A fuzzy differential approach to a two plants production-recycling-disposal inventory problem via genetic algorithms

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Abstract: This paper develops a production, recycling-disposal inventory problem over a finite time horizon. The production and recycling process are performed in separate plants which are located very near to the market. The products are continuously transferred to the market. Here, the dynamic demand is satisfied by production and recycling. Recycling products can be used as new products which are sold again in the market. The rate of production, recycling and disposal are assumed to be function of time. The setup cost, idle cost and environment pollution recovery cost for production-recycling system in industry are also included. Model is formulated using fuzzy differential equation. Two different approaches are used in this model as: 1) modified graded mean integration value (MGMIV); 2) fuzzy preference ordering of intervals (FPOI). A genetic algorithm with binary mode representation, roulette wheel selection and random mutation process are applied. The optimum results are presented in tabular form and graphically.
**Keywords:** fuzzy differential equation; production; recycling-disposal; idle cost; environment protection cost.


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1 **Introduction**

During the last few decades, production-recycling system is an important area of inventory studies due to growing environmental concern and environmental regulations like ‘Kyoto Protocol’in industry. Figure 1 represents a two-plant production-recycling-disposal system. Some units are bought back from market for a recoverable inventory after used by the customers. The serviceable stock is delivered to the customer demand. The serviceable stock can be satisfied by either production or by recycling. The non-recycling items are disposed. The non-serviceable stock is filled up by used products from the customers. The non-serviceable stock is supplied for either recycling or disposal. A number of research papers have already been published on the above type of models by several researchers (cf., Minner and Kleber, 2001; Dobos and
Richter, 2006; Maity et al., 2009; Ilgin and Gupta, 2010; Taleizadeh et al., 2012; and others).

In the classical inventory models, normally static lot size models are formulated. But in the manufacturing environment, the static models are not adequate in analysing the behaviour of such systems and in deriving the policies for their control. Moreover it is usually observed in the market that sales of the fashionable goods, electronic gadgets, seasonable products, food grains, etc., increase with time. For these reasons, dynamic models of production inventory systems have been considered and solved by some researchers (cf., Giri and Chaudhuri, 1998; Maity and Maiti, 2005a; and others). In these models, demand is assumed to be continuous function of time and till now, only a very few researchers (cf., Maity and Maiti, 2005b; Jana et al., 2013) have taken time dependent production function.

**Figure 1** Inventory label for production, recycling and disposal model (see online version for colours)

Here, production cost has four parts.

1. raw material cost which is constant per unit product
2. labour cost which is inversely proportional per unit product
3. wear and tear cost which is proportional to the amount of unit production
4. environmental pollution cost which is proportional to the product.

The last cost is the expenditure due to growing environmental concern (cf., Zhang et al., 1997; Gungor and Gupta, 1999; Ilgin and Gupta, 2010) and according to the rule of environmental regulations like ‘Kyoto Protocol’ for Industry. The ‘Kyoto Protocol’ was established in 1997 at Kyoto, Japan. The purpose of ‘Kyoto Protocol’ are

1. clean development mechanism
2. scientific efforts aimed at understanding detail of total net carbon exchange
3. global-warming potential
4. campus carbon neutrality
5. carbon sequestration in terrestrial ecosystems, etc.
So environmental pollution cost is an important part in production system for industry. Also the recycling process can reduce the environmental pollution, save the crisis of natural resources and raw materials. Therefore, the recycling process can play an important role in industry.

Zadeh (1965) first applied a new concept ‘fuzzy set theory’ to accommodate the uncertainty in non-stochastic sense. After that, Liu and Liu (2003) have developed a class of fuzzy random optimisation: expected value models. Maity et al. (2008) have developed a production recycling model with imprecise holding cost of defective units which is a natural phenomenon in a production process. Bi-fuzzy sets were originally presented by Zadeh (1971) and were further elaborately by Gottwald (1979), Mendel (2002) and Marusak (2009). Till now, none has considered a two-plant optimal production-inventory system producing random defective units of an item with a fuzzy resource constraint and some imprecise inventory costs via optimal control problem.

Genetic algorithms (GAs) are exhaustive search algorithms based on the mechanics of natural selection and genesis (crossover, mutation, etc.) (cf., Goldberg, 1989; Maiti and Maiti, 2007; Jana et al., 2012; and others). Because of its generality, it has been successfully applied to many optimisation problems, for its several advantages over conventional optimisation methods. Holland was inspired by Darwin’s theory about evolution and constructed GAs based upon the fundamental principle of the theory: ‘survival of the fittest’. Some several researchers have applied GA for the optimal solutions of different non-linear models.

Shortcomings in all these studies are summarised below:

- Most of these models considered constant production/ recycling rates and constant production cost. But in many production systems, production cost decreases with increase of production rate (Khouja, 1995; Maiti and Maiti, 2005). None of the above production-recycling models considered production and recycling rates as decision variables.

- Amount of returned units mainly depends on its used units, which again depends on demand. Omar and Yeo (2009) first considered return rate as constant proportion of demand of the item. In reality, demand of an item varies of time. Hence, amount of used units of an item in the society varies over the time. Collection of used units depends on the effect of agents used for this purpose and is not constant, rather is uncertain. Thus it can be concluded that, though return rate depends on demand, it is actually imprecise in nature, which is not considered by any author.

- None of the production-recycling model considered time dependent demand rate which is applicable in most of the real life products.

- Though some authors developed remanufacturing inventory models in fuzzy environment (Roy et al., 2009), none has solved the model using fuzzy differential equation. This approach is required for more accuracy of optimal decision for fuzzy inventory models.

- No researcher consider considered the environmental pollution control costs in production.
In this paper, the production and disposal rates are functions of time and unknown which are taken as control variables. Moreover, the recycling rate is unknown constant and a control variable. Here, production cost is greater than the recycling cost. Also non-serviceable holding cost is less than serviceable holding cost. The total cost is minimised and solved by GA technique. None has considered a two-plants production, recycling-disposal inventory problem with fuzzy differential equation approach. The optimum production, recycling-disposal and stock levels are determined for known dynamic demand function. The model is illustrated through numerical examples and results are also presented graphically.

The imprecise production-inventory-recycling system is formulated as a cost minimisation using fuzzy differential equation and solved using GA with RWS arithmetic, crossover, and random mutation. The present problem can be applicable for the seasonable products with fixed time horizon and can be extended to include random planning horizon. The model can also be formulated with bulk release from plant-1 to plant-2. Here for simplicity fuzzy parameter λ is taken as triangular fuzzy number (TFN) type. But any type of estimation of λ for which α-cut is available, is also applicable for this model. The model presented here is quite realistic and can be used for production-inventory-recycling practitioners.

2 Preliminaries about uncertainty

Uncertainty: There are mainly two types of uncertainty- stochastic and non-stochastic uncertainties. If the sufficient data are not available, then uncertainty can be represented as a random variable. For insufficient data, uncertainty can be expressed by fuzzy variable. Zadeh (1965) represented the fuzzy set theory these imprecise data.

2.1 Zadeh’s extension principle

Let \( \tilde{a} \) and \( \tilde{b} \) be two fuzzy numbers in \( \mathbb{R} \) with membership functions \( \mu_{\tilde{a}}(x) \) and \( \mu_{\tilde{b}}(x) \) respectively, and \( \tilde{c} = f(\tilde{a}, \tilde{b}) \), where \( f: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a binary operation, then according to Zadeh (1978), membership function \( \mu_{\tilde{c}} \) of \( \tilde{c} \) is defined as

\[
\forall z \in \mathbb{R},
\mu_{\tilde{c}}(z) = \sup\{\min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)) \mid x, y \in \mathbb{R} \text{ and } z = f(x, y)\}.
\]

2.1.1 \( \alpha \)-cut of fuzzy number

Let \( \tilde{a} \) be a fuzzy number with membership function \( \mu_{\tilde{a}}(x) \) then its \( \alpha \)-cut is denoted by \( \tilde{a}[\alpha] = [a_L(\alpha), a_R(\alpha)] \), and is defined as

\[
a_L(\alpha) = \inf\{x \in \mathbb{R} : \mu_{\tilde{a}}(x) \geq \alpha\}
\]
\[
a_R(\alpha) = \sup\{x \in \mathbb{R} : \mu_{\tilde{a}}(x) \geq \alpha\}.
\]
2.1.2 Interval number

An interval number is an interval having only the bounds of the parameter concerned. It is denoted by $A$ and defined as

$$A = [a_L, a_R] = \{x : a_L \leq x \leq a_R, x \in \mathbb{R}\},$$

where $a_L, a_R$ are the left and right limit of the interval $A$ on the real line $\mathbb{R}$ respectively. Alternatively, it is represented as

$$A = < m(A), w(A) >,$$

where $m(A) = (a_L + a_R)/2$ and $w(A) = (a_R - a_L)/2$, the mid point and half width of $A$ respectively.

2.2 Fuzzy preference ordering of interval

Any pair of intervals $A = [a_L, a_R] = < m(A), w(A) >$ and $B = [b_L, b_R] = < m(B), w(B) >$ are ordered as $(A, B)$ if $m(A) \leq m(B)$ and classified (Sengupta and Pal, 2009) into two sets $S_1, S_2$ as follows:

1. $(A, B) \in S_1$ if $w(A) \geq w(B)$
2. $(A, B) \in S_2$ if $w(A) < w(B)$.

Against maximisation problem, for 1, $B$ is always the best choice relative to $A$ and for 2, fuzzy preference may be constructed.

Let a set of intervals $(X, B) \in S_2$ where $X = [x_L, x_R] = < m(X), w(X) >$ is a variable interval with changing mean and width. For fuzzy preference, a fuzzy set $B'$ as the rejection of $B$ compared to $X$ is defined as

$$B' = \{(X, B) \in S_2 : m(X) \leq m(B), w(X) < w(B)\}$$

with membership function

$$\mu_{B'}(X, B) = \begin{cases} 
1 & \text{for } m(X) = m(B), \\
\max \{0, \frac{m(X) - (b_L + w(X))}{m(B) - (b_L + w(X))}\} & \text{for } m(B) \geq m(X) \geq (b_L + w(X)), \\
0 & \text{otherwise.}
\end{cases}$$

Similarly, an another fuzzy set, $X'$, rejection of $X$ compared to $B$ in $S_2$ is defined as the complement of $B'$, i.e.,

$$\mu_{X'}(X, B) = 1 - \mu_{B'}(X, B).$$

Therefore $X' \cap B'$ is the set of non-dominated preferences for some pair of intervals $(X, B) \in S_2$ having membership function

$$\mu_{X' \cap B'}(X, B) = \min\{\mu_{X'}(X, B), \mu_{B'}(X, B)\}.$$

Hance a DM may choose the pair of intervals $(X, B)$ as the pair of non-dominated preferences if $\mu_{X' \cap B'}(X, B)$ is approximately equal to its supremum.
2.3 Fuzzy differential equation

General form of first order ordinary differential equation is

$$\frac{dy}{dt} = f(t, y, k), \quad y(0) = c,$$  \hspace{1cm} (2)

where $y$ is dependent variable, $t$ is independent variable, defined on $I$ which contains zero and $k = (k_1, ..., k_n)$ is a vector of constants. Let $\tilde{k} = (\tilde{k}_1, ..., \tilde{k}_n)$ be a vector of TFNs and $\tilde{c}$ be another TFN, then (2) reduces to

$$\frac{d\tilde{y}}{dt} = f(t, \tilde{y}, \tilde{k}), \quad \tilde{y}(0) = \tilde{c},$$  \hspace{1cm} (3)

which is a fuzzy differential equation. According to Buckley and Feuring (2000), to obtain solution of (3), at first find a solution $y = g(t, k, c)$ of (2). Then find

$$\tilde{y}(t) = g(t, \tilde{k}, \tilde{c})$$  \hspace{1cm} (4)

using (1), and its $\alpha$-cut: $\tilde{y}(t)[\alpha] = [y_L(t, \alpha), y_R(t, \alpha)]$ for each $t \in I$ and $\alpha \in (0, 1)$. If the following conditions are satisfied then (4) is the solution of (3).

- $y'_L(t, \alpha)$ and $y'_R(t, \alpha)$ are continuous on $I \times [0, 1]$
- $y'_L(t, \alpha)$ is an increasing function of $\alpha$ for each $t \in I$
- $y'_R(t, \alpha)$ is a decreasing function of $\alpha$ for each $t \in I$
- $y'_L(t, 1) \leq y'_R(t, 1)$ for all $t \in I$.

2.4 Fuzzy Riemann integral

Let $\tilde{f}(x)$ be a closed and bounded fuzzy valued function on $[a, b]$, having $\alpha$-cut $[f_L(x, \alpha), f_R(x, \alpha)]$. According to Wu (2000) if $f_L(x, \alpha)$ and $f_R(x, \alpha)$ are Riemann integrable on $[a,b]$, $\forall \alpha$ then $\tilde{f}(x)$ is fuzzy Riemann integrable on $[a,b]$, and the fuzzy Riemann integral $\int_a^b \tilde{f}(x)dx$ is a closed fuzzy number. Furthermore, the $\alpha$ level set of $\int_a^b \tilde{f}(x)dx$ is given by

$$\left( \int_a^b \tilde{f}(x)dx \right)[\alpha] = \left[ \int_a^b f_L(x, \alpha)dx, \int_a^b f_R(x, \alpha)dx \right]$$  \hspace{1cm} (5)

2.5 Modified graded mean integration representation

If $\tilde{A}$ be a fuzzy number with $\alpha$-cut $\tilde{A}[\alpha] = [A_L(\alpha), A_R(\alpha)]$, then modified graded mean integration value (Chen and Hsieh, 2000) of the fuzzy number $\tilde{A}$ is denoted by $P_k(\tilde{A})$ and is defined as

$$P_k(\tilde{A}) = \frac{\int_0^1 \alpha [kA_L(\alpha) + (1-k)A_R(\alpha)] d\alpha}{\int_0^1 \alpha d\alpha}$$  \hspace{1cm} (6)
where $k \in [0, 1]$ is the DM’s attitude or pessimism parameter. This method is also known as k-preference integration representation.

3 Assumptions and notations

The model under consideration is developed with the following assumptions and notations.

3.1 Assumptions

For the product recycling, disposal model, it is assumed that,

1. demand rate is known and time dependent
2. shortages are not allowed
3. production rate is time dependent and unknown taken as decision variable
4. this is a single item inventory model with finite time length
5. recycling item is same to a new product it is rate is constant and unknown taken as decision variable
6. disposal item is rejected unit, it is rate is constant and unknown taken as decision variable
7. lead time is zero
8. all return units have the same level of quality
9. holding cost of non-serviceable item is less than that for serviceable product
10. holding cost of non-serviceable item is less than that for serviceable product
11. unit production cost is less than the unit recycling cost
12. environment pollution cost for production in industry is also included
13. there are $m$ cycles for production and $n$ cycles for recycling.

3.2 Notations

$x_{Si}(t)$ serviceable stock at time $t$ for $i^{th}$ production cycle
$x_{Sj}(t)$ serviceable stock units at time $t$ for $j^{th}$ production and recycling cycle
$x_{R}(t)$ stock of non serviceable units at time $t$ for production cycles
$x_{Rj}(t)$ stock of non serviceable at time $t$ for $j^{th}$ production and recycling cycle
$u(t)$ $u_0 + u_1 t$, production rate(decision variable) for each production up to $m$ cycles
4 Production-inventory-recycling-disposal model formulation

This paper develops a two-plants production-inventory-recycling-disposal system over a finite time horizon. The production and recycling processes are performed in plants which are located very near to the market and the products of plants are continuously transferred to the market. One plant for production and the other for the recycling. Here, the dynamic demand is satisfied by production and recycling. The used units are bought back and then either recycled or disposed in the said plant. The used units are collected continuously from the customers. Recycling products can be used as new products which are sold again. The rate of production, recycling and disposal are assumed to be function of time. The setup cost, idle cost and environment pollution cost for production in industry are also included. The production cost has four parts.

1 raw material cost which is constant per unit product
2 labour cost which is inversely proportional per unit product
3 wear and tear cost
environmental pollution cost is proportional to the product. The last cost is expenditure due to growing environmental concern and according to the rule of environmental regulations like ‘Kyoto Protocol’ for Industry. At the beginning, production satisfies the demand. After sometime, production and recycling fill up the demand. The first m cycles are presented for production and next n cycles exist both for production and recycling. The period of each of first m cycles and last n cycles are \( t_u \) and \( t_p \) respectively. The time interval of each of first m cycles is equal. Similarly the time interval of each of last n cycles is equal. Production takes \( t_{ui} \) duration in \( i^{th} \), \( i = 1, 2, \cdots, m \) production cycle. Also production and recycling takes \( t_{pj} \) duration in \( j^{th}, j = 1, 2, \cdots, n \) production and recycling cycle. We collect reused products at the rate of \((\alpha_0 + \alpha_1 t) d(t)\) continuously from the market. At the time of collection we also consider the disposal at the rate of \((z_0 + z_1 t)\). The total time horizon \( T = mt_u + nt_p \). The optimisation problem is to minimisation total cost over the finite planning horizon, \( T \) and it is given in Figure 1.

In plant-I, the cost function \( J_1 \) is given below:

\[
J_1 = \text{production costs (PC1) + holding costs for serviceable stocks (HC1)} \\
+ \text{set up costs(SC1) + idle costs(ID1)} \\
\]

\[
= \sum_{i=1}^{m} \int_{(i-1)t_u}^{it_u} C_u u(t) dt + \sum_{i=1}^{m} \int_{(i-1)t_u}^{it_u} h_S x_{S_1}(t) dt + mS_u \\
+ \sum_{i=1}^{m} (t_u - t_{ui}^{i}) I_d u + \sum_{j=1}^{n} \int_{mt_u + (j-1)t_p}^{mt_u + jt_p} C_u u'(t) dt \\
+ \sum_{j=1}^{n} \int_{mt_u + (j-1)t_p}^{mt_u + jt_p} h_S x_{S_2}(t) dt + nS_p + \sum_{j=1}^{n} (t_p - t_{pj}^{j}) I_d p 
\] (7)

In plant-II, the cost function \( J_2 \) is given below:

\[
J_2 = \text{holding costs for NS stocks (HC2) +recycling cost (RC2)} \\
+ \text{collect cost (CC2) + disposal cost (DC2)} \\
\]

\[
= \sum_{i=1}^{m} \int_{(i-1)t_u}^{it_u} h_R x_{R_1}(t) dt + \sum_{j=1}^{n} \int_{mt_u + (j-1)t_p}^{mt_u + jt_p} h_R x_{R_2}(t) dt \\
+ \sum_{j=1}^{n} \int_{mt_u + (j-1)t_p}^{mt_u + jt_p} C_p p(t) dt + \int_{0}^{T} \left[C_r (\alpha_0 + \alpha_1 t) d(t) + C_z z(t)\right] dt 
\] (8)

subject to

\[
\frac{dx_{S_2}(t)}{dt} = \left\{ \begin{array}{ll}
  u(t) - d(t) & \text{if} \quad (i-1)t_u \leq t \leq (i-1)t_u + t_{ui}' \\
  -d(t) & \text{if} \quad (i-1)t_u + t_{ui}' \leq t \leq it_u 
\end{array} \right. 
\] (9)
Using (13) to (15), from (9) the serviceable stock function for the serviceable stock function for the $i^{th}$ production cycle is given by

$$dS_i(t) = \begin{cases} u'(t) + p(t) - d(t) & \text{if} \quad mt_u + (j-1)t_p \\ -d(t) & \text{if} \quad mt_u + (j-1)t_p + t'_{pj} \leq t \leq mt_u + jt_p \end{cases}$$

(10)

$$\frac{dx_{R}(t)}{dt} = \begin{cases} \lambda(\alpha_0 + \alpha_1)t) - p(t) - z(t) & \text{if} \quad 0 \leq t \leq mt_u \\ \lambda(\alpha_0 + \alpha_1)t) - p(t) - z(t) & \text{if} \quad mt_u + (j-1)t_p \leq t \leq mt_u + (j-1)t_p + t'_{pj} \end{cases}$$

(11)

$$\frac{dx_{Rj}(t)}{dt} = \begin{cases} \lambda(\alpha_0 + \alpha_1)t) - p(t) - z(t) & \text{if} \quad mt_u + (j-1)t_p \leq t \leq mt_u + (j-1)t_p + t'_{pj} \end{cases}$$

(12)

where

$$d(t) = d_0 - d_1e^{-\beta t}, \quad x_S(0) = 0 = x_S(it_u), \quad x_S(jt_p) = 0, \quad (13)$$

$$\dot{x}_R(0) = 0, \quad u(t) = u_0 + u_1t, \quad p(t) = p_0, \quad z(t) = z_0 + z_1t \quad (14)$$

and

$$u'(t) = u'_0 + u'_1t, \quad i = 1, 2, \ldots, m, j = 1, 2, \ldots, n. \quad (15)$$

## Solution

Using (13) to (15), from (9) the serviceable stock function for the $i^{th}$ production cycle is given by

$$x_{S_i}(t) = \begin{cases} (u_0 - d_0)(t - (i-1)t_u) + u_1t^2 - d_1e^{-\beta t} + \frac{u_1^2}{2} & \text{if} \quad (i-1)t_u \leq t \leq (i-1)t_u + t'_{ui} \\ \frac{d_1e^{-\beta t}}{\beta} & \text{if} \quad (i-1)t_u + t'_{ui} \leq t \leq it_u \end{cases}$$

(16)

and using (13) to (15), from (10) the serviceable stock function for the $j^{th}$ production and recycling cycle is given by

$$x_{S_j}(t) = \begin{cases} (u_0' + p_0 - d_0)(t - mt_u - (j-1)t_p) + u_1't^2 - d_1'e^{-\beta(t-mt_u+(j-1)t_p)} & \text{if} \quad mt_u + (j-1)t_p \leq t \leq mt_u + (j-1)t_p + t'_{pj} \\ \frac{d_1'e^{-\beta t}}{\beta} & \text{if} \quad mt_u + (j-1)t_p + t'_{pj} \leq t \leq mt_u + jt_p \end{cases}$$

(17)
Again using (13) to (15), from (11) non serviceable stock only production cycles is given by

\[
\tilde{x}_{R}(t) = \begin{cases} 
\lambda \left[ (\alpha_0 + \alpha_1 t)(d_0 t + \frac{d_1 e^{-\beta t}}{\beta}) \right. \\
- \alpha_1 \left( \frac{d_0 t^2}{2} - \frac{d_1 e^{-\beta t}}{\beta^2} \right) - \alpha_0 d_1 + \frac{\alpha_1 d_1}{\beta} \\
- \left( z_0 t + \frac{z_1 t^2}{2} \right) \text{ if } 0 \leq t \leq m t_u 
\end{cases}
\] (18)

and also using (13) to (15), from (12) the non serviceable stock for \( j^{th} \) \((j = 1, 2, \cdots, n)\) production and recycling cycle is given by

\[
\tilde{x}_{R_j}(t) = \begin{cases} 
\lambda \left[ \tilde{x}_{R}(m t_u + (j - 1) t_p) + (\alpha_0 + \alpha_1 t)(d_0 t + \frac{d_1 e^{-\beta t}}{\beta}) \right. \\
- \alpha_1 \left( \frac{d_0 t^2}{2} - \frac{d_1 e^{-\beta t}}{\beta^2} \right) + X_0 \\
- (p_0 - z_0)(t - (m t_u + (j - 1) t_p)) - \frac{z_1}{2}(m t_u + (j - 1) t_p)^2 \\
\text{if } m t_u + (j - 1) t_p \leq t \leq m t_u + (j - 1) t_p + t_{pj} \\
\lambda \left[ \tilde{x}_{R_j}(m t_u + (j - 1) t_p + t_{pj}) + (\alpha_0 + \alpha_1 t)(d_0 t + \frac{d_1 e^{-\beta t}}{\beta}) \right. \\
- \alpha_1 \left( \frac{d_0 t^2}{2} - \frac{d_1 e^{-\beta t}}{\beta^2} \right) + X_1 \\
+ \left( z_0(m t_u + (j - 1) t_p + t_{pj}) + \frac{z_1(m t_u + (j - 1) t_p + t_{pj})^2}{2} \right) \text{ if } m t_u + (j - 1) t_p + t_{pj} \leq t \leq m t_u + j t_p 
\end{cases}
\] (19)

where

\[
X_0 = (\alpha_0 + \alpha_1(m t_u + (j - 1) t_p))(d_0(m t_u + (j - 1) t_p) \\
+ \frac{d_1 e^{-\beta(m t_u + (j - 1) t_p)}}{\beta^2}) - \alpha_1 \left( \frac{d_0 t^2}{2} - \frac{d_1 e^{-\beta(m t_u + (j - 1) t_p)}}{\beta^2} \right)
\]

\[
X_1 = (\alpha_0 + \alpha_1(m t_u + (j - 1) t_p + t_{pj})) \\
(d_0(m t_u + (j - 1) t_p + t_{pj}) + \frac{d_1 e^{-\beta(m t_u + (j - 1) t_p + t_{pj})}}{\beta^2}) \\
- \alpha_1 \left( \frac{d_0(m t_u + (j - 1) t_p + t_{pj})^2}{2} - \frac{d_1 e^{-\beta(m t_u + (j - 1) t_p + t_{pj})^2}}{\beta^2} \right)
\]

In plant-I, from equation (7), the cost function \( J_1 \) is rewritten below:

\[
J_1 = PC1 + HC1 + SC1 + ID1 \\
= \sum_{i=1}^{m} \left[ C_{w0} t_{w_i} \frac{C_{u1}}{u_1} \log(u_0 + u_1 t) \right. \\
+ \left. \frac{C_{w2}}{u_1(\gamma_1 + 1)} \right] \left( u_0 + u_1((i - 1) t_{u} + t_{u,i}) \right)^{(\gamma_1 + 1)}
\]
\[-(u_0 + u_1((i-1)t_u))^{(\gamma_1+1)}\]
\[+ \frac{C_{u3}}{u_1(\gamma_2 + 1)} \left\{ (u_0 + u_1((i-1)t_u + t'_{ui}))^{(\gamma_2+1)} \right\}
- (u_0 + u_1((i-1)t_u))^{(\gamma_2+1)} \]
+ \frac{h_S((u_0 - d_0)((i-1)t_u + t'_{ui})^2 - ((i-1)t_u)^2 - (i-1)t_ut'_{ui})}{2}
+ \frac{u_1}{2} \left( \frac{(i-1)t_u + t'_{ui})^3 - ((i-1)t_u)^3 - (i-1)t_u^3t'_{ui} \right)
+ \sum_{i=1}^m (t_u - t'_{ui})E[Id_u]
+ d_1 \left( \frac{e^{-\beta((i-1)t_u + t'_{ui})} - e^{-\beta((i-1)t_u)}}{\beta^2} \right)
+ d_1 \left( \frac{e^{-\beta((i-1)t_u + t'_{ui})} - e^{-\beta((i-1)t_u + t'_{ui})}}{\beta^2} \right)
+ x_{S_i}((i-1)t_u + t'_{ui})(t_u - t'_{ui}) - d_0 \left( (i-1)t_u - (i-1)t_u + t'_{ui})^2 \right)
+ d_0 \left( (i-1)t_u + t'_{ui})(t_u - t'_{ui}) + \sum_{j=1}^n (t_p - t'_{pj})Id_p \right.
+ d_1 \left( \frac{e^{-\beta((i-1)t_u + t'_{ui})} - e^{-\beta((i-1)t_u + t'_{ui})}}{\beta^2} \right)
+ d_1 \left( \frac{e^{-\beta((i-1)t_u + t'_{ui})} - e^{-\beta((i-1)t_u + t'_{ui})}}{\beta^2} \right)
+ ms_u + ns_p + \sum_{j=1}^n \left[ C_{u0} \left\{ u'_0t'_{pj} + \frac{u'_1}{2} \left( t_u + (j-1)t_p + t'_{pj} \right)^2 \right\} + C_u1t'_{pj} \right.
+ \frac{C_{u2}}{u_1(\gamma + 1)} \left\{ \left( u'_0 + u'_1(t_u + (j-1)t_p + t'_{pj}) \right)^{\gamma+1} \right\}
- \left( u'_0 + u'_1mt_p + (j-1)t_u + t'_{pj} \right)^{\gamma+1} \left\}
+ h_S((u'_0 + p_0 - d_0)((mt_u + (j-1)t_p)^2 - (mt_u + (j-1)t_p)^2)
- (mt_u + (j-1)t_p)t'_{pj})
+ \frac{u'_1}{2} \left( (mt_u + (j-1)t_p + t'_{pj})^3 - (mt_u + (j-1)t_p)^3 \right)
+ \frac{d_1}{\beta^2} \left( e^{-\beta((mt_u + (j-1)t_p + t'_{pj}) - e^{-\beta((mt_u + (j-1)t_p))}} \right)
+ d_1 \left( \frac{e^{-\beta((mt_u + (j-1)t_p + t'_{pj}) - e^{-\beta((mt_u + (j-1)t_p))}} \right)
+ h_S((mt_u + (j-1)t_p)^2 - (mt_u + (j-1)t_p + t'_{pj})^2)
- h_S((mt_u + (j-1)t_p)^2 - (mt_u + (j-1)t_p + t'_{pj})^2) \right)
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\[ + d_0 (mt_u + (j - 1)t_p + t'_{pj})(t_p - t'_{pj}) \]
\[ + d_1 \left( e^{-\beta(mt_u + j t_p)} - e^{-\beta(mt_u + (j-1)t_p + t'_{pj})} \right) \]
\[ + d_1 \frac{e^{-\beta(mt_u + (j-1)t_p + t'_{pj})}(t_p - t'_{pj})}{\beta^2} \]

(20)

In plant-II, the total cost function \( J_2 \) is given below:

\[ J_2 = h_R [HHRRC^2] + CC^2 + DC^2 \]

(21)

where

\[ CC^2 = C_s \left( d_0 T - d_1 \frac{1 - e^{-\beta T}}{\beta} \right) \]

(22)

\[ DC^2 = C_r \left( \alpha_0 d_0 T + \frac{\alpha_1 d_1 T^2}{2} - \frac{\alpha_0 d_1 (1 - e^{-\beta T})}{\beta} \right) \]
\[ + \frac{\alpha_1 d_1 T e^{-\beta T}}{\beta} - \frac{\alpha_1 d_1 (1 - e^{-\beta T})}{\beta^2} \]
\[ - C_z \left( z_0 T + \frac{z_1 T^2}{2} \right) \]
\[ + \sum_{j=1}^{n} C_p t_p \]

(23)

And the total fuzzy holding cost for the collected units is \( HHRRC^2 \) and it’s \( \alpha \)-cut is given by

\[ HHRRC^2[\alpha] = \left[ HHRRC^2_L(\alpha), HHRRC^2_R(\alpha) \right] \]

(24)

\[ HHRRC^2_L(\alpha) = h_R \left[ \sum_{i=1}^{m} (HR2_L(\alpha)) + \sum_{j=1}^{n} (RC2_L(\alpha)) \right] \]

(25)

\[ HHRRC^2_R(\alpha) = h_R \left[ \sum_{i=1}^{m} (HR2_R(\alpha)) + \sum_{j=1}^{n} (RC2_R(\alpha)) \right] \]

(26)

where, following Wu (2000) (cf., §2) we get,

\[ HR2[\alpha] = \left( \int_{(i-1)t_u}^{it_u} \bar{e}_{R_i}(t) dt \right)[\alpha] = \left[ HR2_L(\alpha), HR2_R(\alpha) \right] \]

\[ HR2_L(\alpha) = \lambda_L(\alpha) \left[ \frac{\alpha_0 d_0}{2} ((i-1)t_u)^2 - (it_u)^2 \right] \]
\[ + \frac{\alpha_0 + \alpha_1}{\beta^2} d_1 \left( e^{-\beta((i-1)t_u)} - e^{-\beta(it_u)} \right) \]
\[ + \frac{\alpha_1 d_0}{3} \left( ((i-1)t_u)^3 - (it_u)^3 \right) \]
\[
\begin{align*}
&+ \frac{\alpha_d}{\beta^2} \left( ((i - 1)t_u)e^{-\beta(i - 1)t_u} - (it_u)e^{-\beta(it_u)} \right) - X_0t_u \\
HR2R(\alpha) &= \lambda_R(\alpha) \left[ \frac{\alpha_d}{2} \left( ((i - 1)t_u)^2 - (it_u)^2 \right) \\
&\quad - \left( \frac{\alpha_0 + \alpha_1}{\beta^2} \right) d_1(e^{-\beta(i - 1)t_u} - e^{-\beta(it_u)}) \\
&\quad + \frac{\alpha_1 d_0}{3} ((i - 1)t_u)^3 - (it_u)^3 \\
&\quad + \frac{\alpha_1 d_1}{\beta^2} ((i - 1)t_u)e^{-\beta((i - 1)t_u)} - (it_u)e^{-\beta(it_u)} - X_0t_u \right] \\
HC2[\alpha] &= \left( \int_{mt_u+(j-1)t_p}^{mt_u+jt_p} \tilde{x}_R(t) dt \right)[\alpha] = \left[ HC2L(\alpha), HC2R(\alpha) \right] \\
HC2L(\alpha) &= \lambda_L(\alpha) \left[ x_R(mt_u + (j - 1)t_p) + t_p' \right] \\
&\quad + \frac{\alpha_0 d_0}{2} ((mt_u + (j - 1)t_p + t_p')^2 - (it_u)^2) \\
&\quad - \left( \frac{\alpha_0 + \alpha_1}{\beta^2} \right) d_1(e^{-\beta(mt_u+(j - 1)t_p+t_p')} - e^{-\beta(mt_u+(j - 1)t_p)}) \\
&\quad + \frac{\alpha_1 d_0}{3} ((mt_u + (j - 1)t_p + t_p')^3 - (mt_u + (j - 1)t_p)^3) \\
&\quad + \frac{\alpha_1 d_1}{\beta^2} ((i - 1)t_u)e^{-\beta(mt_u+(j - 1)t_p+t_p')} - (it_u)e^{-\beta(mt_u+(j - 1)t_p)}) \\
&\quad - (p_0 - z_0)\left( \frac{1}{2}((mt_u + (j - 1)t_p + t_p')^2 - (mt_u + (j - 1)t_p)^2) \\
&\quad - (mt_u + (j - 1)t_p)t_p' \right) \\
&\quad - \frac{z_1}{2}((mt_u + (j - 1)t_p)^2 t_p' - X_0t_p') \\
&\quad + x_R(mt_u + (j - 1)t_p + t_p')(t_p' - t_p) \\
&\quad + \frac{\alpha_0 d_0}{2} ((mt_u + jt_p)^2 - (mt_u + (j - 1)t_p + t_p')^2) \\
&\quad - \left( \frac{\alpha_0 + \alpha_1}{\beta^2} \right) d_1(e^{-\beta(mt_u+jt_p)} - e^{-\beta(mt_u+(j - 1)t_p)}) \\
&\quad + \frac{\alpha_1 d_0}{3} ((mt_u + jt_p)^3 - (mt_u + (j - 1)t_p)^3) \\
&\quad + \frac{\alpha_1 d_1}{\beta^2} ((i - 1)t_u)e^{-\beta(mt_u+jt_p)} - (it_u)e^{-\beta(mt_u+(j - 1)t_p)}) \\
&\quad - \left( \frac{z_0}{2}((mt_u + jt_p)^2 - (mt_u + (j - 1)t_p + t_p')^2) \\
&\quad + \frac{z_1}{6}((mt_u + jt_p)^3 \\
&\quad - (mt_u + (j - 1)t_p + t_p')^3) + \{(z_0(mt_u + (j - 1)t_p + t_p') + \} \} (t_p' - t_p) \right] \\
&\quad + \frac{z_1}{2}((mt_u + (j - 1)t_p + t_p')^2 + X_1)(t_p' - t_p) \\
&\quad + \frac{z_1}{2}((mt_u + (j - 1)t_p + t_p')^2 + X_1)(t_p' - t_p)
\end{align*}
\]
\[ HC_{2R}(\alpha) = \lambda_R(\alpha) \left[ x_R(m t_u + (j - 1) t_p) t'_{pj} + \frac{\alpha_0 d_0}{2} ((m t_u + (j - 1) t_p + t'_{pj})^2 - (i t_u)^2) \right. \]
\[ - (\alpha_0 + \alpha_1) d_1 (e^{-\beta (m t_u + (j - 1) t_p + t'_{pj})} - e^{-\beta (m t_u + (j - 1) t_p)}) \]
\[ + \frac{\alpha_1 d_1}{3} ((m t_u + (j - 1) t_p + t'_{pj})^3 - (m t_u + (j - 1) t_p)^3) \]
\[ + \frac{\alpha_1 d_1}{2} e^{-(m t_u + (j - 1) t_p)} - (i t_u) e^{-(m t_u + t'_{pj})} \]
\[ + (p_0 - z_0) \left( \frac{1}{2} ((m t_u + (j - 1) t_p + t'_{pj})^2 - (m t_u + (j - 1) t_p)^2) \right) \]
\[ - (m t_u + (j - 1) t_p) t'_{pj} \left( - \frac{z_1}{2} (m t_u + (j - 1) t_p)^2 t'_{pj} - X_0 t'_{pj} \right) \]
\[ + x_{R2} (m t_u + (j - 1) t_p + t'_{pj}) (t'_{pj} - t_p) \]
\[ + \frac{\alpha_0 d_0}{2} ((m t_u + j t_p)^2 - (m t_u + (j - 1) t_p)^2) \]
\[ - \frac{\alpha_0 + \alpha_1}{\beta^2} d_1 (e^{-\beta (m t_u + j t_p)} - e^{-\beta (m t_u + (j - 1) t_p)}) \]
\[ + \frac{\alpha_1 d_1}{3} ((m t_u + j t_p)^3 - (m t_u + (j - 1) t_p)^3) \]
\[ + \frac{\alpha_1 d_1}{2} ((i - 1) t_u) e^{-\beta (m t_u + j t_p)} - (i t_u) e^{-(m t_u + (j - 1) t_p)}) \]
\[ - \left( \frac{z_0}{2} ((m t_u + j t_p)^2 - (m t_u + (j - 1) t_p + t'_{pj})^2) \right) \]
\[ + \frac{z_1}{6} ((m t_u + j t_p)^3) \]
\[ - (m t_u + (j - 1) t_p + t'_{pj})^3 + \{ z_0 (m t_u + (j - 1) t_p + t'_{pj}) \}
\[ + \frac{z_1}{2} (m t_u + (j - 1) t_p + t'_{pj})^2 + \{ X_1 (t'_{pj} - t_p) \} \]

Here \( \tilde{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \) is a TFN having \( \alpha \)-cut \( \tilde{\lambda}[\alpha] = [\lambda_L(\alpha), \lambda_R(\alpha)] \), where \( \lambda_L(\alpha) = \lambda_1 + \alpha (\lambda_2 - \lambda_1) \) and \( \lambda_R(\alpha) = \lambda_3 - \alpha (\lambda_3 - \lambda_2) \). Also, \( \tilde{x}_R(t)[\alpha] = [x_{RL}(t, \alpha), x_{RR}(t, \alpha)] \).

When \( 0 \leq t \leq T \), it is clear that, \( \frac{dx_{RL}(t, \alpha)}{dt} \) and \( \frac{d}{d\alpha} \left( \frac{dx_{RL}(t, \alpha)}{dt} \right) > 0 \). \( \frac{dx_{RR}(t, \alpha)}{dt} \) and \( \frac{d}{d\alpha} \left( \frac{dx_{RR}(t, \alpha)}{dt} \right) < 0 \). All the four conditions of fuzzy differential equation are satisfied.

6 Formulation for total cost

The \( \alpha \)-cut of total for fuzzy model is given by \([TC_L(\alpha), TC_R(\alpha)]\), where
\[ TC_L(\alpha) = [J_1 + DC_2 + CC_2 + h_R HR_C R_2 L(\alpha)] \]
\[ TC_R(\alpha) = [J_1 + DC_2 + CC_2 + h_R HR_C R_2 R(\alpha)] \]
Approach 1: Using modified graded mean integration value (MGMIV) of fuzzy average profit $FTC$ is

$$
P_k(FTC) = \frac{\int_0^1 \alpha [kTC_L(\alpha) + (1 - k)TC_R(\alpha)] d\alpha}{\int_0^1 \alpha d\alpha}
$$

$$
= 2 \int_0^1 \alpha \left\{ J_1 + DC + CC + [k(FHR_L(\alpha) + FRC_L(\alpha)) + (1 - k)(FHR_R(\alpha) + FRC_R(\alpha)) \right\} d\alpha
$$

$$
= J_1 + DC + CC + \frac{1}{3} \left\{ k(3\lambda_2 + \lambda_1) + (1 - k)(\lambda_3 + 2\lambda_2) \right\}
$$

$$
\times \left[ x_R(mt_u + (j - 1)t_p) t'_p j
+ \frac{\alpha_0 d_0}{2} ((mt_u + (j - 1)t_p + t'_p)^2 - (t_u)^2)
- (\frac{\alpha_0 + \alpha_1}{\beta^2}) d_1 (e^{-\beta(mt_u + (j - 1)t_p + t'_p)} - e^{-\beta(mt_u + (j - 1)t_p)})
+ \frac{\alpha_1 d_0}{3} ((mt_u + (j - 1)t_p + t'_p)^3 - (mt_u + (j - 1)t_p)^3)
+ \frac{\alpha_1 d_1}{\beta^2} ((i - 1)t_u) e^{-\beta(mt_u + (j - 1)t_p + t'_p}) - (t_u) e^{-\beta(mt_u + (j - 1)t_p)})
- (p_0 - z_0) \frac{1}{2} ((mt_u + (j - 1)t_p + t'_p)^2 - (mt_u + (j - 1)t_p)^2)
- (mt_u + (j - 1)t_p t'_p j - \frac{z_1}{2} (mt_u + (j - 1)t_p)^2 t'_p j - X_0 t'_p j)
+ x_R((mt_u + (j - 1)t_p + t'_p)(t'_p - t_p))
+ \frac{\alpha_0 d_0}{2} ((mt_u + j t_p)^2 - (mt_u + (j - 1)t_p + t'_p)^2)
- (\frac{\alpha_0 + \alpha_1}{\beta^2}) d_1 (e^{-\beta(mt_u + j t_p)} - e^{-\beta(mt_u + (j - 1)t_p)})
+ \frac{\alpha_1 d_0}{3} ((mt_u + j t_p)^3 - (mt_u + (j - 1)t_p)^3)
+ \frac{\alpha_1 d_1}{\beta^2} ((i - 1)t_u) e^{-\beta(mt_u + j t_p)} - (t_u) e^{-\beta(mt_u + (j - 1)t_p)})
- (\frac{z_0}{2} ((mt_u + j t_p)^2 - (mt_u + (j - 1)t_p + t'_p)^2)
+ \frac{z_1}{6} ((mt_u + j t_p)^3 - (mt_u + (j - 1)t_p + t'_p)^3)
+ \{(z_0 (mt_u + (j - 1)t_p + t'_p)
+ \frac{z_1}{2} (mt_u + (j - 1)t_p + t'_p)^2) + X_1 \}(t'_p j - t_p) \right]\] (29)
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According to this approach, model reduces to

\[
\begin{align*}
\text{Minimise } Z(u(t), u'(t), z(t), p(t), d(t)) = P_k(TC) \\
\text{subject to } (13) \text{to}(15)
\end{align*}
\]

(30)

Approach 2: The above discussed model can also be reduced to,

\[
\begin{align*}
\text{Minimise } Z(u(t), u'(t), z(t), p(t), d(t)) = [TC_L(\alpha), TC_R(\alpha)] \\
\text{subject to } (13) \text{to}(15)
\end{align*}
\]

(31)

Now, the problems in (30) and (31) are solved using the following GA.

7 GA for proposed model

GAs are exhaustive search algorithms based on the mechanics of natural selection and genesis (crossover, mutation etc.) and have been developed by Holland (1975), his colleagues and his students at the University of Michigan (cf., Goldberg, 1989). A GA normally starts with a set of potential solutions (called initial population) of the decision making problem under consideration. Individual solutions are called chromosome. Selection, crossover and mutation operations take place among the potential solutions to get a new set of solutions and it continues until terminating conditions are encountered. Michalewicz (1992) proposed a GA named contractive mapping genetic algorithm (CMGA) and proved the asymptotic convergence of the algorithm by Banach fixed point theorem. In CMGA, movement from old population to new population occurs only when average fitness of new population is better than the old one. The algorithm is presented below. In the algorithm, \( p_c, p_m \) are probability of crossover and probability of mutation respectively, \( T \) is the generation counter and \( P(T) \) is the population of potential solutions for generation \( T \). \( M \) is iteration counter in each generation to improve \( P(T) \) and \( M_0 \) is the upper limit of \( M \). Initialise \( P(1) \) function to generate the initial population \( P(1) \) (initial guess of solution set). Evaluate \( P(T) \) function to find the fitness of each member of \( P(T) \) which is described after the algorithm.

7.1 GA Algorithm

1. Set generation counter \( T' = 1 \), iteration counter in each generation \( M = 0 \).
2. Initialise probability of crossover \( p_c \), probability of mutation \( p_m \), upper limit of iteration counter \( M_0 \), population size \( N \).
3. Initialise \( P(T') \).
4. Evaluate \( P(T') \).
5. While \( (M < M_0) \).
6. Select \( N \) chromosomes from \( P(T') \) for mating pool using roulette-wheel selection process. Let this set be \( P'(T') \).
7. Select chromosomes from \( P'(T') \), for crossover depending on \( p_c \).
8. Perform crossover on selected chromosomes.
9. Select chromosomes from \( P'(T') \), for mutation depending on \( p_m \).
10. Perform mutation on selected chromosomes to get population \( P_1(T') \).
11. Evaluate\(P_1(T^t)\).
12. Set \(M = M + 1\).
13. If average fitness of \(P_1(T^t) > \) average fitness of \(P(T^t)\) then
   14. Set \(P(T^{t+1}) = P_1(T^t)\).
   15. Set \(T^t = T^t + 1\).
   16. Set \(M = 0\).
17. End If
18. End While
19. Print the best chromosome of \(P(T)\).
20. End algorithm

7.1.1 GA procedures for the proposed model

There are two integer type variables of the proposed model. In order to implement the above GA, binary mode representation, crossover and mutation are used. To implement this all real variables are transferred to integer variable using the following principle:

Let \([a, b]\) be the domain of a real variable \(x\) and required result be correct up to six decimal places. Then the interval \([a, b]\) should be partitioned into \((b - a)10^6\) equal size ranges and value of \(x\) be searched from the set of in-between points. Let \(m\) be the smallest integer for which \((b - a)10^6 \leq 2^m - 1\). Then \(x\) be coded as a binary string of length \(m\). In other words a random integer from the range \([0, 2^m - 1]\) should be assigned to the variable \(x\) and corresponding real value of \(x\) can be obtained from the formula \(a + \frac{(b - a)x}{2^m - 1}\) [cf., Michalewicz, (1992), pp.33–34].

7.1.2 Representation

A \(n\) dimensional integer vector \(X = (x_1, x_2, ..., x_n)\) is used to represent a chromosome, where \(x_1, x_2, ..., x_n\) represent \(n\) decision variables of the problem. For real variable corresponding integer equivalent is obtained following the above approach. Then each \(x_i\) is transformed to their equivalent binary numbers to find binary mode representation of the chromosome.

7.1.3 Initialisation

\(N\) such chromosomes \(X_i = (x_{i1}, x_{i2}, ..., x_{in}), i = 1, 2, ..., N\) are randomly produced by random number generator such that each \(X_i\) conforms the constraints of the problem. The constraints are checked using a separate subfunction named check_constraint(). If a generated solution \(X_i\) satisfies the constraints of the problem then it is considered as feasible solution. Otherwise it is discarded. Selection of GA parameters- \(p_c, p_m, M_0\) is quite difficult and it depends on the nature of the problem. Normally \(p_c, p_m\) are taken in the range \([0.5, 1]\). Set \(p_c = 0.6, p_m = 0.3, T^t = 1, M_0 = 50\).

7.1.4 Evaluation function

This GA can be used to solve optimisation problems with crisp as well as interval objective functions.

- For crisp objective function \(f\), value of objective function due to a solution \(X\), \(f(X)\) is taken as fitness of \(X\). As each components of \(X\) are integers, for a real variable at first integer equivalent \(x_i\) is transferred to corresponding real value by the formula \(\frac{(x_i - a_i)(2^m - 1)}{b_i - a_i}\), where \([a_i, b_i]\) is search interval of \(i\)-th variable and \(m_i\) bits are used to represent \(x_i\). Then these values are taken for evaluation of \(f(X)\).
For interval valued objective function, fitness of a solution $X$ is taken as $d(X) \frac{Popsiz}{e}$, where $d(X)$ is the number of solutions in the population dominated by the solution $X$. In this case also, for a real variable at first its integer equivalent is transferred to corresponding real value and then interval objective $f(X)$ of $X$ is obtained. After that comparison of interval objectives are made using fuzzy preference ordered interval (FPOI) to find the fitness of solutions. A solution $X$ dominates another solution $Y$ if $f(X)$ is better than $f(Y)$ according FPOI.

7.1.5 Selection process for mating pool

The selection scheme in GA determines which chromosomes in the current population are to be selected for recombination. There are several approaches to select chromosomes from the initial population for mating pool. All these approaches has some merits and demerits over the others. Among these approaches RWS process plays a major role. In this study, Roulette wheel selection process is used. The following steps are followed for this purpose

1. Find total fitness of the population $F = \sum_{i=1}^{N} Z(X_i)$.
2. Calculate the probability of selection $p_j$ of each solution $X_j$ by the formula $p_j = \frac{Z(X_j)}{F}$.
3. Calculate the cumulative probability $q_j$ for each solution $X_j$ by the formula $q_j = \sum_{k=1}^{j} p_k$.
4. Generate a random number ‘$r$’ from the range $[0,1]$.
5. If $r < q_1$ then select $X_1$ otherwise select $X_i (2 \leq i \leq N)$ where $q_{i-1} \leq r < q_i$.
6. Repeat steps (4) and (5) $N$ times to select $N$ chromosomes for mating pool. These chromosomes are denoted by $XN_j, j = 1, 2, ..., N$. Clearly one chromosome may be selected more than once and so some members of this set may be identical.
7. Selected chromosomes set is denoted by $P'(T) = \{XN_1, XN_2, ..., XN_N\}$ in the proposed GA algorithm.

7.1.6 Crossover

Crossover operator is mainly responsible for the search of new strings. The exploration and exploitation of the solution space is made possible by exchanging genetic information of the current chromosomes. Crossover operates on two parent chromosomes at a time and generates offspring chromosomes by recombining both parent solution features.

1. Selection for crossover: For each chromosome of $P'$ a random number $r$ has to be generated from the range $[0,1]$. If $r < p_c$ then the chromosome is taken for crossover.
2. Crossover process: Crossover operation has occurred on the selected chromosomes. For each pair of coupled chromosomes $XN_{i1}, XN_{i2}$ a random integer $j$ is generated in the range $[1, n]$. Then another integer $i$ is generated in the range $[1, m_j]$ where $m_j$ is the number of bits used to represent $j^{th}$ variable. Let $xN_{i1j}$ and $xN_{i2j}$ be $j^{th}$ variable of $XN_{i1}, XN_{i2}$ respectively. Then bitwise crossover operation is done on $xN_{i1j}$ and $xN_{i2j}$ at the $i^{th}$ bit (using bitwise operation of C-programming language). Let $xN'_{i1j}$ and $xN'_{i2j}$ be resultant output after crossover. Then $xN'_{i1j}$ is made by first $i$ bits of $xN_{i1j}$ and last $m_j - i$ bits of $xN_{i2j}$ and $xN'_{i2j}$ is made by first $i$ bits of $xN_{i2j}$ and last $m_j - i$ bits of
Then the offsprings $Y_1, Y_2$ replace their parents $X_{N_1}, X_{N_2}$ respectively (if they satisfy constraints of the problem) where

$$Y_1 = (x_{N_11}, x_{N_12}, ..., x_{N_1j-1}, x_{N_1j}', x_{N_1j+1}, x_{N_1j+2}, ..., x_{N_1n})$$

$$Y_2 = (x_{N_21}, x_{N_22}, ..., x_{N_2j-1}, x_{N_2j}', x_{N_2j+1}, x_{N_1j+2}, ..., x_{N_1n})$$

7.1.7 Mutation

Mutation operator is used to prevent the search process from converging to local optima rapidly. It is applied to a single chromosome $X_j$.

1 Selection for mutation: For each chromosome of $P(T)$ a random number $r$ has to be generated from the range $[0,1]$. If $r < p_m$ then the chromosome is taken for mutation.

2 Mutation process: To mutate a chromosome $X_{N_k} = (x_{N_k1}, x_{N_k2}, ..., x_{N_kj})$ a random integer $j$ has to be selected in the range $[1,j]$ where $m_j$ is the number of bits used to represent the $j^{th}$ variable. Now the $i^{th}$ bit of $X_{N_k}$ is flipped (complemented) to get mutated chromosome. If the new chromosome satisfies constraints of the problem, it replaces the parent chromosome.

7.1.8 Implementation

With the above function and values the algorithm is implemented using C-programming language.

8 Numerical experiment

To illustrate the production-recycling model numerically, we consider input data in Table 1 and Table 2 for crisp data. For these input data and by using the above single objective GA technique §7 and using Approach 1 and Approach 2, we solve the problem (23) to (24) and we obtained the optimal productions, optimal recycling and optimal disposal which are $u(t) = 229.13 + 0.6t$, $u_t(t) = 14.8 + 0.4t$, $p(t) = 10$ and $z(t) = 0.71 + 0.16t$. Also the optimal values of $x_{Si}(t), x_{Sj}(t), x_{R}(t), u(t), u_t(t), d(t)$ and $p(t)$ are evaluated using (23) to (24) for different values of $t$. We have shown the optimum results of $x_{Si}(t), x_{R}(t), u(t)$ and $d(t)$ of production-cycle in Table 4. Similarly, the optimum results of $x_{Sj}(t), x_{Rj}(t), u(t), p(t)$ and $d(t)$ for the production and recycling-cycle are presented in Table 5. Here $\lambda = (0.5, 0.8, 1), n = 2, m = 3$.

<table>
<thead>
<tr>
<th>$d_0$</th>
<th>$d_1$</th>
<th>$\delta$</th>
<th>$C_p$ in $$</th>
<th>$C_v$ in $$</th>
<th>$C_r$ in $$</th>
<th>$\gamma_1$</th>
<th>$\gamma_2$</th>
<th>$\alpha_0$</th>
<th>$\alpha_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>0.2</td>
<td>0.05</td>
<td>1.6</td>
<td>0.15</td>
<td>0.11</td>
<td>0.1</td>
<td>0.12</td>
<td>0.12</td>
<td>0.1</td>
</tr>
</tbody>
</table>
A fuzzy differential approach

Table 2 Input data

<table>
<thead>
<tr>
<th>Id_u</th>
<th>S_u</th>
<th>S_p</th>
<th>C_u0</th>
<th>C_u1</th>
<th>C_u2</th>
<th>C_u3</th>
<th>T</th>
<th>h_R</th>
<th>h_S</th>
<th>Id_u</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.5</td>
<td>4.1</td>
<td>5.3</td>
<td>6.4</td>
<td>15.6</td>
<td>1.7</td>
<td>2.3</td>
<td>15</td>
<td>1.3</td>
<td>1.5</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 3 Optimum results for fuzzy model

<table>
<thead>
<tr>
<th>Iteration</th>
<th>u(t)</th>
<th>u'(t)</th>
<th>d(t)</th>
<th>p(t)</th>
<th>z(t)</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fuzzy Model Using MGMIV For k = 0.5</td>
<td>30</td>
<td>36.85</td>
<td>19.952</td>
<td>14.894</td>
<td>12.54</td>
<td>2.7708</td>
</tr>
<tr>
<td>33</td>
<td>36.01</td>
<td>19.081</td>
<td>14.894</td>
<td>12.05</td>
<td>2.0311</td>
<td>812.453</td>
</tr>
<tr>
<td>40</td>
<td>36.85</td>
<td>18.125</td>
<td>14.894</td>
<td>12.54</td>
<td>1.9901</td>
<td>801.156</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Iteration</th>
<th>u(t)</th>
<th>u'(t)</th>
<th>d(t)</th>
<th>p(t)</th>
<th>z(t)</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fuzzy Model Using FPOI For α = 0.5</td>
<td>30</td>
<td>36.01</td>
<td>19.952</td>
<td>14.894</td>
<td>12.54</td>
<td>2.7708</td>
</tr>
<tr>
<td>33</td>
<td>35.45</td>
<td>19.081</td>
<td>14.894</td>
<td>12.05</td>
<td>2.0314</td>
<td>814.674</td>
</tr>
<tr>
<td>40</td>
<td>35.01</td>
<td>18.125</td>
<td>14.894</td>
<td>12.54</td>
<td>1.9901</td>
<td>803.213</td>
</tr>
</tbody>
</table>

Table 4 Optimal values of x_{S1}(t), \tilde{x}_R(t), u(t) and d(t), i = 1, 2

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>1.28</th>
<th>3</th>
<th>3.01</th>
<th>4.01</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>x_{S1}(t)</td>
<td>0</td>
<td>15.33</td>
<td>0</td>
<td>0.16</td>
<td>16.2</td>
<td>0</td>
</tr>
<tr>
<td>\tilde{x}_R(t)</td>
<td>[0, 0]</td>
<td>[1.73, 1.98]</td>
<td>[6.78, 7.98]</td>
<td>[6.86, 7.12]</td>
<td>[8.28, 8.67]</td>
<td>[15.4, 15.98]</td>
</tr>
<tr>
<td>u(t)</td>
<td>29.13</td>
<td>29.77</td>
<td>-</td>
<td>30.53</td>
<td>31.14</td>
<td>-</td>
</tr>
<tr>
<td>d(t)</td>
<td>15.7</td>
<td>15.32</td>
<td>16.63</td>
<td>15.67</td>
<td>15.42</td>
<td>14.97</td>
</tr>
</tbody>
</table>

Table 5 Optimal values of x_{S1}(t), x_{Rj}(t), u'(t), p(t) and d(t), j = 1, 2, 3

<table>
<thead>
<tr>
<th>t</th>
<th>6.01</th>
<th>7.2</th>
<th>9</th>
<th>9.01</th>
<th>9.92</th>
<th>12</th>
<th>12.01</th>
<th>12.88</th>
<th>15</th>
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</thead>
<tbody>
<tr>
<td>x_{S1}(t)</td>
<td>0.04</td>
<td>20.47</td>
<td>0</td>
<td>0.04</td>
<td>22.34</td>
<td>0</td>
<td>0.05</td>
<td>23.56</td>
<td>0</td>
</tr>
<tr>
<td>\tilde{x}_{Rj}(t)</td>
<td>[15.37, [4.3, [10.67, [10.64, [5.34, [15.84, [6.86, [10.45, [15.97]</td>
<td>4.9</td>
<td>10.98</td>
<td>11.0</td>
<td>6.8</td>
<td>15.99</td>
<td>16.1</td>
<td>7.1</td>
<td>11.23</td>
</tr>
<tr>
<td>u'(t)</td>
<td>16.8</td>
<td>17.17</td>
<td>-</td>
<td>17.7</td>
<td>17.95</td>
<td>-</td>
<td>18.12</td>
<td>18.84</td>
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</tr>
<tr>
<td>p(t)</td>
<td>10</td>
<td>12.52</td>
<td>-</td>
<td>10</td>
<td>12.54</td>
<td>-</td>
<td>10</td>
<td>12.54</td>
<td>-</td>
</tr>
<tr>
<td>d(t)</td>
<td>14.97</td>
<td>15.34</td>
<td>14.99</td>
<td>15.34</td>
<td>15.27</td>
<td>15.47</td>
<td>14.99</td>
<td>19.21</td>
<td>16.21</td>
</tr>
</tbody>
</table>

Table 6 shows the experimental results obtained by a GA with different GA parameters. The tested GA parameters contain the population size N_{pop-size}, the probability of crossover P_c and the probability of mutation P_m. We compare these results when different parameters are put with the same or different generations as a stop criterion. It appears that almost all the objective values differ little from each other, which implies that the algorithm is robust to the GA parameters setting and effective to solve multiple objective programming problem. Figure 2 pictorially represents the optimum result for production, recycling, demand and serviceable stock. Figure 2 pictorially also represents the optimum result for non serviceable stock. The increasing demand rate is very small.
Table 6  Comparison of the results obtained with different GA parameters

<table>
<thead>
<tr>
<th>No.</th>
<th>$N_{pop-size}$</th>
<th>$P_c$</th>
<th>$P_m$</th>
<th>Generations</th>
<th>Approach-1</th>
<th>Approach-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>60</td>
<td>0.3</td>
<td>0.2</td>
<td>500</td>
<td>[833.349, 898.56]</td>
<td>865.325</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>0.4</td>
<td>0.3</td>
<td>500</td>
<td>[832.123, 898.12]</td>
<td>865.548</td>
</tr>
<tr>
<td>3</td>
<td>100</td>
<td>0.5</td>
<td>0.4</td>
<td>500</td>
<td>[834.954, 896.18]</td>
<td>864.457</td>
</tr>
<tr>
<td>4</td>
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<td>0.3</td>
<td>0.2</td>
<td>1,000</td>
<td>[834.735, 896.91]</td>
<td>866.627</td>
</tr>
<tr>
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<td>0.4</td>
<td>0.3</td>
<td>1,000</td>
<td>[854.126, 898.34]</td>
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</tr>
<tr>
<td>6</td>
<td>100</td>
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<td>0.4</td>
<td>1,000</td>
<td>[845.543, 898.71]</td>
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<tr>
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<td>$\alpha = 0.8$</td>
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<td></td>
</tr>
<tr>
<td>1</td>
<td>60</td>
<td>0.3</td>
<td>0.2</td>
<td>500</td>
<td>[846.349, 896.16]</td>
<td>871.452</td>
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<tr>
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<td>0.4</td>
<td>0.3</td>
<td>500</td>
<td>[846.349, 896.73]</td>
<td>872.597</td>
</tr>
<tr>
<td>3</td>
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<td>0.4</td>
<td>500</td>
<td>[848.349, 896.05]</td>
<td>872.957</td>
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<tr>
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<td>0.2</td>
<td>1,000</td>
<td>[849.349, 894.78]</td>
<td>873.218</td>
</tr>
<tr>
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<td>0.4</td>
<td>0.3</td>
<td>1,000</td>
<td>[849.349, 894.76]</td>
<td>874.214</td>
</tr>
<tr>
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<td>0.5</td>
<td>0.4</td>
<td>1,000</td>
<td>[850.349, 893.58]</td>
<td>875.451</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>60</td>
<td>0.3</td>
<td>0.2</td>
<td>500</td>
<td>[856.349, 898.16]</td>
<td>877.349</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>0.4</td>
<td>0.3</td>
<td>500</td>
<td>[858.349, 897.76]</td>
<td>877.548</td>
</tr>
<tr>
<td>3</td>
<td>100</td>
<td>0.5</td>
<td>0.4</td>
<td>500</td>
<td>[859.349, 897.79]</td>
<td>878.874</td>
</tr>
<tr>
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<td>0.2</td>
<td>1,000</td>
<td>[861.349, 896.15]</td>
<td>879.658</td>
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<tr>
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<td>1,000</td>
<td>[868.349, 895.84]</td>
<td>881.548</td>
</tr>
</tbody>
</table>

Figure 2  Optimal production, demand, recycling and serviceable stock

9 Conclusions

In this paper, we develop a two plants production, recycling-disposal system over a finite time horizon. Here, the dynamic demand is satisfied by production and recycling. Recycling products can be used as new products which are sold again. The rate of production, recycling and disposal are assumed to be control variables. The cost is expenditure due to growing environmental concern and according to the rule of environmental regulations like ‘Kyoto Protocol’ for Industry. At the beginning, production satisfies the demand. After sometime, production and recycling
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fill up the demand. The total cost is minimised as an optimal control problem. An imprecise production-inventory system, for the first time, solved using fuzzy differential equation. The present problem can be applicable for the seasonable products with fixed time horizon and can be extended to include random planning horizon. The model is illustrated through numerical examples and results are also presented both in tabular form only. The model can be extended for imperfect production, recycling-disposal optimisation problem in uncertain environment.

References


