Stationary distribution and ergodicity of a stochastic single-species model under regime switching in a polluted environment

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Abstract: The long-term statistical rule is one of the important questions for stochastic pollution-population dynamical models, thus it would be worth looking for the stationary distribution as an indicator in analysing the effects of toxicant and noises on the variation of population in evolution process. In present paper, we investigate a stochastic single-species model under regime switching in a polluted environment. By use of the ergodic of Markov chain and constructing Lyapunov function, the sufficient conditions for the positive recurrence and ergodic property are established, which imply the existence of stationary distribution of the model. Moreover, the mean and variance of marginal stationary distribution are estimated. Our analysis indicates that the coloured noise and toxicant may play an important role in determining the shape of stationary distribution and its statistics characteristics. Finally, numerical simulations are carried out to support our theoretical results.

Keywords: environmental pollution; regime switching diffusion; positive recurrence; ergodic property; statistics characteristics.

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1 Introduction

Nowadays, kinds of pollution (air, water and soil pollution), which are responsible for the environmental degradation, have become increasingly serious. Many species are on the verge of an extinction due to toxicant destroy their natural habitats and community structures. Thus, it is important to regulate toxicant suitable and make optimising policy to control ecological unbalance, and we can able to qualitatively estimate the survival risk of the species in a polluted environment. Moreover, the environmental fluctuation, (e.g., humidity, wind speed, climate variability) are also one of the important factors, which can inevitably affect the growth of populations. Just as May (1973) pointed out that owing to random noises, the growth rates in population systems should be stochastic, and as a result, the solution of the system will not tend to a steady positive point, but fluctuate around some average values. There are also experimental evidences (Carpenter et al., 2011) that environmental noises can play a key role in ecological systems. Given this, stochastic dynamical modelling, which is one of the powerful tools to investigate the effect of noise and toxicant on the population in a polluted environment, has attracted many researchers interests (Hallam et al., 1983; Hallam and Ma, 1986; Duan et al., 2004; Yang et al., 2007; Liu and Wang, 2009). On the other hand, due to the nature of Gaussian white noise, we need to consider another noise to reflect phenomena in ecosystems suffering switch abruptly to a contrasting alternative stable state (Scheffer et al., 2001). For example, an abrupt climate change, whether warming or cooling, wetting or drying, could have lasting and profound impacts on natural ecosystems (Scheffer and Carpenter, 2003). These discontinuous environmental factors may seriously affect the evolution of the ecosystem. To model these abrupt nature phenomenons in ecosystem is an interesting problem; recently, the stochastic population models with regime switching have become a research hotspot (Liu and Wang, 2010a, 2010b; Mao and Yuan, 2006; Settati and Lahrouz, 2014; Li and Yin, 2016). In particularly, Liu and Wang (2010a, 2010b) take into account both white noise and colour noise in the investigation of stochastic single-species model under regime switching in a polluted environment:

$$\begin{cases} dx(t) = x(t)\left[r_0(\gamma(t)) - r_1(\gamma(t))C_0(t) - \eta(\gamma(t))x(t)\right]dt + \sigma(\gamma(t))x(t)dB)(t) \\ \dfrac{dC_0(t)}{dt} = KC_e(t) - \dfrac{d_1\theta\beta}{k} - (g+m)C_0(t) \\ \dfrac{dC_e(t)}{dt} = hC_e(t) + u(t) \end{cases} \tag{1.1}$$

where $x(t)$ is the population density at time t, $C_0(t)$ is the concentration of toxicant in the organism at time t, and $C_e(t)$ is the concentration of toxicant in the environment at time t, $\gamma(t)$ is in a finite state space $\mathbb{S} = \{1, 2, ..., N\}$ with generator $\Gamma = (q_{ij})_{N \times N}$ given by

$$\mathcal{P} = \{\gamma(t+\Delta t) = j\gamma(t) = i\} = \begin{cases} q_{ij}\Delta t + o(\Delta t), \text{ if } j \neq i \\ 1 + q_{ii}\Delta t + o(\Delta t), \text{ f } j = i \end{cases} \tag{1.2}$$

where $\Delta > 0$. Here q_{ij} is the transition rate from state i to state j and $q_{ij} \geq 0$ if $i \neq j$ while $q_{ii} = -\sum_{i \neq j} q_{ij}$. The parameters are interpreted as follows: r0 is the intrinsic growth rate of the population without toxicant, r_1 is the dose response rate; η is the coefficient of intraspecific competition; k is the net organismal uptake rate of toxicant from the environment; d_1 is the uptake rate of toxicant in food per unit mass organism; θ is the concentration of toxicant in the resources; β is the average rate of food intake per unit mass organism; g is the net organismal excretion rate of toxicant; m is depuration rate of toxicant due to metabolic process and other losses; h is the total loss rate of the toxicant from the environment of the system; $u(t)$ is the exogenous total toxicant input at time t. In addition, σ is the intensity of the white noise in different regimes. $r_0(i)$, $r_1(i)$, $\eta(i) \geq 0$ and σ(i) > 0 for all $i \in \mathbb{S}$. $B(t)$ is a standard Brownian motion defined in complete probability space (Ω, $\mathcal{F}$, $\{\mathcal{F}_t\}_{t \geq 0}$, P)which is independent from Markov chain $\gamma(t)$. Note that model (1.1) is a three-dimensional hybrid system and $C_0(t)$ and $C_e(t)$ can be explicitly solved [as functions of $u(t)$] from the last two equations. We thus only need consider the following hybrid system

$$dx(t) = x(t)\left[r_0(\gamma(t)) - r_1(\gamma(t))C_0(t) - \eta(\gamma(t))x(t)\right]dt + \sigma(\gamma(t))x(t)dB)(t) \tag{1.3}$$

To keep it simple, we define

$$b(\gamma(t)) = r_0(\gamma(t)) - 0.5\sigma^2(\gamma(t)), \bar{b} = \sum_{i=1}^{N} b(i)\pi_i$$

$$\langle f \rangle = \frac{1}{t}\int_0^t f(s)ds, \quad (f)_* = \liminf_{t \to \infty} f(\gamma(t)), \quad (f)_* = \limsup_{t \to \infty} f(\gamma(t))$$

For model (1.3), the authors obtained the following threshold result:

Lemma 1.1: (See Liu and Wang , 2010a, 2010b) For model (1.3):

1 if $\bar{b} < \langle r_1 C_0 \rangle_*$, then the population $x(t)$ will go to extinction

2 if $\bar{b} = \langle r_1 C_0 \rangle_*$, then $x(t)$ will be stochastic non-persistent in the mean

3 if $\bar{b} > \langle r_1 C_0 \rangle_*$, then $x(t)$ will be stochastic weak persistent in the mean

4 if $\bar{b} > \langle r_1 C_0 \rangle^*$, then $x(t)$ will be stochastic strongly persistent in the mean

5 if $(r_0)_* - 0.5\,(\sigma^2) > (r_1 C_0)^*$, then $x(t)$ will be stochastically permanent.

The above survival analysis results reflect the asymptotic behaviours of sample paths; however, the long-term statistical characters of a stochastic system can be seen from the existence of its stationary distribution. Because of the complexities of multi-noise system, it is not easy to prove the existence of the stationary distribution. Recently, thanks to the theoretical technique proposed by Zhu and Yin (2007, 2009), there are some works focus on the positively recurrence and ergodic property, for example, Settati and Lahrouz (2014) established the sufficient conditions that ensuring the model is positive recurrent and showed the existence of a unique ergodic stationary distribution. Liu et al. (2013) investigated the ergodic property and positive recurrence of the model by stochastic Lyapunov functions under small perturbation. Li and Yin (2016) considered a stochastic logistic model with regime switching and proved the existence and uniqueness of stationary distribution. We also refer the readers to Zhao et al. (2016), Hu et al. (2014), Yang and Yin (2012) and the references therein in this respect.

To the best of our knowledge, there is rare result about the stationary distribution of the stochastic pollution-population model under regime switching (1.3). The main aim of this paper is to investigate the stationary distribution and ergodic of model (1.3). This paper is organised as follows. By use of ergodic of Markov chain and constructing Lyapunov function, in Section 2, the sufficient condition for the existence of stationary distribution as the main result of this paper is established, moreover, the mean and variance of marginal stationary distribution are estimated. In Section 3, some numerical simulations are carried out to illustrate our theoretical results. Finally, a brief discussion is given in Section 4.

2 Main results

First of all, let $\mathbb{R}_+ = \{x \in R: x > 0\}$, and $|x| = \sqrt{\sum_{i=1}^{n} x_1^2}$. Note that the Markov chain can be rewritten as $\gamma(t) = \gamma(0) + \sum_{n=1}^{\infty} Z_n I\,(\tau_n \le t)$ where τ_n, Z_n have the following conditional distributions. For given $\gamma(\tau_k) = i$, $\tau_k + 1 - \tau_k$ follows exponential distributed with the mean $\frac{1}{q_{ii}}$, and the jump $Z_k + 1 = \gamma(\tau_k + 1 - \tau_k)$ is independent of the past and has a probability of $-\frac{q_{ij}}{q_{ii}}$.

The Markov process $(x(t), \gamma(t)) \in \mathbb{R}_+^n \times \mathbb{S}$, consisting of a diffusion component $x(t)$ and a jump component $\gamma(t)$, can be described by

$$\begin{cases} dx(t) = f(x(t), (\gamma(t))dt + f(x(t), (\gamma(t))dB(t) \\ x(0) = x_0 \in \mathbb{R}_+^n, \xi(0) = \varpi \in \mathbb{S} \end{cases} \tag{2.1}$$

where $B(t)$ is a d-dimensional Brownian motion, and $f(\cdot,): \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$, $g(\cdot,): \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^{n \times d}$. For any twice continuously differentiable function $V(x, k) \in C^2(\mathbb{R}^n \times \mathbb{S})$, we can define linear operator $\mathcal{L}$ by

$$\begin{aligned} \mathcal{L}V(x,k) = \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}(x,i)\frac{\partial^2 V(x,i)}{\partial x_i \partial x_j} + \sum_{j=1}^{n} f_j(x,i)\frac{\partial V(x,i)}{\partial x_j} \\ + \sum_{j \ne k \in \mathbb{S}} q_{kj}(x)(V(x,j) - V(x,k)) \end{aligned} \tag{2.2}$$

where $a(x, i) = g(x, i)g^T(x, i)$ with the superscript T stands for the transpose of a matrix or vector.

According to the work of Zhu and Yin (2007), the existence of stationary distribution of (1.3) can be determined by the positive recurrent over some non-empty bounded open subset of $\mathbb{R}^n$, so we first give some fundamental results.

Lemma 2.1: (Zhu and Yin, 2007) (Theorem 3.13) If the following conditions are satisfied

1 for $u \neq v$, $q_{uv} > 0$, $u, v \in \mathbb{S}$

2 for each $k \in \mathbb{S}$, $\lambda|\xi|^2 \leq \xi^T a(x, k)\xi \leq \lambda^{-1}|\xi|^2$, for all $\xi \in$ R $\mathbb{R}^n$, with some constant $\lambda \in (0, 1]$ for all $x \in \mathbb{R}^n$

3 there exists a bounded open subset D of $\mathbb{R}^n$ with a regular, (i.e., smooth) boundary satisfying that, for each $k \in \mathbb{S}$ there exists a non-negative function $V(\cdot, k)$: $\mathcal{D}^C \to \mathbb{R}$ such that $V(\cdot, k)$ is twice continuously differentiable and that for some $\varsigma > 0$,

$$\mathcal{L}V(x, k) \leq -\varsigma, \text{ for any } (x, k) \in \mathcal{D}^C \times \mathbb{S}$$

then the solution process $(x(t), \xi(t))$ of model (2.1) is positive recurrent, i.e., there exists a unique stationary distribution $\nu(\cdot,\cdot) = (\nu(\cdot, i) : i \in \mathbb{S})$ of $((x(t), \xi(t))$.

Lemma 2.2: For model (1.3), we have $\limsup_{t \to \infty} E[x^p(t)] \leq L(p)$.

In fact, we can obtain the following result by solving the second and third equations of model (1.1):

Lemma 2.3: For model (1.1), if $0 < k + \dfrac{d_1\theta\beta}{k} < g + m, \max\limits_{t\in\mathbb{R}_+} u(t) < h,$ then $0 \leq C_0(t) < 1$, $0 \leq C_e(t) < 1$, for any $t \in \mathbb{R}_+$

From now on, we impose $0 < k + \dfrac{d_1\theta\beta}{k} < g + m, \max\limits_{t\in\mathbb{R}_+} u(t) \leq h,$ on model (1.1) or (1.3).We then give the following two assumptions

Assumption 1: $q_{uv} > 0$, $u \neq v$, $u, v \in \mathbb{S}$ and

Assumption 2: $\Phi = \sum\limits_{i=1}^{N} \pi_i \left\{ r_0(i) - r_1(i) - \dfrac{\sigma^2(i)}{2} = \dfrac{[r_0(i) - r_1(i) - \varepsilon\eta(i)]^2}{4\varepsilon\eta(i)} \right\} > 0$

where Markov chain has a unique stationary distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_N)$, which is the solution of the system of linear equation $\pi\Gamma = 0$ subject to $\sum_{i=1}^{N} \pi_i = 1$, $(\pi_i > 0)$ and $\in$ is a sufficiently small positive number satisfying $0 < \in \leq \dfrac{\max\limits_{i\in\mathbb{S}} r_0(i) - \min\limits_{i\in\mathbb{S}} r_1(i) - 4\min\limits_{i\in\mathbb{S}} \sqrt{\eta(i)}}{\min\limits_{i\in\mathbb{S}} \eta(i)}$.

Remark 2.1: Note that the Assumption 1 implies that the Markov chain is irreducible. Moreover, we can see that $\gamma(t)$ with finite state $\mathbb{S}$ is ergodic.

Next, we can prove

Theorem 2.1: If Assumptions 1–2 hold, then for any initial value $(x(0), \gamma(0)) \in \mathbb{R}_+ \times \mathbb{S}$, the solution process $x(t)$ of model (1.3) is positive recurrent and has a unique ergodic stationary distribution.

Proof: From Lemma 2.1, we only need to verify conditions (1)–(3) hold to prove Theorem 2.1. Assumption 1 implies that condition (1) in Lemma 2.1 holds. Next, the diffusion matrix of model (1.3) $a(x, k) = \sigma^2(k) > 0$, which can verify that condition (2) in Lemma 2.1 holds. Now, we are in a position to prove condition (3). Define a C^2-function

$$V(x,k) = x - 1 - \in \ln x + \in (\bar{\omega}_k + \bar{\omega}) = V_1(x) + V_2(x) \tag{2.3}$$

where $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_N)^T$, and $|\bar{\omega}| = \sqrt{\bar{\omega}_1^2 + \bar{\omega}_2^2 + \cdots + \bar{\omega}_N^2}$, and $\bar{\omega}_k$, $i = 1, \ldots, N$ are to be determined later. $|\bar{\omega}|$ is used to guarantee $\bar{\omega}_k + |\bar{\omega}|$ is non-negative. Making use of the generalised Itô formula (2.2), we have

$$\begin{aligned}
\mathcal{L}V_1(x - \in) &= [r_0(k) - r_1(k)C_0 - \eta(k)x]\frac{\in}{2}\sigma^2(k) \\
&= \eta(k)x^2 + [r_0(k) - r_1(k)C_0 - \in \eta(k)]x - \left[r_0(k) - r_1(k)C_0 - \frac{1}{2}\sigma^2(k)\right]\in \\
&\leq -\eta(k)\left\{x - \frac{[r_0(k) - r_1(k)C_0 - \in \eta(k)]}{2\eta(k)}\right\}^2 \\
&\quad - \in \left\{r_0(k) - r_1(k) - \frac{1}{2}\sigma^2(k) - \frac{[r_0(k) - r_1(k)C_0 - \in \eta(k)]^2}{4 \in \eta(k)}\right\}
\end{aligned} \tag{2.4}$$

and

$$\mathcal{L}V_2(k) = \in \sum_{j \neq k \in \mathbb{S}} q_{kj}(\bar{\omega}_j - \bar{\omega}_k) \tag{2.5}$$

Next, we define a vector $C = (C_1, C_2, \ldots, C_N)^T$ with $C_k = r_0(k) - r_1(k) - \frac{1}{2}\sigma^2(k) - \frac{[r_0(k) - r_1(k)C_0 - \in \eta(k)]^2}{4 \in \eta(k)}$. It follows from the irreducibility of generator matrix Γ and Lemma 2.3 in Khasminskii et al. (2007), for C_k there exists a solution $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_N)^T$, for the following Poisson system:

$$\Gamma\bar{\omega}_k = C_k - \left(-\sum_{i=1}^{N}\pi_i C_i\right)e = \Phi e, 1 \leq k \leq N \tag{2.6}$$

where e is the column vector with all its entries equal to 1. So we can obtain

$$\sum_{j \neq k \in \mathbb{S}} q_{kj}(\bar{\omega}_j - \bar{\omega}_k) - \left[\begin{array}{c} r_0(k) - r_1(k) - \frac{1}{2}\sigma^2(k) \\ -\frac{[r_0(k) - r_1(k)C_0 - \in \eta(k)]^2}{4 \in \eta(k)} \end{array}\right] = -\Phi \tag{2.7}$$

Combining (2.4), (2.5), (2.6) and (2.7) results in

$$\mathcal{L}V(x,k) \leq -\eta(k)\left\{x - \frac{[r_0(k) - r_1(k) - \in \eta(k)]}{2\eta(k)}\right\}^2 - \in \Phi \triangleq \Lambda(x,k) \tag{2.8}$$

Case 1: if $x \to \infty$, then we have $\Lambda(x, k) \to -\infty$.

Case 2: if $x \to 0$, then we have $\Lambda(x, k) \to -\dfrac{[r_0(k) - r_1(k) - \in \eta(k)]^2}{4\eta(k)} - \in \Phi < -1.$

Consequently, there exists a sufficiently large κ and let $D = \left[\dfrac{1}{\kappa}, \kappa\right]$, then we can obtain that $\mathcal{L}V(x, k) \leq -1$ for any $(x, k) \in D^c \times \mathbb{S}$. It therefore follows from Lemma 2.1 that the solution process of model (1.3) is positive recurrent with respect to the domain D and model (1.3) has a unique stationary distribution $\mu(\cdot,)$ with ergodic property.

Remark 2.2: Theorem 2.1 implies that under some reasonable conditions the population density will fluctuate in some non-empty domain, and the statistical character of model may show that it has a unique stationary distribution with ergodic property, which observed the evolution of the population from another point of view.

Now, we are in a position to establish the statistical characteristics (the mean and variance) of the marginal stationary distribution.

Theorem 2.2: Under assumptions of Theorem 2.1, we have

$$\sum_{k\in\mathbb{S}} \eta(k) \int_{\mathbb{R}_+} x\mu(dx, k) = \sum_{k\in\mathbb{S}} \eta(k) \left[r_0(k) - \frac{1}{2}\sigma^2(k) \right] - \lim_{t\to\infty} \langle r_1 C_0 \rangle \tag{2.9}$$

Proof: Form Lemma 2.2, we know that for any $p > 0$, there exists a positive constant $L(p)$ such that $\limsup_{t\to\infty} \mathbb{E}|x(t)|^p \leq L(p)$. According to the ergodic property of $(x(t), \gamma(t))$, for any $m > 0$, we can obtain $\mathcal{P} = \left\{ \lim_{t\to\infty} \frac{1}{t} \int_0^t (x(s) \wedge m) ds = \sum_{k\in\mathbb{S}} \int_{\mathbb{R}_+} (x \wedge m) \mu(x, k) dx \right\} = 1.$ By using the dominated convergence theorem, we have $\mathbb{E} = \left\{ \lim_{t\to\infty} \frac{1}{t} \int_0^t (x(s) \wedge m) ds \right\} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E}(x(s) \wedge m) ds = \sum_{k\in\mathbb{S}} \int_{\mathbb{R}_+} (x \wedge m) \mu(x, k) dx.$ Note that $\sum_{k\in\mathbb{S}} \int_{\mathbb{R}_+} x\mu(x, k) dx = \lim_{t\to\infty} \sum_{k\in\mathbb{S}} \int_{\mathbb{R}_+} (x \wedge m) \mu(x, k) dx \leq \limsup_{t\to\infty} \mathbb{E}|x(t)| < \infty,$ which means that the function $f(x) = x$ is integrable with respect to measure μ.

Define $V_3 = \ln x$, we can obtain

$$\begin{aligned} \frac{1}{t}\ln x(t) &= \frac{1}{t}\ln x(0) + \frac{1}{t}\int_0^t \left[r_0(\gamma(s)) - \frac{1}{2}\sigma^2(\gamma(s)) \right] ds \\ &- \frac{1}{t}\int_0^t r_1(\gamma(s)) C_0 s(ds) - \frac{1}{t}\int_0^t \eta(\gamma(s)) x(s) ds + \frac{1}{t}\int_0^t \sigma(\gamma(s)) dB(s) \end{aligned} \tag{2.10}$$

Since $\int_0^t \sigma(\gamma(s)) dB(s)$ is a real-valued continuous local martingales, whose quadratic variations are $\left\langle \int_0^t \sigma(\gamma(s)) dB(s), \int_0^t \sigma(\gamma(s)) dB(s) \right\rangle_t = \int_0^t \sigma(\gamma(s)) ds \leq \hat{\sigma}^2 t < \infty.$ Make use of strong law of large numbers for local martingales (Lipster, 1980) results in $\lim_{t\to\infty} \int_0^t \sigma(\gamma(s)) dB(s) = 0.$

It follows from the ergodic property of $(x(t), \gamma(t))$ that $\lim_{t\to\infty} \frac{\ln x(t)}{t} = 0$, a.s. Otherwise, if $\lim_{t\to\infty} \frac{\ln x(t)}{t} > 0$, a.s. or $\lim_{t\to\infty} \frac{\ln x(t)}{t} < 0$, a.s., we can know that $\lim_{t\to\infty} x(t) = +\infty$ or $\lim_{t\to} x(t) = 0$, which contradicts with the fact that its stationary density lies in R+. Thus, we have $\sum_{k\in\mathbb{S}} \eta(k)\int_{\mathbb{R}_+} x\mu(dx,k) = \sum_{k\in\mathbb{S}} \pi(k)\left[r_0(k) - \frac{1}{2}\sigma^2(k)\right] - \lim_{t\to\infty}\langle r_1 C_0\rangle.$

Remark 2.3: Theorem 2.2 can provide some useful information of the variation of the population in evolution process. In addition, we can see that the mean toxicant concentration is detrimental to the survival level of population.

Remark 2.4: Theorems 2.1–2.2 are an extension of the related works presented in Liu and Wang (2010a), where the existence of stationary distribution is not discussed.

3 Numerical simulations

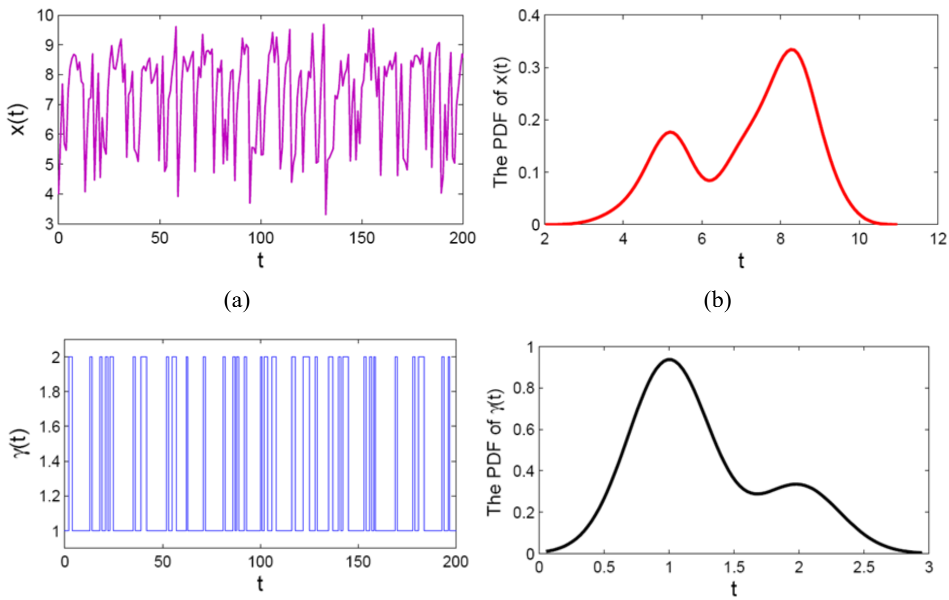
In this section, we shall carry out some numerical simulations to illustrate the different switching regimes (Markov chain with different generators Q_{ij}, in practice, it can be estimated from the observed data (Flavia et al., 2014; Yasunari, 2006) play an important role in determining the large time evolution behaviours of model (1.3).

Example 3.1: Let us assume that Markov chain $\gamma(t)$ is in the state space $\mathbb{S} = \{1, 2\}$ with the generator $\Gamma_1 = \begin{pmatrix} -3 & 3 \\ 7 & -7 \end{pmatrix}$. We can get the stationary distribution of $\gamma(t)$ is $\pi^{(1)} = (0.7, 0.3)$. We set $C_0(t) = 0.3 + 0.1\,\frac{\sin t}{t}$, $r_0(1) = 1.1$, $r_1(1) = 0.3$, $\eta(1) = 0.12$, $\sigma(1) = 0.06$, $r_0(2) = 0.85$, $r_1(2) = 0.2$, $\eta(2) = 0.15$, $\sigma(2) = 0.05$. We can calculate that $\Phi = 0.1344 > 0$. It follows from Theorem 2.1 that the population density $x(t)$ has a unique ergodic stationary distribution, Figure 1 confirms this.

To highlight the effect of state space of Markov chain on the distribution of $x(t)$, we perform the following another example.

Example 3.2: Assume that Markov chain $\gamma(t)$ is in the state space $\mathbb{S} = \{1, 2, 3\}$ with the generator $\Gamma_2 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -3 & 2 \\ 1 & 2 & -3 \end{pmatrix}$, here states 1, 2, 3 can represent the sunny, rain and cloudy days in a year. We can calculate the stationary distribution of $\gamma(t)$ is $\pi^{(2)} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. We set $C_0(t) = 0.3 + 0.1\,\frac{\sin t}{t}$, $r_0(1) = 1.3$, $r_1(1) = 0.35$, $\eta(1) = 0.16$, $\sigma(1) = 0.01$, $r_0(2) = 1.15$, $r_1(2) = 0.25$, $\eta(2) = 0.15$, $\sigma(2) = 0.02$, $r_0(3) = 1$, $r_1(3) = 0.11$, $\eta(3) = 0.14$, $\sigma(3) = 0.015$. By direct calculation, we can get $\Phi = 0.063 > 0$. It then follows from Theorem 2.1 that $x(t)$ has a unique ergodic stationary distribution, which consistent with Figure 2. In addition, we can estimate the mean and variance of $x(t)$ are 6.1988 and 5.4032 respectively, which show that the variation of the population density is significant.

Figure 1 Panels, (a) sample trajectory of $x(t)$ with initial value $x(0) = 4$ (b) probability density function of solution process of $x(t)$ (c) sample trajectory of Markov chain $\gamma(t)$ (d) probability density function of $\gamma(t)$ (see online version for colours)

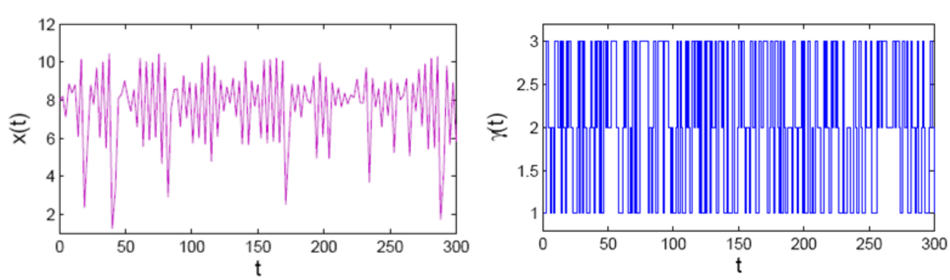


(a) (b)

(c) (d)

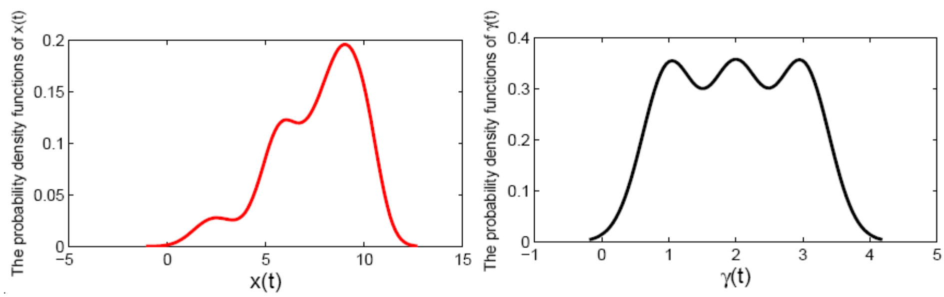
It's worth noting that the stationary distribution of $x(t)$ has a different peak pattern: $x(t)$ is distributing in two peaks of Figure 1 and that of is in a triple peaks of Figure 2. This may be due to the number of state space of Marked chain is different. Thus, we can draw the conclusion form above examples that the Markov chain exert great influence on the shape of stationary distribution of $x(t)$ in a polluted environment.

Figure 2 Panels, (a) sample trajectory of $x(t)$ with initial value $x(0) = 8$ (b) probability density function of solution process of $x(t)$ (c) sample trajectory of Markov chain $\gamma(t)$ (d) probability density function of $\gamma(t)$ (see online version for colours)



(a) (b)

Figure 2 Panels, (a) sample trajectory of $x(t)$ with initial value $x(0) = 8$ (b) probability density function of solution process of $x(t)$ (c) sample trajectory of Markov chain $\gamma(t)$ (d) probability density function of $\gamma(t)$ (continued) (see online version for colours)



(c) (d)

4 Discussion

The ecological problems of environmental fluctuation of toxicant-population interaction have attracted many researchers' attention recently (Liess, 2002). It is therefore a meaningful topic to investigate the combined effect of toxicant and environmental fluctuation on the evolution of population. In this paper, we addressed the statistical characteristic of the population density $x(t)$ in a polluted environment. We prove the existence of unique ergodic stationary distribution under some conditions and estimated the mean and variance of marginal stationary distribution. Comparing with the existing works, we further extended the survival results (see Lemma 1.1) from a statistical viewpoint, more precisely, the stationary distribution (see Theorem 2.1) and its statistical characteristic (see Theorem 2.2) can reflect more information, which is an important measure of evolution process for the stochastic models due to the uncertainty. Also, it is difficult to directly prove the existence of stationary distribution of model (1.3), we consider the limit system, which can show the limit dynamic behaviours, to illustrate the effect of environmental fluctuation and toxicant on the statistical characteristic of model (1.3). Our theoretical and numerical simulation result has shown that:

1 As mentioned above, stationary distribution is one of the most important statistical properties in ecotoxicology, our results suggested that the Markov chain may affect the shape of the stationary distribution significantly. More precisely, the state number of the generator of Markov chain is corresponding to that of peaks of stationary distribution for the population density $x(t)$ (see Figures 1–2).

2 The Markov chain can suppress the extinction of the species. In panel (d) of Figure 2, the stationary distribution of Markov chain $\gamma(t)$ implies that the species have a equality opportunity to live in the three state space, however, due to the balance effect of Markov chain may reduce the extinction risk in 'bad' environment, the survival in total population scale exists [see Panel (a) in Figure 2].

3 The pollution may affect the fate of the population, the final size of the population $x(t)$ is also determined by the limit mean of toxicant concentration and dose-response rate $\lim_{t\to\infty} \langle r_1 C_0 \rangle$, which is negative association with the mean of population density [see (2.12)].

Thus, our result may provide some formulas to estimate the statistical characteristics of population density in evolution process and access the effect of toxicant on the survival level of population. In future, one may consider a multi-species toxicant-population interaction model under regime-switching and discuss the statistical characteristic, or the random disturbance of the toxicant (Wei et al., 2017).

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