol{\Theta}) & \cdots & a_{2N}(\boldsymbol{\Theta}) \\
  \vdots & \vdots & & \vdots \\
  a_{M1}(\boldsymbol{\Theta}) & a_{M2}(\boldsymbol{\Theta}) & \cdots & a_{MN}(\boldsymbol{\Theta})
\end{pmatrix}_{M \times N}$$

where $\boldsymbol{\Theta} = (r_i, q_i, k_i)$ was defined as the diffusive source parameters vector of its diffusion distribution, $a_{mj}(\boldsymbol{\Theta})$ was the signals that could be observed by the $i$th sensor node when the target source in the $j$th grid.

In this paper, the compressive sensing matrix $\mathbf{A}(\boldsymbol{\Theta})$ was generated indirectly according to the diffusion distribution of the diffusive target source, so $\mathbf{A}(\boldsymbol{\Theta})$ could be used to describe the real-time diffusion distribution state indirectly.

(5) Measurements vector $\mathbf{Y} \in \mathbb{R}^M$

$$\mathbf{Y} = (y_1, y_2, \ldots, y_M)^T$$

was the $M \times 1$ dimensional vector with the element $y_i$, which was the signals superposition of multiple target sources measured by the $m$th sensor node. The model of compressive sensing $\mathbf{Y} = \mathbf{A} \mathbf{X}$ could be described as follows:

$$\begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_M
\end{pmatrix} =
\begin{pmatrix}
a_{i1}(\boldsymbol{\Theta}) & a_{i2}(\boldsymbol{\Theta}) & \cdots & a_{iN}(\boldsymbol{\Theta}) \\
a_{21}(\boldsymbol{\Theta}) & a_{22}(\boldsymbol{\Theta}) & \cdots & a_{2N}(\boldsymbol{\Theta}) \\
\vdots & \vdots & & \vdots \\
a_{M1}(\boldsymbol{\Theta}) & a_{M2}(\boldsymbol{\Theta}) & \cdots & a_{MN}(\boldsymbol{\Theta})
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_M
\end{pmatrix}$$

The problem of multi-target source localisation in sensor networks could be transformed into a compressive sensing sparse recovery problem of the $N$ sparse target source vector with $M$ measurements vector, which could be solved by the $l_1$-norm optimisation algorithm. The parameters approximate estimation of the target source could be obtained from the compressive sensing matrix $\mathbf{A}(\boldsymbol{\Theta})$, which usually be formed as a finite discrete over complete dictionary with target grids. And the compressive sensing matrix $\mathbf{A}(\boldsymbol{\Theta})$ also contained the parameters information of the diffusion distribution state in the environment space. So, the diffusion distribution reconstruction or target source parameters $\boldsymbol{\Theta} = (r_i, q_i, k_i)$ estimation could be realised by the sparse recovery of the dictionary $\mathbf{A}(\boldsymbol{\Theta})$.

As for the compressive sensing sparse recovery problem, the composition of $\mathbf{A} \in \mathbb{R}^{M \times N}$ would be determined with the grids division size or density, and the maximum target number $K$ could be reconstructed with a satisfied RIP constraint condition. In order to reconstruct the target vector $\mathbf{X}$, the number of sensors usually needed to be satisfied with $M \geq O(K \lg (N/K))$.

3 Variational Bayesian expectation maximisation algorithm

In the multi-target localisation problem, it was usually necessary to determine the unknown parameters of source position, release rate, diffusion coefficient and reconstruct the diffusion distribution state at the same time. In this paper, a variational Bayesian expectation maximisation (VB-EM) algorithm was proposed for the joint optimal estimation problem based on the Gaussian Mixture prior model.

3.1 Multi-dimensional mixture Gaussian prior distribution

According to the compressive sensing system model of multi-target localisation problem, the number of grids $N$ was far greater than the number of targets $K$, and the targets vector $\mathbf{X}$ had $K$ sparsity. Suppose the noise vector $\mathbf{W}$ in the measurement model (1) with a Gaussian distribution of variance $\lambda = \sigma^2$:

$$p(\mathbf{W}) = N(0, \lambda I)$$

Then, the likelihood function of $\mathbf{Y}$ was:

$$p(\mathbf{Y} | \mathbf{X}, \mathbf{W}) = \left(\frac{\lambda}{2\pi}\right)^{\frac{M}{2}} \exp \left( -\frac{1}{2\lambda} \| \mathbf{Y} - \mathbf{AX} \|_2^2 \right)$$

Considering the sparsity of $\mathbf{X} \in \mathbb{R}^N$, it was assumed to be a vector with multi-dimensional mixture Gaussian distribution, and its prior distribution was as follows:

$$p(\mathbf{X} | \mathbf{A}) = \prod_{i=1}^{N} N(0, \gamma_i \mathbf{A})$$

$$= \prod_{i=1}^{N} \left( \left( \gamma_i \right)^{-1} \exp \frac{-\mathbf{X}^T \text{diag}(\gamma_i \mathbf{A}) \mathbf{X}}{2} \right)$$

where, $\gamma_i$ was a nonnegative parameter for the variance characteristic and represented the statistical characteristic of the sparse target source vectors. $\mathbf{A} = (\gamma_i \mathbf{A})_{i \times i}$ was the covariance matrix of sparse target vector $\mathbf{X}$, it was a positive definite matrix with symmetry property and mainly used to represent the correlation structure of sparse source $\mathbf{X}$ and the compressive sensing matrix $\mathbf{A}(\boldsymbol{\Theta})$. $\text{det}(\gamma_i \mathbf{A})$ was a determinant of matrix, in which, $\mathbf{A}$ could be also regarded as the hidden random variables in the system.

The conditional probability of $\mathbf{X}$ could be informed from Bayesian theorem:

$$p(\mathbf{X} | \mathbf{Y}, \lambda, \gamma_i, \mathbf{A}) = \frac{p(\mathbf{Y} | \mathbf{X}, \lambda) p(\mathbf{X} | \gamma_i, \mathbf{A})}{p(\mathbf{Y} | \lambda, \gamma_i, \mathbf{A})}$$

$$= \left( \frac{\lambda}{2\pi} \right)^{\frac{M}{2}} \det(\Sigma)^{-\frac{1}{2}} \exp \left( -\frac{(\mathbf{X} - \mu)^T (\Sigma)^{-1} (\mathbf{X} - \mu)}{2} \right)$$

The mean and covariance were as follows:

$$\mu = (\lambda I^{-1} + \mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{Y}$$

$$= \mathbf{I} \mathbf{A}^T (\lambda I + \mathbf{A} \mathbf{A}^T)^{-1} \mathbf{Y}$$
\[ \Sigma = \left( \Gamma^{-1} + \frac{1}{\lambda} A^T A \right)^{-1} \]  
\[ = \Gamma - \Gamma A^T \left( \lambda I + A \Gamma A^T \right)^{-1} A \Gamma \]

where, \( \Gamma = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_N, \lambda) \) and the sparsity of \( X \) was determined by \( \gamma_i \) in \( \Gamma \). If \( \gamma_i = 0 \), then \( \chi_i = 0 \), and the corresponding \( i \) column in the perception matrix \( A(\theta) \) would be deleted in the Bayesian inference algorithm. If the parameters \( \lambda \) and \( \Gamma \) were accurately estimated, the estimation of \( X \) could be obtained with the posterior probability function.

### 3.2 Variational Bayesian inference

Assuming \( A \) was the vector of the system hidden parameter, the noise vector was \( W \) and \( Y \) was the observation vector, as for the variational Bayesian inference algorithm, it has two steps: one was the selection of the system state model with the solution of the likelihood function \( p(Y|A,W) \), and the other was to get the posterior probability distribution \( p(A|Y,W) \) for the unknown parameters prediction.

The variational Bayesian inference could be as follows:

\[
\ln p(Y|W) = L(q(A,W)) + KL(q(A) \parallel p(A|Y,W)) 
\]

where, the \( L(q(A,W)) \) represented the lower variation confidence bound of variational Bayesian inference, and \( KL(q(A) \parallel p(A|Y,W)) \) was the differences asymmetric quantity of \( p(A|Y,W) \) and \( q(A) \), which was named as KL divergence.

Both of \( L(q(A,W)) \) and \( KL(q(A) \parallel p(A|Y,W)) \) could be described respectively as follow:

\[
L(q(A,W)) = \ln \int q(A) p(Y,A|W) \frac{q(A)}{q(A)} dA 
\]

\[
KL(q|p) = -\ln \int q(A) p(Y,A|W) \frac{q(A)}{q(A)} dA 
\]

When the KL divergence was zero, the \( L(q(A,W)) \) could get the maximum value for any variation distribution of \( q(A,W) \) and the variation distribution \( q(A,W) \) would be equivalent to the posterior distribution \( p(A|Y,W) \).

### 3.3 Diffusion distribution state reconstruction and parameters estimation of diffusive sources

As for the multiple diffusive sources localisation in this paper, the estimation of the hidden parameters vector \( A \) and the noise vector \( W \) needed to be completed with the known observation vector \( Y \), in which, the hidden variable vector \( A \) was contained with unknown parameters and the state diffusion distribution.

\[
p(A|Y,W) = q(A,W) 
\]

The maximum value of \( L(q(A,W)) \) would be obtained by iterative updating of \( q(A,W) \) with the variational Bayesian inference. By introducing the average field hypothesis, the decomposition of \( q(A,W) \) could be replaced by:

\[
q(A,W) = \sum_{i=1}^{N} q(\gamma_i,A,\lambda) = \sum_{i=1}^{N} q(\gamma_i) q(A_i) q(\lambda) 
\]

where, the global optimisation could be get with each \( q(\gamma_i,A,\lambda) \).

The corresponding general solution could be get with the partial derivation of \( q(\gamma_i,A,\lambda) \).

\[
q(\gamma_i,A,\lambda) = \frac{\exp\left(E_{\gamma_i}[\ln p(Y_i|\gamma_i,A,\lambda)]\right)}{\prod_{i=1}^{N} \exp\left(E_{\gamma_i}[\ln p(Y_i|\gamma_i,A,\lambda)]\right)} d\gamma_i dA d\lambda 
\]

where, \( E_{\gamma_i}[\cdot] \) was an expectation function, and it could be get from equation (17) with a logarithmic operation:

\[
\ln q(\gamma_i,A,\lambda) = E_{\gamma_i}[\ln p(Y_i|\gamma_i,A,\lambda)] 
\]

The VB-EM algorithm has two steps. In the VB-E step, the posterior probability distribution would be approximately estimated with the expectation operation \( \ln q(\gamma_i,A,\lambda) \) according to equation (18), and in VB-M step, the parameters \( \lambda \) and \( \gamma_i \) would be updated for maximising the expectations of \( \ln q(\gamma_i,A,\lambda) \), and determined whether the variation lower-bound reached the pre-set threshold value.

**VB-E step:** firstly update \( q(\gamma_i,A,\lambda) \), and it was described as:

\[
\ln q(\gamma_i,A,\lambda) = E_{\gamma_i}[\ln p(Y_i|\gamma_i,A,\lambda)] 
\]

\[
\approx -\frac{1}{2} (\Gamma_i - u_i)^T \Sigma_i^{-1} (\Gamma_i - u_i) 
\]

Therefore, \( q(\gamma_i,A) \) was a Gaussian distribution with mean \( u_i \) and variance \( \Sigma_i \), and the mean \( u_i \) and variance \( \Sigma_i \) could be respectively expressed as:

\[
\Sigma_i = \left( \frac{1}{\lambda} A^T A + \Gamma_i^{-1} \right)^{-1} 
\]

\[
= \Gamma_i - \Gamma_i A^T \left( \lambda + A \Gamma_i A^T \right)^{-1} A \Gamma_i \]

\[
u_i = \left( \lambda \Gamma_i^{-1} + A^T A \right)^{-1} A^T Y_i 
\]

\[
= \Gamma_i A^T \left( \lambda + A \Gamma_i A^T \right)^{-1} Y_i 
\]
The $\gamma, A$ could be updated and estimated approximately as follows:

$$\begin{align*}
\hat{\gamma}, \hat{A} &= \arg \max E_{\gamma, A}[\ln p(\gamma, A | Y)]_{\hat{\gamma}, \hat{A}} \\
&\propto -\ln \left( \prod_{i=1}^{N} |A_i| \right) - \frac{1}{2} tr \left( \Sigma_i + u_i u_i^T \right) \\
&\propto -\frac{1}{2} \sum_{i=1}^{N} \ln(\gamma_i) - \frac{1}{2} \sum_{i=1}^{N} \ln |A_i| - \frac{1}{2} tr \left( \Sigma_i + u_i u_i^T \right)
\end{align*}$$

Then:

$$\begin{align*}
\hat{\gamma} &= tr \left( A^T (\Sigma_i + u_i u_i^T) \right) \\
\hat{A} &= \frac{1}{\gamma} (\Sigma_i + u_i u_i^T)
\end{align*}$$

Secondly, $\lambda$ could be approximately iterative estimated and updated as follows:

The posterior probability distribution of $W$ could be get as follow:

$$\begin{align*}
p(\lambda | Y, \gamma, A) &= \frac{p(Y, \gamma, A) p(\lambda)}{\int p(Y, \gamma, A) p(\lambda) d\lambda} \\
&\propto \int p(Y, \gamma, A | \lambda) d\lambda \\
&= \frac{1}{\sqrt{2\pi\lambda}} \exp \left( -\frac{(Y - \gamma A)^2}{2\lambda} \right) d\lambda
\end{align*}$$

Assumed $u_i = Y - \gamma A A_i$, it could be get:

$$\int p(Y, \gamma, A | \lambda) d\lambda = cdf(u_i)$$

where, $cdf(•)$ represented the cumulative distribution function of standard normal distribution.

$$cdf(u_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u_i} \exp \left( -\frac{u_i^2}{2} \right) du_i$$

and

$$p(\lambda | Y, \gamma, A) = \frac{1}{cdf(u_i)} p(Y, \gamma, A | \lambda)$$

$$E_{\gamma, A}[\ln p(\lambda | Y, \gamma, A)]_{\hat{\gamma}, \hat{A}} = \frac{1}{cdf(u_i)} \int \lambda p(Y, \gamma, A | \lambda) p(\lambda) d\lambda$$

Then:

$$L(q(\lambda)) = E_{\gamma, A}[\ln p(\lambda | Y, \gamma, A)]_{\hat{\gamma}, \hat{A}}$$

$$\propto \left\{ -\frac{1}{2\lambda} |Y - \gamma A A_i| \right\}$$

The $\lambda$ could be approximately estimated and updated as follows:

$$\hat{\lambda} = \arg \max E_{\gamma, A}[\ln p(\lambda | Y, \gamma, A)]_{\hat{\gamma}, \hat{A}}$$

$$\propto -\frac{M}{2} \ln(\hat{\lambda}) - \frac{1}{2} \hat{\lambda}^{-1} W^T W$$

By derivation of the above formula (23):

$$\hat{\lambda} = \frac{tr (\Sigma + \mu^T \mu)}{M}$$

**VB-M step:**

By substituting $q(\gamma, A, \lambda)$ for $L(q(\gamma, A, \lambda))$, the optimal configuration of the observation vector $A, (\Theta)$ could be realised as follow:

$$\hat{A}, (\Theta) = \arg \max E_{\gamma, A}[\ln p(A, \gamma, A, \lambda)]_{\hat{\gamma}, \hat{A}, \hat{\lambda}}$$

$$= \arg \min E_{\gamma, A}[|Y - \gamma A A_i|]_{\hat{\gamma}, \hat{A}, \hat{\lambda}}$$

(33)

where $\hat{A}, (\Theta)$ was the objective optimisation function of the observation vector $A$, which was mainly related to the parameters $\Theta$ of the diffusion distribution, as well as the grids number $N$ and grid divisions $\gamma$. The parameters vector $\Theta$ of the system state distribution could be obtained by partial derivative solution of equation (33):

$$\hat{\Theta} = \arg \min \left( \prod_{i=1}^{N} \gamma_i \det(A_i) \right) - \frac{1}{2} tr (\Sigma_i^{-1} (\Sigma_i + u_i u_i^T))$$

(34)

In this paper, the compressive sensing diffusive source localisation problem was implemented based on the Variational Bayesian Expectation Maximisation algorithm was shown in Figure 1.

It was as follows:

1. Initialising the system variables and obtaining the environment compressive sensing samples $Y$ with $A_0 (\Theta)$.

2. The diffusion state vector reconstruction $A$ and source parameters estimation $\Theta$ were fulfilled with the joint VB-EM estimation algorithm, in which, the grid division parameters $\gamma$, optimisation was also given for the compressive sensing dictionary. Updating the value of $\lambda, \gamma$, and $A$ with equations (23) and (32) based on the calculation of mean $u_i$ and variance $\Sigma_i$ with equation (20) and (21), at last gave the estimation of parameters vector $\Theta$.

3. Determine whether the variance of the estimator was satisfied with the threshold, if the requirement is not met, the dictionary parameters would be modified to obtain new information for a new updating, otherwise, the algorithm ended.
Figure 1  The estimation of the diffusion state and parameters vector with the VB-EM algorithm (see online version for colours)

4 Simulation

4.1 Simulation environment initialisation

Simulations of the proposed estimation with VB-EM algorithm were given for the diffusive source localisation problem. It was assumed that the target monitoring space was divided into $N = 200 \times 200$ grids. The sensor nodes number $M = 100$ and each sensor node with a known position $r_i$. The measurements vector $Y$ was with mean value $0$ and variance $\lambda = \sigma^2$, and the signal-to-noise ratio (SNR) was defined as:

$$SNR = 10 \log_{} \left( \frac{\| A(\Theta) \|}{\lambda} \right)$$  \hspace{1cm} (35)

Two performance indexes were defined: the average target recovery accuracy rate and the mean recovery error of target position vector. It was considered that the target localisation successful when the Euclidean distance between the sparse target recovery position $\hat{r}_i$ and real position $r_i$ was less than one grid width. The ratio of the successfully recovery targets number and the whole targets was defined as the average target recovery accuracy rate:

$$\text{The Average Target recovery Accuracy Rate} = \frac{\text{The Number of Successfully Recovery Targets}}{\text{The Total Targets}}$$  \hspace{1cm} (36)

The mean value of Euclidean distance between all sparse recovery target position and real target position was defined as the mean recovery error of target position:

$$\text{The Mean Recovery Error of Target Position} = \frac{1}{\hat{K}} \sum_{i=1}^{\hat{K}} | \hat{r}_i - r_i |$$  \hspace{1cm} (37)

where, $\hat{K}$ was the number of sparse recovery targets in the $i$ th simulation.

4.2 Simulation analysis

The performance of the proposed VB-EM algorithm was compared with the OMP, BP and GMP sparse recovery algorithms for the compressive sensing diffusive source localisation. Simulation results were given in Figures 2–4.

The different compression sensing sparse recovery algorithms performance with different SNR values were shown in Figure 2.

In which, the target number was $K = 1$ and the sensor nodes number was $M = 100$. Figure 2(a) described the relationship between the average target recovery accuracy rate of different sparse recovery algorithms with different SNRs. It could be seen that with the SNRs increasing, all the sparse recovery algorithms performance would be improved. And the performance of various algorithms would be indistinguishable when the SNR more than 20 dB or less than 5 dB. When the SNR value between 5 dB and 15 dB, the performance of different algorithms was changed distinctly, the proposed VB-EM algorithm has the similar performance with OMP and GMP, which is better than the BP compression reconstruction algorithm. Figure 2(b) described the mean target position recovery error of different compression sensing sparse recovery algorithms with different SNR values. It could be seen that with the SNR value increasing, the localisation average error of all algorithms could decrease, and the most important thing was that the proposed VB-EM algorithm was similar to all the traditional compressive sensing sparse recovery methods in reconstruction error, which was due to the Bayesian algorithm could not show its superiority with a lower dimension of the target sparsity.

The performance of different sparse recovery algorithms with different sensor node samples was shown in Figure 3.

In which, the target sparsity was set to $K = 5$, and the signal-to-noise ratio was set to $SNB = 20$ dB, and Figure 3(a) described the average target recovery accuracy rate of different sparse recovery algorithms with the number of node sensing samples increasing from 60.
Compressive sensing multi-target diffusive source localisation using sparse recovery algorithms

Figure 2 Different compression sensing algorithms performance comparisons with different SNRs: (a) the average target recovery accuracy rate and (b) the mean recovery error of target position (see online version for colours)

![Figure 2](image)

It could be seen from Figure 3(a): as the number of sensor node samples was less than 80, and the average target recovery accuracy rate of all the algorithms was almost equal to zero. When the number of samples increasing from 80 to about 135, the average target recovery accuracy rate of traditional OMP, BP, GMP algorithms and VB-EM algorithm would be also increasing with a certain rate, and in which, the OMP algorithm was the slowest, followed by BP and GMP algorithm, but the VB-EM algorithm was fastest among all the compressive sensing algorithms. When the number of measurements reaches about 135, VB-EM would achieve the average target recovery accuracy rate of 1 firstly. And the GMP algorithm could also achieve 1 when the number of measurements reached about 145. However, the OMP and BP algorithm still failed to achieve the average target recovery accuracy rate of 1 when the maximum number of measurements was 180. It could be seen that VB-EM algorithm could achieve a higher average target recovery accuracy rate with fewer samples, it was mean that the smallest number of samples was needed for the VB-EM algorithm at the same time.

Figure 3 Performance comparison of different sparse recovery algorithms with different sensor node samples: (a) the average target recovery accuracy rate and (b) the mean recovery error of target position (see online version for colours)

![Figure 3](image)

Figure 3(b) described the mean recovery error of target position performance comparison of different compressive sensing algorithms with the same sensor nodes samples. With the increasing of samples, the mean recovery error of target position of all compressive sensing algorithms could be gradually reduced. Among them, the proposed VB-EM algorithm had a faster convergence speed, and its location error performance was better than other traditional compressive sensing sparse recovery algorithms.

Figure 4 described the different compression sensing reconstruction algorithms performance with different target sparsity (the number of targets), in which, the sensor nodes sampling times was fixed to 150 and the signal-to-noise ratio was set to 20 dB. Figure 4(a) showed the different compression sensing algorithms performance comparison of
reconstruction accurate probability with the sparsity increasing from 1 to 16, and 100 repeated experiments times.

**Figure 4** Performance comparison of different compression sensing algorithms with different target source sparsity: (a) the average target recovery accuracy rate and (b) the mean recovery error of target position (see online version for colours)

It could be seen that the average target recovery accuracy rate of all compressive sensing algorithms would be equal to 1 with about $K = 1$ sparsity. With the sparsity $K$ increasing, the average target recovery accuracy rate of the OMP algorithm would decrease rapidly, and when the sparsity $K = 6$, the target source vector could not be reconstructed accurately for the OMP algorithm. Secondly, as for the BP algorithm and GMP algorithm, the original vectors would not be reconstructed accurately when the sparsity greater than 9. The simulation showed that the VB-EM algorithm could not accurately reconstruct the original signal until the sparsity was more than 12. It could be seen that the VB-EM algorithm could have a higher average target recovery accuracy rate than other compressive sensing algorithms with the same sparsity, and it was more suitable for accurate vector recovery with a relatively higher sparsity dimensions.

Figure 4(b) described the relationship between the target source sparsity (the number of targets) and the mean recovery error of target position with different sparse recovery algorithms. As the value of the sparsity increasing, the mean target position recovery error of all algorithms would increase with the RIP characteristics of the compressive sensing matrix, which would change at the same time, so the recovery performance would become poor. However, the simulation results showed that the VB-EM algorithm had higher robustness, and its recovery performance was still better than other algorithms.

**5 Summary**

In this paper, a compressive sensing diffusive source localisation method was given using the variational Bayesian expectation maximisation (VB-EM) sparse recovery algorithm, in which, the joint optimisation of target source parameters estimation and diffusion distribution state reconstruction in the dynamic environment was implemented and the compressive sensing dictionary mismatch problem was also solved with the estimation results by changing the grids division parameters. Simulation results show that the proposed sparse recovery algorithm could effectively fulfil the sparse target vector recovery and parameters estimation of the diffusion distribution state simultaneously. Compared with the traditional compressive sensing sparse recovery algorithms, it could obtain a higher effectiveness and robustness performance.

**Acknowledgements**

This work was supported by the financial support of Natural Science Foundation of China (61573253), Tianjin Science and Technology Project of Special Correspondent (19JCTPJJC51600,19JCTPJJC52300), National Natural Science Foundation of China (No. U1813222), Tianjin Natural Science Foundation (No. 20JCYBJC21900, No. 18JCYBJC16500) and Key Research and Development Project from Hebei Province (No.19210404D), Ministry of Education Industry University Cooperation Education Project (201901198040), National Training Programs of Innovation and Entrepreneurship for Undergraduates (201910069045).

**References**
