A fuzzy robust programming approach to multi-objective portfolio optimisation problem under uncertainty

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Abstract: Portfolio selection is one of the most important problems in financial markets. This paper proposes a novel robust flexible portfolio optimisation model based on possibilistic mean and variance and flexible constraints, to cope with inherent uncertainty of such problem. The proposed model is extended by introducing a modified robust flexible approach. The developed models are evaluated and validated by using the real data of Tehran stock exchange. The obtained results show that in higher violation penalties, the proposed models outperform the deterministic model. In addition, experimental analyses are provided to compare the performance of the robust flexible portfolio optimisation model to the modified version.

Keywords: fuzzy mathematical programming; robust optimisation; portfolio problem; possibilistic mean; possibilistic variance; flexible constraints.


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1 Introduction

Financial planning problems, such as asset allocation under risk, are among the most attractive problems in the area of uncertain programming. Recently, we can see a notable increase in financial mathematics related researches where portfolio optimisation problem has become an interesting topic in financial studies. Markowitz (1952) proposed one of the first models for this problem. He explained that a reasonable investor not only focuses on maximising expected return of an asset, but also concentrates on minimising relevant risk. In the mathematical model proposed by Markowitz, variance of the portfolio return demonstrates the risk of the selected portfolio. This model is known as Markowitz model or mean-variance model. As a good selection of assets can decrease the risk, it is necessary to study portfolio selection problem (Ahmed and El-Alem, 2005).

A useful advantage of the mean-variance model is that it decreases the risk to a convex quadratic program, which can be solved in an efficient manner (Zymler et al., 2011). This model has a significant effect on the modern portfolio management.

Here we describe the mean-variance model; Assume a portfolio is composed of $n$ assets with returns $r_i$, $i = 1, \ldots, n$. Consider $\text{cov}(r_i, r_j)$, $i, j = 1, \ldots, n$, be the covariance of these assets returns. Also, asset weights in the portfolio are $x_i$, $i = 1, \ldots, n$. Finally, $r_p$ is total return of the portfolio and $\text{Var}(r_p)$, is its variance (that can be used as a measure of risk). Thus, the mathematical portfolio optimisation model is:

Choose $x_1, \ldots, x_n$, where:

$$
\sum_{i=1}^{n} x_i = 1, \quad 0 \leq x_i \leq 1
$$

Such that maximises the return of portfolio:

$$
r_p = \sum_{i=1}^{n} r_i x_i
$$

And also such that minimises the risk of portfolio:

$$
\text{Var}(r_p) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \text{cov}(r_i, r_j).
$$

As seen above, the portfolio optimisation problem is a multiple-objective program.

Numerous models in literature have been proposed for portfolio optimisation by using different approaches. For instance, Deng et al. (2005) proposed a min-max model to portfolio selection problem under uncertainty that exists in inputs. In the another research, Dentcheva and Ruszczyński (2006) considered the problem of assembling a portfolio of limited assets whose return rates were defined by a discrete joint distribution function and proposed a new model of portfolio optimisation including stochastic dominance constraints on return rate of portfolio. Ustun and Kasimbeyli (2012) formulated an expanded mean-variance-skewness model with multi objective functions. In their model, both returns and return errors were achieved using different forecasting methods and the objective function was to minimise the mean absolute forecast error. Ma et al. (2012) considered a portfolio optimisation model with cardinality constraints, in which the smallest amount of value-at-risk (VaR) was considered as the objective function. Considering the security returns with interval expected returns as uncertain variables, Li and Qin (2014) proposed a mean semi-absolute deviation model within the structure of uncertainty theory, which could be handled by the classical optimisation
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algorithms. Deng and Zhao (2013) attempted to present some practical methods to achieve new and precise results on risks value ranges for traditional Markowitz portfolio model. Huang and Qiao (2012) discussed a multi-period portfolio selection problem in which asset returns were obtained by exports’ evaluations. In their work, the asset return rates were considered as uncertain variables and researchers proposed an uncertain risk indicator adjustment model. Li et al. (2012) proposed an expected regret minimisation model. Their model was to minimise the expected value of difference between the maximum asset return and the obtained asset return associated with each portfolio.

In recent years, more attention has been paid to incorporating uncertainty in portfolio planning models where fuzzy models have been widely researched. Fuzzy set theory, introduced by Zadeh (1973), handles uncertainties with reasoning that are approximate and not fixed and exact. The factors such as complexity and uncertainty of reasons are using fuzzy set theory (Pishvaee et al., 2008). Liu and Zhang (2013) presented a multi-objective portfolio optimisation problem for portfolio selection in fuzzy environments in which both the asset return rates and the turnover rates were considered as fuzzy variables. Applying the possibility theory, return and liquidity were considered as fuzzy variables and their quantities were determined by possibilistic mean. Similarly, quantities of risk of market and liquidity risk were determined by lower possibilistic semi-variance. Li et al. (2013) proposed a possibilistic portfolio model considering risk-averse constraint and investment without risk established upon the possibilistic mean and variance, while supposing that the expected assets rate of returns was a fuzzy number. Their model presented obviously that financial market is influenced by several non-probabilistic factors, risk-averse indexes which not exclusively to obtain the returns expected rate in their real investment, but also to guarantee that the greatest amount of their possible risk is less than an expected loss. Zhang and Zhang (2014) considered a multi-period fuzzy portfolio selection problem that tried to maximise final wealth enforced by risk control, in which the returns of assets were described by possibilistic mean values. They specified a possibilistic absolute deviation as the portfolio risk control. Also, authors proposed a multi-period mean absolute deviation model of fuzzy portfolio selection while considering transaction cost, threshold constraints, borrowing constraints and cardinality constraints. Fasanghari and Montazer (2010), Liu (2011) and Yunusoglu and Selim (2013) are three other researches of portfolio problem which used fuzzy decisions making methods.

In 2008, when global financial crisis occurred, financial managers placed more attention on managing portfolios under uncertainty. Uncertainty occurs when knowledge about a future event is not enough and it can be decreased (but not necessarily removed) by collecting more information (Vose, 2008). In real world, it is necessary to consider some variables as uncertain ones, because we do not have enough knowledge about them. In portfolio optimisation problem, returns of assets are uncertain variable; because decision maker cannot predict the future return of assets precisely. In addition, he or she cannot use stochastic techniques to handle this variable. Robust optimisation is an effective modelling tool associated with decision problems subject to non-stochastic variables uncertainty. Robust optimisation models aim to achieve the best decision by considering worst-case parameters amounts within these sets (Zymler et al., 2011). Ben-Tal and Nemirovski (1999) proposed a robust optimisation model to protect portfolio against the uncertainty in the assets returns. They showed that when the assets returns can be changed within an ellipsoidal uncertainty set that is determined through their means and covariance, the optimisation problem results are remembering of the
Mean-variance model. Therefore, as resulting of 2008–2010 global financial crisis showed obviously the extreme importance of robustness of portfolios and therefore a more comprehensive understanding of robust portfolios is needed to cause its suitable use. In recent years, attention to robust optimisation has been increased and several models have been proposed. Liesiö et al. (2007) developed a methodology that generalises preference programming techniques into portfolio problems, if a subset of project suggestions is financed by considering multiple evaluation criteria. They also developed an algorithm that computes all non-dominated portfolios, subject to lack of information about criteria weights and project-precise execution levels. Gregory et al. (2011) derived a robust model from a minimum-regret view and tested the robust model properties with respect to portfolio construction. They investigated the effect of different meanings of the bounds on the uncertainty sets and presented that robust models give well-variegated portfolios in connection with the number of assets and weights of asset. Liesiö et al. (2008) extended robust portfolio modelling to explain project mutual dependences, uncertain cost information and variable budget levels. These extensions led to a multi-objective binary linear programming problem that contained interval-valued objective function coefficients. They also realised all not under control solutions by a tailored algorithm. Their extended robust portfolio modelling structure allowed more complete modelling of portfolio problems and prepared support for cost-benefit examinations. Zymler et al. (2011) proposed a new robust optimisation model for constructing portfolios that contain European-style possibilities. Their model traded off weak and strong and gave a guarantee on the worst-case return of portfolio. The weak guarantee was implemented during the time that the asset returns are understood within the recommended uncertainty set, while the strong guarantee implemented for all possible asset returns. The result of model established a convex second-order cone program, which was tractable to efficient numerical solution processes. Shen and Zhang (2008) implemented the idea of robust optimisation to the portfolio selection problem on multi-stage scenario trees. The objective of their proposed model was to maximise expected value of utility function as in a classical stochastic programming problem, except that they incorporated ambiguities in the probability distributions of the scenario tree. They showed that such a problem can be expressed in a formula as a finite convex program in the conic form, on which general convex optimisation methods can be implemented. Fertis et al. (2012) defined the idea of robust risk measure as the worst possible risks when probability measures are likely to happen. They also introduced robust forms of conditional value-at-risk (CVaR) and entropy-based risk measures. In another paper, Kim et al. (2014) investigated the behaviour of robust portfolios by depicting how robustness conducts to higher reliance on factor movements. Concentrating on the robust formulation with an ellipsoidal uncertainty set for expected returns, they showed that as the portfolio robustness increases, its optimal weights approach the portfolio with a variance that is maximally described by factors. Zhu et al. (2014) developed a new portfolio selection structure with a feature of two parts robustness in both return distribution modelling and portfolio optimisation. DeMiguel and Nogales (2009) proposed a class of portfolios that have better firmness properties than the traditional portfolios with minimum variance. The proposed portfolios were constructed using specific robust estimators and could be calculated by solving a nonlinear program, where robust estimation and portfolio optimisation were presented in
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a single step. Sadjadi et al. (2012) proposed a novel portfolio modelling approach with uncertain data and analysed it using different robust optimisation methods. To solve the proposed formulations, they used genetic algorithm.

In this paper, we consider returns of assets as fuzzy numbers, because fuzzy mathematical programming is a flexible method for dealing with uncertainties that arise from incomplete knowledge of decision maker about the variables (Pishvaee et al., 2012b). In addition, fuzzy mathematical programming gives us a structure to deal with the sources of uncertainty such as ambiguity, imprecision and vagueness, at the same time (Nguyen et al., 2007). In this paper, portfolio optimisation problem is considered as a multi-objective problem; we propose a new robust flexible programming (RFP) that includes possibilistic programming. Another property of this problem is that it is a cardinality constrained problem. Robust possibilistic programming is a new method to face problems with incomplete knowledge on parameters. There are no widely researches about this method in the literature. Zhang et al. (2009) proposed a robust chance-constrained fuzzy possibilistic programming model for management of water quality with agricultural systems. Pishvae et al. (2012a) proposed a robust possibilistic programming model for effective coping with uncertain parameters. They applied the proposed approach to design a socially responsible supply chain network. Finally, Inuiguchi et al. (1994) investigated robust possibilistic programming model in a multi-period location-allocation problem. It should be noted that the main property of robust possibilistic programming models is considering variability of confidence levels. The decision maker do not need to determine confidence levels in the. This method can be classified as proactive.

Concluding from literature, robust fuzzy method has not been widely studied and this method has not applied to portfolio problem. However, papers frequently considered fuzzy or robust method independently. Also, RFP approach has not been studied widely in articles. In other hand, portfolio problems that are proposed earlier have not considered flexible constraints. Therefore, we can summarise the contributions of current paper as follows:

1 proposing a novel RFP approach for the portfolio optimisation problem
2 considering the flexibility of stock weight constraints
3 applying the proposed method to the emerging Tehran stock exchange for the first time.

The proposed robust programming approach is a proactive method which enables the decision maker to determine the confidence level of uncertain constraints.

The rest of the paper is organised as follows; In Section 2, the concerned portfolio problem is defined and formulated. Proposed RFP models are presented in Section 3. In addition, some definitions and classification of fuzzy programming are presented in this section. The proposed RFP models are implemented for a case study and results of models compared in Section 4. Finally, Section 5 includes the conclusion of the paper and some suggestions for future researches.
2 Problem description and formulation

2.1 Problem description

A portfolio is any selection of financial assets. Portfolios may be designed and managed by banks, financial professionals and even by individual investors and other financial institutions. After Markowitz’s (1952) research, which introduced expected return as a desirable factor and variance of return as undesirable factor in portfolio selection, it is widely accepted that portfolio designing relates to investors’ risk behaviour. It is worthy to note that portfolio theory originates from this Markowitz’s work in 1952. In addition, portfolio designing may depend on other factors such as time frame and investment objectives. Therefore in simple state, a proper asset selection and allocation’s goal is to maximise the expected return and to minimise the risk of assets in the same time. Hence, to obtain proper solutions, trade-off between return and risk have should be made such that indicates we deal with a multi-objective problem. The experience in finance shows that investment in several funds and varying investment can help us to reduction of portfolio’s risk, it means that is better if we determine bounds for assets weights in portfolio. Three important aspects of portfolio theory are:

1. it describes a set of assets by their expected returns and risks
2. it determines how much an investor should invest in a certain asset
3. it indicates which assets have best returns at a given level of the risk.

2.2 Notations and assumptions

The multi-objective portfolio optimisation with flexible constraint is proposed under following notations and assumptions:

2.2.1 Notations

- parameters:
  \( \tilde{k} \) uncertain expected return of the portfolio
  \( \tilde{r}_i \) uncertain expected return of asset \( i \)
  \( \text{cov}(\tilde{r}_i, \tilde{r}_j) \) covariance between the expected returns of assets \( i \) and \( j \)
  \( l \) constant lower bound of assets weights in the portfolio
  \( u \) constant upper bound of assets weights in the portfolio
  \( N \) constant number of assets that we want to have in portfolio.

- variables:
  \( x_i \) Weight of asset \( i \) in the portfolio
  \( m_i \) binary variable that equals to 1, if asset \( i \) exist in portfolio, otherwise, equals to 0.
2.2.2 Assumptions
1. no short selling is allowed in order to make the problem more realistic
2. portfolio includes determined number (L) of n available risky assets
3. assume that upper and lower bound constraints are flexible.

2.3 Model formulation

A single investment period and n risky assets with uncertain return for investment have been assumed in the standard portfolio optimisation problem. Assumptions pertaining to the multi-objective portfolio optimisation problem are:

\[
\text{max } E\left(\tilde{k}\right) = \sum_{i=1}^{n} x_i \tilde{r}_i \tag{4}
\]

\[
\text{min } \text{Var}\left(\tilde{k}\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \text{cov} \left(\tilde{r}_i, \tilde{r}_j\right) \tag{5}
\]

subject to:

\[
\sum_{i=1}^{n} x_i = 1 \tag{6}
\]

\[
lm_i \leq x_i \leq um_i \tag{7}
\]

\[
\sum_{i=1}^{n} m_i = N \tag{8}
\]

\[
x_i \geq 0, \quad i = 1, \ldots, n \tag{9}
\]

\[
m_i \in \{0, 1\}, i = 1, \ldots, n \tag{10}
\]

That objective function (4) aims at maximising the expected return of the portfolio and objective function (5) aims at minimising the variance of portfolio that represents the risk of portfolio. In other word, this relation represents the risk minimisation goal. Constraint (6) is that budget constraint which assures sum of all assets weights in the portfolio is equal to one. Any asset that exists in portfolio has upper and lower bounds for its weight (i.e., Constraint 7). We consider these constraints as flexible constraints, since they assume that assets weights are allowed to have a little perturbation within their bounds. Constraint (8) shows constant cardinality of portfolio optimisation problem. Finally, relations (9) and (10) enforce the non-negativity and binary limitations of the decision variables. The non-negativity constraints show that short sales are not allowed.

3 Fuzzy mathematical programming

In lights of the types of uncertainties, fuzzy mathematical programming can be divided into two categories (Pishvaee et al., 2012a; Inuiguchi and Ramik, 2000; Mula et al., 2006; Torabi and Hassini, 2008; Zahiri et al., 2014):
1. fuzzy mathematical programming with vagueness, which is called flexible programming. Goals and constraints in this category are flexible on target values of objective functions and constraints elasticity.

2. fuzzy mathematical programming with ambiguity, which is called possibilistic programming.

This type of programming deals with ambiguous coefficients of constraints and objective functions. In this paper, we deal with both categories of fuzzy mathematical programming. In the next sub-sections, we will discuss about both flexible and possibilistic programming approaches as we face them in proposed portfolio problem.

3.1 Flexible programming approach

In flexible programming, flexibility exists in either constraints satisfaction or in target values of objective functions. This programming implies that uncertainty can be found in all parameters, constraints and even in relationships. Unfortunately, few researches have been done regarding this class of problems. In equation (7), we assume that the assets weights in portfolio have flexibility which means little perturbations are allowed. This equation is composed from two parts as follows:

\[ x_i \leq \mu m_i \]

and

\[ x_i \geq \lambda m_i \]

To handle the vagueness, we use method introduced by Peidro et al. (2009) which converts the flexible constraints to single ones. The general form of flexible constraint with fuzzy relationship is:

\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \]

Suppose that \( \alpha, \beta \in F(\mathbb{R}) \) are fuzzy numbers, a simple approach to rank these numbers is to use a ranking function that map fuzzy numbers into \( \mathbb{R} \), \( g:F(\mathbb{R}) \rightarrow \mathbb{R} \), where \( g \) operates as following form:

\[
\begin{align*}
    g(\alpha) \leq g(\beta) & \iff \alpha \leq \beta \\
    g(\alpha) \geq g(\beta) & \iff \alpha \geq \beta \\
    g(\alpha) = g(\beta) & \iff \alpha = \beta
\end{align*}
\]

If following equations regarding \( g \) be correct, then \( g \) is a linear ranking function.

\[
\begin{align*}
    g(\alpha + \beta) &= g(\alpha) + g(\beta), \quad \forall \alpha, \beta \in F(\mathbb{R}) \\
    g(r \alpha) &= rg(\alpha), \quad \forall r \in \mathbb{R}, r > 0, \forall \alpha \in F(\mathbb{R})
\end{align*}
\]

To solve equation (12), assume that \( g \) is a linear ranking function and \( \phi \) is a given function as follows:
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$$\varphi(a,x,b) = \begin{cases} \tilde{p}_i, & a_i x \leq b_i \\ \tilde{p}_i - a_i x + b_i, & b_i \leq a_i x \leq b_i + \tilde{p}_i \\ 0, & a_i x \geq b_i + \tilde{p}_i \end{cases}$$

where \( \tilde{p}_i \in F(\mathbb{R}) \) is a fuzzy number and its support is in \( \mathbb{R}^+ \). It denotes allowed perturbations in relationships. Membership function that deals with \( i \)'th constraint \((u^i)\) can be yielded by Cadenas and Verdegay (1997) method as following form:

$$u^i(a,x,b) = \frac{\varphi(a,x,b)}{g(\tilde{p}_i)}, \quad \mu^i : F(\mathbb{R}) \rightarrow [0,1]$$

Now, we can define a \( \gamma \)-cut set on membership function to know the minimum degree of constraints satisfaction. So, we have:

$$\mu^i(a,x,b) \geq \gamma \iff \frac{\varphi(a,x,b)}{g(\tilde{p}_i)} \geq \gamma \iff \frac{\tilde{p}_i - a_i x + b_i}{g(\tilde{p}_i)} \geq \gamma$$

$$\iff g(\tilde{p}_i) - g(a,x) + g(b_i) \geq g(\tilde{p}_i) \gamma$$

$$\iff g(a,x) \leq (b_i + \tilde{p}_i (1-\gamma)) \iff a_i x + \tilde{p}_i (1-\gamma)$$

Therefore, equation (12) can be converted to following form:

$$a_i x_j \leq b_i + \tilde{p}_i (1-\gamma)$$

$$x_j \geq 0, i, j \in N, \gamma \in [0,1]$$

(13)

we take into account \( \tilde{p}_i \) as a triangular fuzzy numbers. In addition, considering first index of Yager (1981), we can transform the fuzzy linear problem defined in equation (13) to a crisp linear problem defined in the following equation:

$$a_i x_j \leq b_i + \left(p_i(2) + \frac{d_{p_i} - d'_{p_i}}{3}\right) (1-\gamma)$$

where \( p_i(2) \) is the central point of the triangular fuzzy number and \( d'_{p_i} \), and \( d_{p_i} \), are left and right lateral margins respectively (see Figure 1). We can apply this method to constraints too. Therefore, equation (11) converts to following equation:

$$x_i \leq \text{um}_i \rightarrow x_i \leq \text{um}_i + \left(p_i(2) + \frac{d_{p_i} - d'_{p_i}}{3}\right) (1-\gamma)$$

and

$$x_i \leq \text{lm}_i \rightarrow -x_i \leq -\text{lm}_i \rightarrow x_i \geq \text{lm}_i - \left(p_i(2) + \frac{d_{p_i} - d'_{p_i}}{3}\right) (1-\gamma)$$

(14)
3.2 Possibilistic programming

As mentioned earlier, possibilistic programming deals with ambiguity in coefficients of constraints and objective functions. In this type of fuzzy mathematical programming, fuzzy coefficients are considered as possibility distributions. These distributions can be derived from available data about fuzzy parameters.

3.2.1 Possibilistic mean and variance

Carlsson and Fullér (2001) proposed a calculation method for possibilistic mean and variance. Here we describe their method; if $A$ is a fuzzy number with $A(\gamma) = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$, in order to obtain the possibilistic mean value and variance of $A$:

$$E(A) = \int_0^1 \gamma (a_1(\gamma) + a_2(\gamma)) d\gamma$$

and

$$Var(A) = \frac{1}{2} \int_0^1 \gamma (a_2(\gamma) - a_1(\gamma))^2 d\gamma$$

If $\xi = (\xi(1), \xi(2), \xi(3), \xi(4))$ is a trapezoidal fuzzy number, we have following result that are given in (Carlsson et al., 2002):

$$E(A) = \int_0^1 \gamma [(\xi(1) - (1 - \gamma))(\xi(2) - \xi(1)) + \xi(2) + (1 - \gamma)(\xi(4) - \xi(3))] d\gamma$$

$$= \frac{\xi(1) + \xi(2)}{2} + \frac{\xi(4) - \xi(3) - \xi(2) + \xi(1)}{6}$$

and

$$Var(A) = \frac{(\xi(2) - \xi(1))^2}{4} + \frac{(\xi(2) - \xi(1))(\xi(2) - \xi(1) + \xi(4) - \xi(3))}{6}$$

$$+ \frac{(\xi(2) - \xi(1) + \xi(4) - \xi(3))^2}{24}$$

$$= \left[ \frac{\xi(2) - \xi(1)}{2} + \frac{\xi(2) - \xi(1) + \xi(4) - \xi(3)}{6} \right]^2 + \frac{(\xi(2) - \xi(1) + \xi(4) - \xi(3))^2}{72}.$$
3.2.2 Possibilistic approach toward portfolio problem

In Section 3, we described a portfolio problem with some constraints. In this problem, we consider uncertain returns of assets as trapezoidal fuzzy numbers that are given for asset $i$ by $r_i = (r_i(1), r_i(2), r_i(3), r_i(4))$, $i = 1, \ldots, n$. Therefore considering previous section, we state a possibilistic program for portfolio problem. The following results are given at (Carlsson et al., 2002), too:

$$E\left(\sum_{i=1}^{n} r_i x_i\right) = \sum_{i=1}^{n} \frac{1}{2} \left[ r_i(1) + r_i(2) + \frac{1}{3} (r_i(4) - r_i(3) - r_i(2) + r_i(1))\right] x_i$$

(15)

and

$$\text{Var}\left(\sum_{i=1}^{n} r_i x_i\right) = \left(\sum_{i=1}^{n} \frac{1}{2} \left[ r_i(2) - r_i(1) + \frac{1}{3} (r_i(4) - r_i(3) + r_i(2) - r_i(1))\right] x_i \right)^2 + \frac{1}{72} \left(\sum_{i=1}^{n} (r_i(2) - r_i(1) + r_i(4) - r_i(3)) x_i \right)^2$$

(16)

Consider $h: \mathbb{R}^n \to \mathbb{R}$, $h(x) = \sum_{i=1}^{n} a_i x_i$, and $j: \mathbb{R} \to \mathbb{R}$, $j(x) = x^2$. The first one is a convex function and the second one is a univariate and non-decreasing convex function. According to these considerations and Bazaraa et al. (2006), the composite function $f: \mathbb{R}^n \to \mathbb{R}$ that is defined as $f(x) = j[h(x)]$ is a convex function. Similarly, considering that sum of two convex functions are a convex function, the variance function that introduced before in equation (16) is a convex functions too.

3.3 Goal programming

Idea of goal programming (GP) first developed by Charnes et al. (1955) and then extended to management and financial decision making by Ijiri (1965). GP is a special type of mathematical programming with multi objective functions. Despite from that linear programming minimises or maximises objective functions; GP minimises the distance between desired and real results. GP has a vast application in management and financial decisions. In this paper, we use GP to handle the multi-objective portfolio problem. Considering two main objective functions [i.e., equations (15), (16)] in our model, we can convert them to a single objective function and constraints as follows:

$$\text{Max } E\left(\sum_{i=1}^{n} r_i x_i\right) - \rho \left(\sum_{i=1}^{n} d_i^+ + d_i^-\right)$$

s.t.

$$\text{Var}\left(\sum_{i=1}^{n} r_i x_i\right) + \sum_{i=1}^{n} d_i^- - \sum_{i=1}^{n} d_i^+ = G$$

$$d_i^- \cdot d_i^+ = 0$$

(17)

Variance is always a positive value and our desire goal of variance is zero. We set $G = 0$, $d_i^- = 0$, and $d_i^+ \geq 0$, therefore we can rewrite equation (17) as following form:
MaxE\left( \sum_{i=1}^{n} r_{xi} \right) - \rho \left( \sum_{i=1}^{n} d_{i}^* \right)
\text{s.t.}
\text{var}\left( \sum_{i=1}^{n} r_{xi} \right) - \sum_{i=1}^{n} d_{i}^* = 0
\quad d_{i}^*, x_i \geq 0

3.4 RFP models

As mentioned earlier, robust programming deals with optimisation problems in uncertain situations. A robust solution of optimisation problem must be has both feasibility robustness and optimality robustness (see Pishvaee et al., 2012a, for definitions of feasibility robustness and optimality robustness). According to the previous sections, we can formulate the basic flexible and possibilistic programming (BFPP) model as:

MaxE\left( \sum_{i=1}^{n} r_{xi} \right) - \rho \left( \sum_{i=1}^{n} d_{i}^* \right)
\text{s.t.}
\text{var}\left( \sum_{i=1}^{n} r_{xi} \right) - \sum_{i=1}^{n} d_{i}^* = 0
x_i \leq um_i + \left( p_i(2) + \frac{d_{pi} - d_{p}^*}{3} \right)(1 - \gamma)
x_i \geq lm_i - \left( p_i(2) + \frac{d_{pi} - d_{p}^*}{3} \right)(1 - \gamma)
\sum_{i=1}^{n} x_i = 1
\sum_{i=1}^{n} m_i = N
x_i \geq 0, i = 1, \ldots, n
m_i \in \{0, 1\}, i = 1, \ldots, n
0.5 < \gamma \leq 1

where \gamma is confidence level of chance constraints in above-mentioned formulation and is a parameter that decision maker must determine it. It is possible that selected value was not the best choice for confidence level; this method is a reactive method. If the decision maker determines several confidence level values and tests them, then he can select best value; this method is called interactive method. In addition, with increase of number of chance constraints, determining the proper value of confidence level becomes more and more hard and expensive.
3.4.1 RFP model

The proposed RFP model is defined as follows:

$$\text{MaxE} \left( \sum_{i=1}^{n} r_{xi} \right) - \rho \left( \sum_{i=1}^{n} d_{i}^{*} \right) - \delta \left( \sum_{i=1}^{n} \left( p_{(2)} + \frac{d_{(2)} - d_{(1)}}{3} \right) (1 - \gamma) - p_{(1)} \right)$$

s.t.

$$\text{var} \left( \sum_{i=1}^{n} r_{xi} \right) - \sum_{i=1}^{n} d_{i}^{*} = 0$$

$$x_{i} \leq um_{i} + \left( p_{(2)} + \frac{d_{(2)} - d_{(1)}}{3} \right) (1 - \gamma)$$

$$x_{i} \geq ln_{i} - \left( p_{(2)} + \frac{d_{(2)} - d_{(1)}}{3} \right) (1 - \gamma)$$

$$\sum_{i=1}^{n} x_{i} = 1$$

$$\sum_{i=1}^{n} m_{i} = N$$

$$x_{i} \geq 0, i = 1, \ldots, n$$

$$m_{i} \in \{0, 1\}, i = 1, \ldots, n$$

$$0.5 < \gamma \leq 1$$

The first term in objective function indicates the expected value of $k$, which results in maximisation of expected return of the portfolio. This term also exists in BFPP model. The second term relates to GP approach that was discussed in Section 3.5. $\rho$ demonstrates the weight of term against other terms in objective function. This term causes the objective function be sensitive to the aberration of objective function value over the optimal expected value. Also, this term controls the optimality robustness of the solution.

The third term, $\delta \left[ \left( p_{(2)} + \frac{d_{(2)} - d_{(1)}}{3} \right) (1 - \gamma) - p_{(1)} \right]$, represents the confidence level of chance constraints, where $\left[ \left( p_{(2)} + \frac{d_{(2)} - d_{(1)}}{3} \right) (1 - \gamma) - p_{(1)} \right]$ is difference between value of worst case of uncertain parameter and used value in chance constraint and $\delta$ is the penalty value of allowed violation in each constraint that has uncertain parameter(s). This term controls the feasibility robustness of solution.

In RFP model, the minimum confidence level ($\gamma$) is a variable that must be determined according to the constraints and the objective function. Therefore, judgment on confidence levels of chance constraints is removed in this model. In addition, with increasing the number of chance constraints, RFP model can obtain the optimal values of confidence levels conveniently. Therefore, this model does not need processes such as simulation experiments to determine value of confidence levels.

It is obvious that RFP model tries to trade-off three terms of objective function:

1. expected return
2. optimality robustness
3. feasibility robustness.
3.4.2 Modified robust flexible programming (MRFP) model

The RFP model has no significant attention on possibility of constraints violation. In fact, it is conservative against calculation of constraints violation. Hence, we propose the MRFP model that is a less conservative approach:

\[
\text{Max } E(k) - pd^+ - (1-\gamma)\delta \left[ \left( p(2) + \frac{d_p - d_p'}{3} \right)(1-\gamma) - p(1) \right]
\]

s.t. \( x, \gamma \in F \)

where \( F \) is the feasible region of RFP model. The MRFP model is a nonlinear program, which its complexity is more than linear ones.

4 Implementation and evaluation

The implementation and evaluation of the proposed robust portfolio optimisation model is done through three steps as follows.

- **Step 1**: the real data is derived from Tehran stock exchange market that is the most important financial market of Iran in order to evaluate proposed RFP model and compare it to BFPP. Here, we consider 30 stocks of this market and use historical data pertaining returns of stocks from November to September 2014. This historical data are then converted to trapezoidal fuzzy number using method proposed in Azadeh et al. (2010). Perturbation of stocks' weight in portfolio is estimated as a triangular fuzzy number according to experts' opinions. Fuzzy numbers representing stocks' returns and stocks' weights in portfolio are represented in Table 1.

- **Step 2**: the CONOPT solver of GAMS 24.1.2 optimisation software is used to solve proposed models on a core i5 PC with 3 GB of RAM. To solve the models, nominal data given in Table 1 are used and results are given in Table 2. In BFPP model, the minimum confidence level of chance constraints is constant and the model is not able to adjust it. Hence, results of this model are represented in 0.7 and 0.8 confidence levels (0.9 confidence level is not reported because its result was not better than reported confidence levels.).

- **Step 3**: in order to evaluate performance of the proposed models and test robustness and desirability of their solutions, realisation is needed. Here, we generate ten random realisation uniformly and then test solutions performance under each realisation. When \( \xi = (\xi(1), \xi(2), \xi(3), \xi(4)) \) is a trapezoidal fuzzy number, we can generate a random number between \( \xi(1) \) and \( \xi(2) \). Note that these are two extreme points of possibility distribution function (i.e., \( \xi \sim [\xi(1), \xi(2)] \)), so with generating a random number uniformly, the realisation can be produced. The obtained solutions under nominal data (\( x^* \)) will be replaced in the model that it converts as following form:
A fuzzy robust programming approach

\[
\text{Max } \sum_{i=1}^{n} r_{i}^{\text{real}} x_{i}^{*} - \rho \left( \sum_{i=1}^{n} d_{i}^{*} \right) - \lambda \left( \sum_{i=1}^{n} r_{i}^{*} \right) - \epsilon \omega \left( \sum_{i=1}^{n} R_{i}^{*} \right)
\]

s.t.

\[
\text{var} \left( \sum_{i=1}^{n} t_{i}^{\text{real}} x_{i}^{*} \right) - d_{i}^{*} = 0
\]

\[
x_{i}^{*} \leq u_{i}^{*} + p_{i}^{\text{real}} + R_{i}^{*}
\]

\[
x_{i}^{*} + R_{i}^{*} \geq l_{i}^{*} - p_{i}^{\text{real}}
\]

\[
\sum_{i=1}^{n} x_{i}^{*} = 1
\]

\[
\sum_{i=1}^{n} m_{i}^{*} = N
\]

\[
R_{i}^{*}, R_{i}^{*} \geq 0, 1 = 1, \ldots, n
\]

where \( R_{i}^{*} \) and \( R_{i}^{*} \) indicate the violation of constraints under realisation. Therefore these are decision variables of above-mentioned formulation. For evaluation of the proposed models under realisation, the standard deviation and average of values of objective function are used as performance indexes. Results of realisation are given in Table 3.

Table 1  Fuzzy stock’s return and stock’s weight perturbation

<table>
<thead>
<tr>
<th>Stock (i)</th>
<th>Stock’s return (%) ((r(1), r(2), r(3), r(4)))</th>
<th>Perturbation of stock’s weight in portfolio ((p(1), p(2), p(3)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(3.26, 5.2, 6.4, 8.95)</td>
<td>(0, 0.01, 0.05)</td>
</tr>
<tr>
<td>(2)</td>
<td>(17.04, 22.82, 23.63, 29.25)</td>
<td>(0, 0.05, 0.1)</td>
</tr>
<tr>
<td>(3)</td>
<td>(–0.69, 0.67, 1.74, 3.7)</td>
<td>(0, 0.02, 0.03)</td>
</tr>
<tr>
<td>(4)</td>
<td>(–0.74, 1.09, 4.61, 6.37)</td>
<td>(0, 0.05, 0.07)</td>
</tr>
<tr>
<td>(5)</td>
<td>(5.74, 12.43, 16.88, 22.23)</td>
<td>(0, 0.04, 0.06)</td>
</tr>
<tr>
<td>(6)</td>
<td>(–0.19, 4.04, 4.10, 8.27)</td>
<td>(0, 0.03, 0.08)</td>
</tr>
<tr>
<td>(7)</td>
<td>(–1.69, 3.82, 4.05, 9.34)</td>
<td>(0, 0.05, 0.09)</td>
</tr>
<tr>
<td>(8)</td>
<td>(–9.92, –7.1, –2.32, –1.33)</td>
<td>(0, 0.03, 0.07)</td>
</tr>
<tr>
<td>(9)</td>
<td>(1.63, 7.65, 29.67, 33.53)</td>
<td>(0, 0.06, 0.1)</td>
</tr>
<tr>
<td>(10)</td>
<td>(–5.67, –2.6, 8.77, 11.85)</td>
<td>(0, 0.04, 0.06)</td>
</tr>
<tr>
<td>(11)</td>
<td>(21.42, 27.42, 54.87, 64.67)</td>
<td>(0, 0.01, 0.04)</td>
</tr>
<tr>
<td>(12)</td>
<td>(–10.32, –6.91, –5.8, –1.29)</td>
<td>(0, 0.07, 0.09)</td>
</tr>
<tr>
<td>(13)</td>
<td>(–9.24, –7.39, –1.42, 0.12)</td>
<td>(0, 0.04, 0.07)</td>
</tr>
<tr>
<td>(14)</td>
<td>(4.05, 11.72, 13.41, 19.4)</td>
<td>(0, 0.05, 0.11)</td>
</tr>
<tr>
<td>(15)</td>
<td>(–4.23, –0.13, 1.69, 5.77)</td>
<td>(0, 0.04, 0.05)</td>
</tr>
<tr>
<td>(16)</td>
<td>(8.71, 14.39, 19.55, 21.8)</td>
<td>(0, 0.06, 0.1)</td>
</tr>
<tr>
<td>(17)</td>
<td>(–3.73, 1.24, 13.39, 16.45)</td>
<td>(0, 0.03, 0.09)</td>
</tr>
<tr>
<td>(18)</td>
<td>(–4.14, 1.54, 6.27, 7.52)</td>
<td>(0, 0.07, 0.09)</td>
</tr>
<tr>
<td>(19)</td>
<td>(16.05, 26.3, 26.8, 37.63)</td>
<td>(0, 0.04, 0.07)</td>
</tr>
<tr>
<td>(20)</td>
<td>(6.64, 14.55, 19.54, 25.52)</td>
<td>(0, 0.07, 0.15)</td>
</tr>
<tr>
<td>(21)</td>
<td>(1.26, 5.73, 8.15, 10.72)</td>
<td>(0, 0.03, 0.08)</td>
</tr>
</tbody>
</table>
Table 1  Fuzzy stock’s return and stock’s weight perturbation

<table>
<thead>
<tr>
<th>Stock (i)</th>
<th>Stock’s return (%) (r(1), r(2), r(3), r(4))</th>
<th>Perturbation of stock’s weight in portfolio (p(1), p(2), p(3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(22)</td>
<td>(3.1, 10.34, 21.64, 25.49)</td>
<td>(0, 0.05, 0.09)</td>
</tr>
<tr>
<td>(23)</td>
<td>(–10.69, –0.17, 9.57, 14.8)</td>
<td>(0, 0.02, 0.06)</td>
</tr>
<tr>
<td>(24)</td>
<td>(–0.47, 4.23, 6.11, 9.23)</td>
<td>(0, 0.04, 0.09)</td>
</tr>
<tr>
<td>(25)</td>
<td>(–1.59, 2.47, 5.67, 10)</td>
<td>(0, 0.04, 0.12)</td>
</tr>
<tr>
<td>(26)</td>
<td>(21.79, 53.61, 128.31, 139.52)</td>
<td>(0, 0.07, 0.1)</td>
</tr>
<tr>
<td>(27)</td>
<td>(5.23, 6.93, 9.48, 10.11)</td>
<td>(0, 0.05, 0.1)</td>
</tr>
<tr>
<td>(28)</td>
<td>(–9.69, –7.2, –5.04, –1.07)</td>
<td>(0, 0.06, 0.08)</td>
</tr>
<tr>
<td>(29)</td>
<td>(–1.53, 4.14, 15.96, 20.55)</td>
<td>(0, 0.03, 0.07)</td>
</tr>
<tr>
<td>(30)</td>
<td>(–9.18, –6.75, –4.33, 0.36)</td>
<td>(0, 0.07, 0.14)</td>
</tr>
</tbody>
</table>

Table 2  Performance of models under nominal data

<table>
<thead>
<tr>
<th>BFPP model</th>
<th>RFP model</th>
<th>MRFP model</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\gamma = 0.7)</td>
<td>(\gamma = 0.8)</td>
<td>(\gamma = 0.8)</td>
</tr>
<tr>
<td>(i)</td>
<td>(x(i))</td>
<td>(Z = 22.364)</td>
</tr>
<tr>
<td>2</td>
<td>0.229</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>0.471</td>
<td>(Z = 22.311)</td>
</tr>
<tr>
<td>19</td>
<td>0.155</td>
<td>(Var = 22.001)</td>
</tr>
<tr>
<td>26</td>
<td>0.139</td>
<td>26</td>
</tr>
</tbody>
</table>

Results of realisation shows that BFPP (\(\gamma = 0.7\)) and RFP models have higher averages under an acceptable standard deviation than other models. It is obvious that proposed models are not the best for all fields and conditions and each of these models has higher performance in special conditions. Performance of models can be showed better by conducting sensitivity analyses on models parameter(s). Sensitivity analyses on penalties of violation in realisation model (i.e., \(\lambda\), \(\omega\)) have been done on BFPP (\(\gamma = 0.7\)), RFP and MRFP models and results are given in Figures 2 and 3. These figures demonstrate that at lower amounts of violation penalties, BFPP has more objective function values than other models. But at higher amounts of violation penalties, RFP and MRFP models have better performance than BFPP model.
Table 3  Performance of models under realisation

<table>
<thead>
<tr>
<th>Number of realisation</th>
<th>BFPP ($\gamma = 0.7$)</th>
<th>BFPP ($\gamma = 0.7$)</th>
<th>RFP</th>
<th>MRFP</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30.443</td>
<td>21.426</td>
<td>20.927</td>
<td>36.775</td>
</tr>
<tr>
<td>3</td>
<td>26.481</td>
<td>31.703</td>
<td>28.89</td>
<td>23.172</td>
</tr>
<tr>
<td>4</td>
<td>30.246</td>
<td>17.428</td>
<td>9.779</td>
<td>14.732</td>
</tr>
<tr>
<td>5</td>
<td>20.564</td>
<td>16.044</td>
<td>21.923</td>
<td>33.605</td>
</tr>
<tr>
<td>6</td>
<td>32.031</td>
<td>13.916</td>
<td>27.055</td>
<td>19.098</td>
</tr>
<tr>
<td>7</td>
<td>18.443</td>
<td>24.082</td>
<td>36.141</td>
<td>25.736</td>
</tr>
<tr>
<td>8</td>
<td>26.86</td>
<td>23.832</td>
<td>35.322</td>
<td>31.062</td>
</tr>
<tr>
<td>9</td>
<td>16.571</td>
<td>27.299</td>
<td>21.799</td>
<td>8.83</td>
</tr>
<tr>
<td>10</td>
<td>27.132</td>
<td>12.857</td>
<td>20.447</td>
<td>8.261</td>
</tr>
<tr>
<td>Average</td>
<td>25.043</td>
<td>21.000</td>
<td>24.774</td>
<td>22.31</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>4.876</td>
<td>5.443</td>
<td>6.996</td>
<td>8.908</td>
</tr>
</tbody>
</table>

Figure 2  Objective function of BFPP, RFP and MRFP models under values of $\lambda$ (see online version for colours)

Figure 3  Objective function of BFPP, RFP and MRFP models under values of $\omega$ (see online version for colours)
5 Conclusions

Portfolio problem is one of the most attractive problems in financial planning area that deals with selecting and managing assets in a portfolio. This paper considers a single-period multi-objective portfolio optimisation problem and applies cardinality constraint and flexibility of assets weights to it. Assets returns of portfolio are imprecise, i.e., the problem has been considered under uncertainty. To model uncertainty, a novel robust flexible model and its modified approaches are proposed and compared to deterministic model. A sample of stocks from Tehran stock exchange market that is the most important Iranian financial market is used to show the performance of proposed model. Results show that at lower amounts of violation penalties, deterministic model has better performance, while by increasing the violation penalties, robust models outperforms the deterministic model.

Various future research can be suggested as:
1. extending the proposed models to other types of portfolio problems, models such as multi-period portfolio or portfolio with transaction costs
2. applying the proposed models to other optimisation problems
3. applying other possibilistic and flexible programming approaches to current portfolio problem
4. comparing proposed models performance to meta-heuristic algorithms when solving the given problem.

References

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Appendix

Preliminaries

We explain some preliminaries on fuzzy theory in this section. $\zeta$ is a fuzzy number ($\zeta \in F(\mathbb{R})$) if be a normal and convex fuzzy set of $\mathbb{R}$. This numbers refer to a connected set of possible values, where possible values domain is a weight of $[0, 1]$ interval. This weight is called membership function and membership function of $\zeta$ is denoted by $\mu_\zeta(x)$. $\gamma$-level set of fuzzy number is showed by $[\zeta]_\gamma = \{x \in \mathbb{R} | \mu_\zeta(x) \geq \gamma\}$, $\gamma \in [0, 1]$. $\zeta$ is a trapezoidal fuzzy number if it is denoted by $\zeta = \zeta(1), \zeta(2), \zeta(3), \zeta(4)$ and its membership function is in following form

$$
\mu_\zeta(x) = \begin{cases} 
0, & x \leq \zeta(1), x \geq \zeta(4) \\
\frac{x - \zeta(1)}{\zeta(2) - \zeta(1)}, & \zeta(1) \leq x \leq \zeta(2) \\
1, & \zeta(2) \leq x \leq \zeta(3) \\
\frac{\zeta(4) - x}{\zeta(4) - \zeta(3)}, & \zeta(3) \leq x \leq \zeta(4)
\end{cases}
$$

Figure 4 shows a trapezoidal fuzzy number. If $\zeta(2) = \zeta(3)$, then we have a triangular fuzzy number. These two types of fuzzy numbers are common ones.

Figure 4  Trapezoidal fuzzy number possibility distribution