Thermal stress analysis of thin films in the context of generalised thermoelasticity

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Abstract: This study deals with the three-dimensional generalised thermoelasticity based on Lord and Shulman’s theory and Green and Lindsay’s theory. The fundamental equations which include both theories, are used. The thermoelastic problems for a homogeneous and isotropic film whose surfaces are traction free and subjected to a partial heating are analysed by means of the Laplace and Fourier transforms. The state space approach is used in the transformed domains. The inversion are carried out numerically. The numerical calculations for temperature and stresses are carried out. The influence of the finite velocity of the thermal wave grows as the film thins.

Keywords: thin film; thermal stress; elasticity; generalised thermoelasticity; thermal wave; relaxation time; thermomechanical coupling; Lord-Shulman theory; Green-Lindsay theory; integral transform; numerical inversion; state space approach.


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1 Introduction

The dynamical coupled theory takes into account the coupling between temperature and strain fields (e.g., Furukawa and Irschik, 2005). However, the theory involves the contradiction that thermal wave propagates at an infinite velocity. The theory of generalised thermoelasticity has been developed in an attempt to eliminate this paradox. There are several theories of the generalised thermoelasticity. First theory has modified Fourier's law and introduced one relaxation time and is named as the Lord-Shulman (L-S) theory (Lord and Shulman, 1967). Second theory has modified energy equation and stress-strain-temperature equation and introduced other two relaxation times and is named as the Green-Lindsay (G-L) theory (Green and Lindsay, 1972). Recently, other theories have been proposed. The progress on the generalised thermoelasticity was reviewed (e.g., Hetnarski and Ignaczk, 1999), and the technical book related to the generalised thermoelasticity was published (Ignaczk and Ostoja-Starzewski, 2010). We used the fundamental equations of generalised thermoelasticity, that is, heat conduction equation, equation of motion, and stress-strain-temperature equation, introduced by Noda et al. (1989), which include the L-S theory and G-L theory, and analysed a lot of problems (e.g., Furukawa et al., 1991, 1999; Furukawa, 2010).

This study deals with the generalised thermoelasticity for thin films based on L-S theory and G-L theory. It is assumed that the medium is initially in a natural state. The surfaces of the medium are traction free and subjected to sudden partial heating. The temperature, displacement and stresses are obtained by means of integral transforms, that is, Laplace transform and Fourier transform. The state space approach (Bahar, 1975; Bahar and Hetnarski, 1978; Furukawa, 2001) is used in the transformed domains. This approach is very familiar in the field of modern control theory. Analytical inversions are much difficult, so the inversions are carried out numerically. The influence of the finite velocity of the thermal wave grows as the film thins.

2 Analysis

2.1 Fundamental equations

The fundamental equations for a homogeneous and isotropic medium combined with two generalised thermoelasticity theories, L-S and G-L theories, are expressed in the following.

a Strain-displacement relation

\[ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \]  \hspace{1cm} (1)

b Equation of motion

\[ \sigma_{i,j} + F_i = \rho u_{i,i} \]  \hspace{1cm} (2)

c Energy equation

\[ -q_{i,i} + W = \rho c(T + \delta_{i} \delta_{i} T_{s}) + (3\lambda + 2\mu)\alpha T_{i} T_{i} \]  \hspace{1cm} (3)
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d  Stress-strain-temperature relation

\[ \sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij} - (3\lambda + 2\mu)\alpha(T + \delta_{ik}t_iT_j)\delta_{ij} \]  

(4)

e  Heat equation

\[ q_{ij} + \delta_{ik}t_iq_{ij} = -\lambda T_j. \]  

(5)

Here \( u_i \) is displacement component; \( \varepsilon_{ij} \), strain component; \( \varepsilon_{kk} \), volumetric strain; \( \sigma_{ij} \), stress component; \( F_i \), body force component; \( \rho \), density; \( t \), time; \( q_j \), heat flux; \( T \), temperature; \( W \), internal heat supply; \( \alpha \), coefficient of linear thermal expansion; \( c_v \), specific heat at constant volume; \( \lambda, \mu \), Lamé’s constants; \( T_0 \), initial absolute temperature; \( t_0, t_1 \), relaxation times; \( \lambda \), thermal conductivity; and \( \delta_{ik} \), Kronecker’s delta whose subscript \( k \) denotes the number of relaxation times. When we put \( k \) as 0, 1, and 2, these equations coincide with classical, L-S, and G-L theories, respectively. The comma denotes the differentiation with following variable.

From equations (3) and (5), heat conduction equation is rewritten as

\[ \kappa T_{ik} - (T + t_0T_j)_{ik} + \frac{1}{\rho c_v} (W + \delta_{ik}W_j) = \delta \left( \varepsilon_{ik} + \delta_{ik}t_i\varepsilon_{ik} \right), \]  

(6)

where

\[ m_i = \frac{3\lambda + 2\mu}{\lambda + 2\mu} = 1 - \frac{1}{\nu}, \quad \delta = \frac{(3\lambda + 2\mu)\alpha^2 T_0}{(\lambda + 2\mu)\rho c_v}. \]  

(7)

\( \kappa \) is thermal diffusivity, \( \nu \) is Poisson’s ratio, and \( \delta \) is thermomechanical coupling parameter.

We consider the three-dimensional generalised thermoelasticity for a homogeneous and isotropic film of thickness \( l \). The body force and internal heat supply are absent here. The fundamental equations, which include the L-S and G-L theories, consist of the heat conduction equation

\[ \kappa \nabla^2 T - (T + t_0T_j)_{,i} + \frac{1}{\rho c_v} (W + \delta_{ik}W_j) = \delta \left( \varepsilon_{ik} + \delta_{ik}t_i\varepsilon_{ik} \right), \]  

(8)

the equation of motion

\[ \begin{align*}
\sigma_{xx,x} + \sigma_{yx,y} + \sigma_{zx,z} &= \rho u_{,xx} \\
\sigma_{yx,y} + \sigma_{yy,y} + \sigma_{zy,z} &= \rho v_{,yy} \\
\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} &= \rho w_{,zz}
\end{align*} \]  

(9)

and the stress-strain-temperature relations

\[ \begin{align*}
\sigma_{xx} &= 2\mu \varepsilon_{xx} + \lambda (T + \delta_{ik}t_iT_j) \varepsilon_{xx} \\
\sigma_{yx} &= 2\mu \varepsilon_{yx} + \lambda (T + \delta_{ik}t_iT_j) \varepsilon_{yx} \end{align*} \]  

(10)
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\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad e = e_{xx} + e_{yy} + e_{zz} \]  

(11)

and u, v, and w are displacement components in the x, y and z directions, respectively.

The strain-displacement relations are

\[
\begin{align*}
\varepsilon_{xx} &= u_x, \\
\varepsilon_{yy} &= v_y, \\
\varepsilon_{zz} &= w_z \\
\varepsilon_{xy} &= \frac{1}{2}(u_y + v_x), \\
\varepsilon_{yz} &= \frac{1}{2}(v_z + w_y), \\
\varepsilon_{zx} &= \frac{1}{2}(w_x + u_z).
\end{align*}
\]

(12)

When we introduce equation (12) into equations (9) and (10), we obtain the equations of motion represented by displacements as follows.

\[
\begin{align*}
\mu \nabla^2 &\begin{bmatrix} u \\ v \\ w \end{bmatrix} + (\lambda + \mu) \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} - (3\lambda + 2\mu)\alpha^2 \begin{bmatrix} (T + \delta_{xx} T_0) \\ (T + \delta_{yy} T_0) \\ (T + \delta_{zz} T_0) \end{bmatrix} = \rho \begin{bmatrix} u \\ v \\ w \end{bmatrix}.
\end{align*}
\]

(13)

We introduce the following dimensionless quantities:

\[
\begin{align*}
(x, y, z) &= \frac{v}{\kappa}(x, y, z), \\
\theta &= \frac{T}{T_0}, \\
\tilde{\theta} &= \frac{\theta}{\kappa \alpha T_0}, \\
\sigma &= \frac{\sigma_{\tilde{\theta}}}{\mu \alpha T_i}, \\
\tilde{\sigma} &= \frac{\sigma}{\alpha T_i}, \\
\tilde{u} &= \frac{v}{\kappa}(u, v, w), \\
\tilde{v} &= \frac{\nabla^2 \theta - (1 + \tilde{\theta}_0) s \theta'}{m_1} + (m_2 - 1) \frac{\tilde{\sigma}^2}{m_3} + m_1 (1 + \delta_{xx} \tilde{\sigma}) \frac{\tilde{\theta}}{s^2} - m_1 \delta_{xx} \tilde{\theta}.
\end{align*}
\]

(14)

where \( v \) is the velocity of longitudinal wave given by

\[ v = \sqrt{\frac{\lambda + 2\mu}{\rho}} \]

and \( T_0 \) is the reference temperature.

2.2 Laplace and Fourier transforms

Substituting these quantities and applying the Laplace transform to equations (8), (10) and (13), we have three equations:

\( a \) Heat conduction equation

\[ \nabla^2 \theta' - (1 + \tilde{\theta}_0) s \theta' = \frac{\delta}{m_1} + (1 + \delta_{xx} \tilde{\theta}) \tilde{\theta} \]

(15)

\( b \) Equation of motion

\[ \begin{align*}
\nabla^2 &\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} + (m_2 - 1) \begin{bmatrix} \tilde{\sigma}_x \\ \tilde{\sigma}_y \\ \tilde{\sigma}_z \end{bmatrix} - m_1 (1 + \delta_{xx} \tilde{\sigma}) \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} = m_3 s^2 \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix}.
\end{align*}
\]

(16)
c Stress-strain-temperature equation

\[
\begin{align*}
\begin{bmatrix}
\sigma_{xx}^* \\
\sigma_{yy}^* \\
\sigma_{zz}^*
\end{bmatrix} &= 2 \begin{bmatrix}
\bar{\nu}_{xx}^* \\
\bar{\nu}_{yy}^* \\
\bar{\nu}_{zz}^*
\end{bmatrix} + \left( \frac{1}{m_2} - 2 \right) \bar{\sigma}' - \frac{m_1}{m_2} (1 + \delta_{xx} \bar{\theta}) \\
\sigma_{xy}^* &= \begin{bmatrix}
\bar{\nu}_{xy}^* \\
\bar{\nu}_{yx}^* \\
\bar{\nu}_{xy}^* + \bar{\nu}_{yx}^*
\end{bmatrix}, \\
\sigma_{yz}^* &= \begin{bmatrix}
\bar{\nu}_{yz}^* \\
\bar{\nu}_{zy}^* \\
\bar{\nu}_{yz}^* + \bar{\nu}_{zy}^*
\end{bmatrix}, \\
\sigma_{zx}^* &= \begin{bmatrix}
\bar{\nu}_{zx}^* \\
\bar{\nu}_{xz}^* \\
\bar{\nu}_{zx}^* + \bar{\nu}_{xz}^*
\end{bmatrix}
\end{align*}
\]

(17)

Here superscript asterisk (*) is Laplace transform, \( s \) is Laplace parameter and

\[
\begin{align*}
\bar{\nabla}^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \\
m_2 &= \frac{\lambda + 2\mu}{\mu} = \frac{2(1-\nu)}{1-2\nu}.
\end{align*}
\]

(19)

The homogeneous initial conditions are used to this transform.

Applying the Fourier transforms concerned with \( y \) and \( z \) directions to equations (15)–(18), the following relations are obtained.

\[
\begin{align*}
\hat{\theta'}_{xx} - (\eta^2 + \zeta^2) \hat{\theta'} - (1 + \bar{\lambda}s) s \hat{\theta'} &= \frac{\delta}{m_1} (1 + \delta_{xx} \bar{\theta}) s (\hat{\tilde{u}}_{xx} + i\eta \hat{\tilde{v}}' + i\zeta \hat{\tilde{w}}') \\
\hat{\tilde{u}}'_{xx} &= (\eta^2 + \zeta^2) \hat{\tilde{u}}' - m_1 (1 + \delta_{xx} \bar{\lambda}) \left( i\eta \hat{\tilde{v}}' - \hat{\tilde{w}}' \right), \\
\hat{\tilde{v}}'_{xx} &= (\eta^2 + \zeta^2) \hat{\tilde{v}}' - (1 + \delta_{xx} \bar{\lambda}) \hat{\tilde{u}}' - m_1 \eta \hat{\tilde{v}}' - i\zeta \hat{\tilde{w}}', \\
\hat{\tilde{w}}'_{xx} &= (\eta^2 + \zeta^2) \hat{\tilde{w}}' - m_1 \eta \hat{\tilde{v}}' + i\zeta \hat{\tilde{w}}' \\
&\quad + (m_2 - 1) \left( i\eta \hat{\tilde{u}}_x + i\zeta \hat{\tilde{w}}_x \right) - m_2 \left( \hat{\tilde{u}}' + i\eta \hat{\tilde{v}}' + i\zeta \hat{\tilde{w}}' \right) \\
\hat{\tilde{u}}'_{yy} &= (\eta^2 + \zeta^2) \hat{\tilde{u}}' - m_2 \left( \hat{\tilde{u}}' + i\eta \hat{\tilde{v}}' + i\zeta \hat{\tilde{w}}' \right)
\end{align*}
\]

(20)

where over hat (^) denotes Fourier transform and \( \eta \) and \( \zeta \) are Fourier parameters related to \( y \) and \( z \) directions, respectively.
2.3 State space approach

Modifying equations (20)–(23) and adding the identity equations relating to displacements and temperature, we have

\[
D\hat{\tau} = \frac{d\hat{\tau}}{dx}, \quad D\hat{\nu} = \frac{d\hat{\nu}}{dx}, \quad D\hat{\phi} = \frac{d\hat{\phi}}{dx}, \quad D\hat{\theta} = \frac{d\hat{\theta}}{dx}
\]

with \( D = \frac{d}{dx} \), we obtain the following state space equation

\[
\frac{dV(\eta, \zeta, s)}{dx} = A(\eta, \zeta, s)V(\eta, \zeta, s),
\]

where

\[
V(\eta, \zeta, s) = \left\{ \hat{\tau}, \hat{\nu}, \hat{\phi}, D\hat{\tau}, D\hat{\nu}, D\hat{\phi}, D\hat{\theta} \right\}
\]

\[
A(\eta, \zeta, s) = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
a_{31} & 0 & 0 & 0 & a_{56} & a_{57} & a_{58} & 0 \\
a_{62} & a_{63} & a_{64} & a_{65} & 0 & 0 & 0 & 0 \\
a_{72} & a_{73} & a_{74} & a_{75} & 0 & 0 & 0 & 0 \\
a_{82} & a_{83} & a_{84} & a_{85} & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
a_{31} = s^2 + \eta^2 + \zeta^2, \quad a_{32} = i\left( \frac{1}{m_2} - 1 \right) \eta, \quad a_{33} = i\left( \frac{1}{m_2} - 1 \right) \zeta, \quad a_{56} = m_1
\]

\[
a_{62} = m_2(s^2 + \eta^2) + \zeta^2, \quad a_{63} = (m_2 - 1)\eta \zeta, \quad a_{64} = im_2 \eta, \quad a_{65} = i(1 - m_2)\eta
\]

\[
a_{72} = a_{31}, \quad a_{73} = m_2(s^2 + \zeta^2) + \eta^2, \quad a_{74} = im_2 \zeta, \quad a_{75} = i(1 - m_2)\zeta
\]

\[
a_{82} = \frac{i}{m_1} \delta_s (1 + \delta_3 \bar{\xi}_s) \eta, \quad a_{83} = i\frac{\delta_s}{m_1} (1 + \delta_3 \bar{\xi}_s) \zeta, \quad a_{84} = \delta_s n(1 + \bar{\xi}_s) + \eta^2 + \zeta^2,
\]

\[
a_{85} = \frac{\delta_s}{m_1} (1 + \delta_3 \bar{\xi}_s).
\]

The solution to equation (25) is easily obtained as

\[
V(\eta, \zeta, s) = \exp[A(\eta, \zeta, s)\eta]V(0, \eta, \zeta, s).
\]

The characteristic equation of matrix \( A(\eta, \zeta, s) \) is

\[
p^8 + b_1 p^7 + b_2 p^6 + b_3 p^5 + b_4 = 0.
\]
In equation (29), the terms of odd power do not appear in this case. Therefore, the eigenvalues are written as $\pm p_1, \pm p_2, \pm p_3$, and $\pm p_4$.

From Cayley-Hamilton theorem, the matrix $A(\eta, \zeta, s)$ satisfies the characteristic equation (29). Therefore, we obtain

$$A^8 + b_1A^6 + b_2A^4 + b_3A^2 + b_4I = 0,$$

(30)

where $I$ is identity matrix.

From equation (30), the exponential matrix $\exp(A\tau)$ can be truncated at the seventh order without approximation, that is,

$$\exp(A\tau) = \sum_{k=0}^{\infty} \frac{(A\tau)^k}{k!} = I + \frac{A\tau}{1!} + \frac{(A\tau)^2}{2!} + \cdots$$

$$= c_0I + c_1A\tau + c_2(A\tau)^2 + c_3(A\tau)^3 + c_4(A\tau)^4 + c_5(A\tau)^5 + c_6(A\tau)^6 + c_7(A\tau)^7.$$  

(31)

When the matrix $A$ is replaced with the eigenvalue $p$, unknowns $c_i (i = 0, 1, 2, \ldots, 7)$ are obtained.

3 Numerical calculations

3.1 Analytical model

We consider an infinite plate (thin film) having the thickness $l$ under the partial heating $f(y, z, t)$ on the upper surface ($x = l$). The relative heat transfer coefficients of bottom and upper surfaces are $h_a$ and $h_b$, respectively.

3.2 Boundary conditions

The temperature boundary condition of upper surface is

$$q = \lambda h_b \{ T - f(y, z, t) \}.$$  

(32)

Substituting this equation into the modified Fourier equation and applying Laplace and Fourier transforms, we obtain

$$\bar{\tau} = \bar{T} : \hat{D}\hat{\vartheta} + (1 + \delta_{1}\bar{r}_0s)H_a\hat{\vartheta} = (1 + \delta_{1}\bar{r}_0s)H_a\hat{\vartheta},$$  

(33)

where

$$\bar{f} = \frac{f(y, z, t)}{T_s}, \quad H_a = \frac{K}{V_a}h_a, \quad H_s = \frac{K}{V_s}h_s.$$  

(34)

Similarly, the temperature boundary condition of bottom surface ($x = 0$) is

$$\bar{\tau} = 0 : \hat{D}\hat{\vartheta} - (1 + \delta_{1}\bar{r}_0s)H_a\hat{\vartheta} = 0.$$  

(35)
The traction boundary conditions of bottom and upper surfaces are obtained from traction-free conditions

\[
\begin{align*}
\mathbf{r} = 0: \bar{\sigma}_{xz} &= \bar{\sigma}_{yz} = \bar{\sigma}_{zx} = 0, \\
\mathbf{r} = \mathbf{r}_b: \sigma_{xz} &= \sigma_{yz} = \sigma_{zx} = 0
\end{align*}
\] (36)

Using temperature and traction boundary conditions, we can obtain the temperature, displacements and thermal stresses in the transformed domains. Since it is difficult to obtain the solutions in the original domain analytically, we use the numerical inversions.

### 3.3 Calculations

In the calculations, we use the dimensionless thickness \( \bar{T} = 1 \), which means thin film, relaxation time \( \bar{T}_r = 0.01 \), Poisson’s ratio \( \nu = 0.3 \), and thermomechanical coupling parameter \( \delta = 0.03 \).

For temperature conditions, we use dimensionless relative heat transfer coefficients \( h_a = 0 \) and \( h_b = 10 \), and \( f(y, z, t) = T_i H(1-|y|)H(1-|z|)H(t) \).

Figure 1 shows the temperature distribution at the thickness direction at the centre of heating \((y = z = 0)\). The alternate long and short dash line, the short dashed line and the solid line indicate times \( \bar{T} = 0.6 \), \( \bar{T} = 1.0 \), and \( \bar{T} = 2.0 \), respectively. The appearance where the temperature rises is understood as time passes.

**Figure 1** Temperature distribution at the thickness direction \((y = z = 0)\)

![Temperature distribution at the thickness direction](image)

Figure 2 shows the comparison of generalised theory and classical theory for thermal stress distribution \( \sigma_{xz} \). The short dashed line and the solid line indicate the classical theory and generalised theory, respectively. The peak stress has occurred at the stress wave front. The maximum stress of generalised theory is slightly greater than that of classical one. It is important that the consideration of the finite velocity of thermal wave enlarges the maximum thermal stress. Generally, large relaxation time enlarges this difference.

**Figure 2** Comparison of generalised theory and classical theory for thermal stress distribution \( \sigma_{xz} \)

![Comparison of generalised theory and classical theory for thermal stress distribution](image)
Figure 2  Thermal stress $\sigma_y$ distribution at the thickness direction ($y = z = 0$)

Figure 3 shows the temperature change at the centre of heating according to the width of heating. The alternate long and short dash line and the solid line indicate one-dimensional case and three-dimensional case (present case), respectively. It is understood that this result approaches the one-dimensional case as the width of heating extends.

Figure 3  Temperature change of at the centre of heating ($x = 1, y = z = 0$)

4 Conclusions

In the present study, the three-dimensional thermal stress of thin film under a partial heating was analysed in the context of generalised thermoelasticity. The integral transforms, that is, Laplace transform and Fourier transform, were used, and state space approach was adopted. From numerical calculation, the maximum stress at the stress wave front of generalised theory was slightly greater than that of classical one.
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