A new look at compactly supported biorthogonal Riesz bases of wavelets

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Abstract: In this paper, we give two algorithms to compute compactly supported biorthogonal Riesz basis of wavelets. The input to these algorithms are filters of the transfer and the dual transfer functions, which are obtained by solving the Bezout equation. This Bezout equation arises from biorthogonality of the scaling function and the dual scaling function. We solve the Bezout equation in a simple and algebraic way. We also give a case study of the biorthogonal wavelets showing their detail construction. Some references to Sobolev regularity of the wavelets which is a qualitative property of the wavelet is also made by us.

Keywords: biorthogonal wavelet; Bezout equation; transfer function; transition operator; Riesz bases; Sobolev regularity.


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1 Introduction

The term wavelet was introduced by Grossmann and Morlet (1984) in their work on constant shapes. It denotes a univariate function \( \psi \), defined on \( \mathbb{R} \), which when subjected to the fundamental operations of dyadic dilations and integer translations yields an orthogonal basis of \( L_2(\mathbb{R}) \). That is, the functions \( \psi_{j,k} := 2^j \psi(2^j \cdot - k) \), for \( j, k \in \mathbb{Z} \), form a complete orthonormal basis for \( L_2(\mathbb{R}) \). One of the pioneer work in this field is the orthogonal wavelets by Daubechies (1988) (see also Daubechies, 1992).

In many numerical applications, instead of orthogonality, it is desired that the translated dilates \( \psi_{j,k} \), \( k \in \mathbb{Z} \) form a Riesz basis for \( L_2(\mathbb{R}) \). Note that, an orthogonal wavelet basis is itself a Riesz basis. Some other important Riesz bases of \( L_2(\mathbb{R}) \) are semiorthogonal wavelet bases of Chui and Wang (1992) (CW-wavelet), biorthogonal wavelet bases of Cohen et al. (1992) (CDF-wavelet), and compactly supported wavelet bases of Jia et al. (2003) (JWZ-wavelet). Construction of most of the wavelets follow multiresolution analysis (MRA) (Christian, 1998; Han and Jia, 2007; Chui, 1992; Cohen, 2003; Daubechies, 1992; Goswami and Chan, 1999; Mallat, 1989, 1998; Meyer, 1992).

**Definition (Cohen, 2003):** A multiresolution analysis is a sequence of closed subspaces of \( L_2(\mathbb{R}) \), such that the following properties are satisfied:

1. the sequence is nested, i.e., for all \( j \), \( V_j \subset V_{j+1} \)
2. the spaces are related to each other by dyadic scaling, i.e.,
   \[ f \in V_j \iff f(2^j \cdot) \in V_{j+1} \iff f(2^{-j} \cdot) \in V_0 \]
3. the union of the spaces is dense, i.e., for all \( f \) in
   \[ L_2(\mathbb{R}) \lim_{j \to \infty} \| f - P_j f \|_{L_2} = 0, \]
   where \( P_j \) is the orthogonal projection onto \( V_j \)
4. the intersection of the spaces is reduced to the null function, i.e.,
   \[ \lim_{j \to \infty} \| P_j f \|_{L_2} = 0 \]
5. there exist a function \( \phi \in V_0 \) such that the family \( \{ \phi \cdot (\cdot - k) \mid k \in \mathbb{Z} \} \) is a Riesz basis of \( V_0 \).

In MRA of biorthogonal wavelet bases (Cohen et al., 1992), instead of having one hierarchy of approximation spaces, we have two hierarchies of approximation spaces:

\[ \cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \]

and \( \cdots \tilde{V}_{-2} \subset \tilde{V}_{-1} \subset \tilde{V}_0 \subset \tilde{V}_1 \subset \tilde{V}_2 \subset \cdots \).
The detail space $W_j$ and $\tilde{W}_j$ are defined by $V_j + W_j$ and $V_{j+1} = V_j + \tilde{V}_j$, respectively. In general, $W_j$ is not orthogonal to $V_j$ and $\tilde{W}_j$ is not orthogonal to $\tilde{V}_j$. In biorthogonality, we need

$$\tilde{W}_j \perp V_j \quad \text{and} \quad W_j \perp \tilde{V}_j.$$  \hfill (1.1)

Let $\phi$, $\tilde{\phi}$, $\psi$ and $\tilde{\psi}$ be such that

$$V_j := \text{span}\left\{ \phi_{j,k} := 2^j \phi(2^j k); \quad j, k \in \mathbb{Z} \right\},$$

$$\tilde{V}_j := \text{span}\left\{ \tilde{\phi}_{j,k} := 2^j \tilde{\phi}(2^j k); \quad j, k \in \mathbb{Z} \right\},$$

$$W_j := \text{span}\left\{ \psi_{j,k} := 2^j \psi(2^j k); \quad j, k \in \mathbb{Z} \right\}$$

and

$$\tilde{W}_j := \text{span}\left\{ \tilde{\psi}_{j,k} := 2^j \tilde{\psi}(2^j k); \quad j, k \in \mathbb{Z} \right\}.$$

Then the functions $\phi$ and $\tilde{\phi}$ are called the scaling function and the dual scaling function, respectively. Similarly, $\psi$ and $\tilde{\psi}$ are called the wavelet and the dual wavelet, respectively. The biorthogonality condition is

$$\langle \psi_{j,k}, \tilde{\psi}_{j',k'} \rangle = \delta_{j,j'} \delta_{k,k'}, \quad j, k, j', k' \in \mathbb{Z}. \quad \hfill (1.2)$$

In this case, the pair $(\psi, \tilde{\psi})$ is called a biorthogonal wavelet pair.

Here, we give a little discussion on $CW$, $JWZ$ and $CDF$ wavelets. In all these cases, the scaling functions as well as the wavelets are compactly supported, and are symmetric or antisymmetric. The dual scaling functions and the dual wavelets exist in both $CDF$ and $CW$ cases. While they are compact in $CDF$ cases, they are non-compactly supported in $CW$ cases. The biorthogonality property (1.2) is clear in case of $CDF$ and $CW$-wavelets while this property is not so far established in case of $JWZ$-wavelets which are recently introduced.

The pair $\psi, \tilde{\psi}$ satisfy the two scale relations

$$\hat{\phi}(\xi) = m_0 \left( \frac{\xi}{2} \right) \hat{\phi} \left( \frac{\xi}{2} \right) \quad \text{and} \quad \hat{\tilde{\phi}}(\xi) = \tilde{m}_0 \left( \frac{\xi}{2} \right) \hat{\phi} \left( \frac{\xi}{2} \right), \quad \xi \in \mathbb{R}, \quad \hfill (1.3)$$

for some trigonometric polynomials $m_0(\xi)$ and $\tilde{m}_0(\xi)$ which are called as transfer functions of $\phi$ and $\tilde{\phi}$, respectively. The necessary and sufficient condition (Cohen et al., 1992) for exact reconstruction is
This is also one of the necessary conditions (Cohen et al., 1992) for biorthogonality of the pair \((\psi, \tilde{\psi})\). It is also shown (Cohen et al., 1992) that for suitable choice of \(x\), the above equation reduces to the Bezout equation

\[
(1 - x^k) P(x) + x^k P(1 - x) = 1,
\]

where \(P(x)\) is a polynomial of degree \(k - 1\) (this is briefly outlined in Section 2).

The paper consists of five sections. The construction of CDF wavelets is briefly discussed in Section 2. Their qualitative properties are briefly discussed in Section 3. In Section 4, we have attempted to give a new look at these constructions, at least computationally and provide two algorithms for this. Note that computing all CDF wavelets is space and time consuming. Therefore, in this section, we provide a case study of these wavelets. In particular, we find all the transfer functions for these cases by solving Bezout equation in a simple and algebraic way, then find the corresponding Lawton matrices, their eigenvalues, and the trigonometric polynomial which is invariant under the action of the transition operator. In Section 5, we give a conclusion.

## 2 Biorthogonal Riesz bases of wavelets

In this section, we briefly outline the construction of the CDF wavelets. These wavelets form Riesz bases of \(L_2(\mathbb{R})\). A Riesz basis can be defined in several ways (Cohen et al., 1992; Chui, 1992). The following definition is proposed in Chui (1992) and is based on Fourier transform which is used in several literatures.

**Definition:** The following two conditions define a Riesz basis.

- a sequence \((2^j u(2^j \cdot -k))_{j,k \in \mathbb{Z}}\) is said to be a Riesz sequence if and only if for all \(\xi\) \(\in \mathbb{R}\) there exists constants \(0 < A < B < \infty\), such that

\[
A \sum_k |\hat{u}(\xi + 2k\pi)|^2 < B
\]

(2.1)

- a Riesz sequence \((2^j u(2^j \cdot -k))_{j,k \in \mathbb{Z}}\) is a Riesz basis if and only if \((2^j u(2^j \cdot -k))_{j,k \in \mathbb{Z}}\) are linearly independent.

In our construction, we are considering compactly supported wavelets. Therefore, our transfer functions \(m_0\) and \(\tilde{m}_0\) are polynomials of exponential types. That is,

\[
m_0(\xi) = \frac{1}{2} \sum_{n=N_1}^{N_2} h_n e^{-i\omega_n \xi},
\]

(2.2)

and

\[
\tilde{m}_0(\xi) = \frac{1}{2} \sum_{n=N_1}^{N_2} \tilde{h}_n e^{-i\omega_n \xi}
\]

(2.3)
where \( m_0(0) = \tilde{m}_0(0) = 1 \). The two scale equations are

\[
\phi(x) = \sum_k h_k \phi(2x - k)
\]

(2.4)

and

\[
\tilde{\phi}(x) = \sum_k \tilde{h}_k \tilde{\phi}(2x - k),
\]

(2.5)

which in Fourier domain are equivalent to (1.3). Parallel to orthonormal case (3.47) of Daubechies (1988) or (6.2.6) of Daubechies (1992), we define

\[
\psi(x) := \sum_k g_{k+1} \phi(2x - k) = \sum_k (-1)^k \tilde{h}_{k-1} \phi(2x - k),
\]

(2.6)

And

\[
\tilde{\psi}(x) := \sum_k \tilde{g}_{k+1} \tilde{\phi}(2x - k) = \sum_k (-1)^k h_{k-1} \tilde{\phi}(2x - k).
\]

(2.7)

Equivalently,

\[
\tilde{\psi}(\xi) = e^{\frac{i}{2} \tilde{m}_0 \left( \frac{\xi}{2} + \pi \right)} \tilde{\phi}\left( \frac{\xi}{2} \right),
\]

(2.8)

and

\[
\tilde{\psi}(\xi) = e^{\frac{i}{2} \tilde{m}_0 \left( \frac{\xi}{2} + \pi \right)} \tilde{\phi}\left( \frac{\xi}{2} \right).
\]

(2.9)

The function \( \psi \) and \( \tilde{\psi} \) can only be candidates for Riesz bases of wavelets if they satisfy \( \psi(0) = 0 \) and \( \tilde{\psi}(0) = 0 \). In terms of \( m_0 \) and \( \tilde{m}_0 \) these are equivalent to

\[
\tilde{m}_0(\pi) = 0 = m_0(\pi).
\]

(2.10)

The following Lemma (Cohen and Daubechies, 1992) gives a necessary and sufficient condition for linear independence of \( \psi_{j,k} \) and \( \tilde{\psi}_{j,k} \).

**Lemma 1:** The sequences \((\psi_{j,k})\) and \((\tilde{\psi}_{j,k})\) are linearly independent if and only if

\[
\langle \psi_{j,k}, \tilde{\psi}_{j',k'} \rangle = \delta_{j,j'} \delta_{k,k'}.
\]

The above biorthogonality condition between \( \psi_{j,k} \) and \( \tilde{\psi}_{j,k} \) is equivalent to a biorthogonality condition between \( \phi_{0,\ell} \) and \( \tilde{\phi}_{0,\ell} \).

**Lemma 2** (Cohen et al., 1992):

\[
\langle \psi_{j,k}, \tilde{\psi}_{j',k'} \rangle = \delta_{j,j'} \delta_{k,k'} \text{ if and only if } \langle \phi_{0,\ell}, \tilde{\phi}_{0,\ell} \rangle = \delta_{\ell,\ell}.
\]
The stability of the biorthogonal wavelets and the condition \( \langle \phi_{0,k}, \tilde{\phi}_{0,j} \rangle = \delta_{k,l} \) are shown with the help of two operators \( T_0 \) and \( \tilde{T}_0 \) called as transition operators which act upon \( 2\pi \)-periodic functions. They are defined by

\[
(T_0 f)(\xi) = m_0 \left( \frac{\xi}{2} \right)^2 f\left( \frac{\xi}{2} \right) + m_0 \left( \frac{\xi}{2} + \pi \right)^2 f\left( \frac{\xi}{2} + \pi \right)
\]

and

\[
(\tilde{T}_0 f)(\xi) = \tilde{m}_0 \left( \frac{\xi}{2} \right)^2 f\left( \frac{\xi}{2} \right) + \tilde{m}_0 \left( \frac{\xi}{2} + \pi \right)^2 f\left( \frac{\xi}{2} + \pi \right).
\]

**Theorem 1** (Cohen et al., 1992): Let \( m_0 \) and \( \tilde{m}_0 \) satisfy (1.4). Then the following two conditions are equivalent

a. \( \phi, \tilde{\phi} \in R \) and \( \langle \phi_{0,k}, \tilde{\phi}_{0,j} \rangle = \delta_{k,l} \)

b. there exist strictly positive trigonometric polynomials \( f_0 \) and \( \tilde{f}_0 \) so that \( T_0 f_0 = f_0 \) and \( \tilde{T}_0 \tilde{f}_0 = \tilde{f}_0 \) and they are the only trigonometric polynomials (up to normalisation) invariant under \( T_0 \) and \( \tilde{T}_0 \), respectively.

Let us begin by designing linear phase filters associated with the transfer functions \( m_0 \) and \( \tilde{m}_0 \) that lead to symmetric scaling functions \( \phi \) and \( \tilde{\phi} \). The transfer functions are symmetric in nature and satisfy (1.4). Moreover, they are polynomials. They are either in form

a. \( m_0(\xi) = \cos \left( \frac{\xi}{2} \right)^{2l} P_0(\cos \xi), \quad \tilde{m}_0(\xi) = \cos \left( \frac{\xi}{2} \right)^{2l} \tilde{P}_0(\cos \xi) \)

b. \( m_0(\xi) = e^{-i\frac{\xi}{2}} \cos \left( \frac{\xi}{2} \right)^{2l+1} P_0(\cos \xi), \quad \tilde{m}_0(\xi) = e^{-i\frac{\xi}{2}} \cos \left( \frac{\xi}{2} \right)^{2l+1} \tilde{P}_0(\cos \xi) \)

where \( l, \tilde{l} \in N, \; P_0 \) and \( \tilde{P}_0 \) are polynomials with \( P_0(-1) \neq 0 \) and \( \tilde{P}_0(-1) \neq 0 \). Substituting \( m_0 \) and \( \tilde{m}_0 \) in (1.4), we get

\[
\left( \cos \frac{\xi}{2} \right)^{2K} P_0(\cos \xi) \tilde{P}_0(\cos \xi) + \left( \sin \frac{\xi}{2} \right)^{2K} P_0(-\cos \xi) \tilde{P}_0(-\cos \xi) = 1
\]

with \( K = l + \tilde{l} \) in case (a) and \( K = l + \tilde{l} + 1 \) in case (b).

Since, \( \sin^2 \frac{\xi}{2} = \frac{1 - \cos \xi}{2} \), (2.13) reduces to the Bezout equation

\[
(1 - x)^K P(x) + x^K P(1 - x) = 1
\]

(2.14)
where \( P(x) = p_0(\cos \xi) p_0(\cos \xi) \) is a polynomial in \( x \). This equation has a polynomial solution \( P(x) \) of degree \( K - 1 \) which is guaranteed by Theorem 6.3 of Cohen et al. (1992).

Let

\[
P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{K-1}x^{K-1}.
\] (2.15)

Substituting this in (2.14), collecting all the coefficients of \( x^j, j = 0, 1, 2, \ldots, K - 1 \) together and comparing it with the right hand side of the same equation, we get

\[
Aa = e
\] (2.16)

where

\[
a = (a_0, a_1, \cdots, a_{K-1})^T, \quad A = (a_{i,j})_{i,j=0}^{K-1}, \quad a_{i,j} = (-1)^{i-j} \binom{K}{i-j}
\]

and \( e = (1, 0, 0, \cdots, 0)^T \).

It is easy to observe that \( A \) is a lower triangular matrix with

\[
\det A = \binom{K}{0}^K = 1 \neq 0.
\]

Therefore, unique solution of (2.16) exists and the solution is

\[
a_0 = 1
\]

\[
a_1 = \binom{K}{1}a_0
\]

\[
a_2 = \binom{K}{1}a_1 - \binom{K}{2}a_0
\]

\[\vdots\]

\[
a_{K-1} = \binom{K}{1}a_{K-2} - \binom{K}{2}a_{K-3} + \cdots + (-1)^{K} \binom{K}{K-1}a_0.
\]

It is easy to observe that all \( a_i \)'s are integers.

Checking condition (b) of Theorem 1 needs transition operators \( T_0 \) and \( \tilde{T}_0 \) which are defined in (2.11) and (2.12), respectively. Let

\[
f_0(\xi) = b_N e^{-iN\xi} + b_{N+1} e^{-i(N-1)\xi} + \cdots + b_{N+1} e^{-iN\xi} + b_N e^{iN\xi}.
\] (2.17)

\[
\tilde{f}_0(\xi) = \tilde{b}_N e^{-iN\xi} + \tilde{b}_{N+1} e^{-i(N-1)\xi} + \cdots + \tilde{b}_{N+1} e^{-iN\xi} + \tilde{b}_N e^{iN\xi}.
\] (2.18)

Then, the function \( T_0f_0 \) can be written in matrix form as

\[
T_0f_0(\xi) = e_T M_T \beta_T
\] (2.19)
where
\[ e_\beta = (e^{-iN\xi}, e^{-i(N-1)\xi}, \ldots, e^{i(N-1)\xi}, e^{iN\xi}), \]
\[ \beta_\beta = (b_{-N}, b_{-N+1}, \ldots, b_{N-1}, b_N)^T, \]
and \( M_\beta \) is a suitable matrix of order \((2N + 1) \times (2N + 1)\). Let
\[
|m_0(\xi)|^2 = \sum_{k=-N}^{N} H_k e^{-iK\xi}. \tag{2.20}
\]

It is easy to verify that \( M_\beta \) is the Lawton matrix \( H \) (Cohen et al., 1992; Curran and McDarby, 2003; Lawton, 1991), where
\[
H = \left(2H_{i-jz}^N\right)_{i,j=-N} \quad H_K = \sum_r h_r h_{r+N}. \tag{2.21}
\]

Similarly, the function \( \tilde{T}_0 \tilde{f}_0 \) can be written in matrix form as
\[
\tilde{T}_0 \tilde{f}_0(\xi) = e_\beta M_\beta \beta_\beta \tag{2.22}
\]
where
\[ e_\beta = (e^{-i\tilde{N}\xi}, e^{-i(\tilde{N}-1)\xi}, \ldots, e^{i(\tilde{N}-1)\xi}, e^{i\tilde{N}\xi}), \]
\[ \beta_\beta = (b_{-\tilde{N}}, b_{-\tilde{N}+1}, \ldots, b_{\tilde{N}-1}, b_{\tilde{N}})^T, \]
and \( M_\beta \) is the Lawton matrix \( \tilde{H} \) where
\[
\tilde{H} = \left(2\tilde{H}_{i-jz}\right)_{i,j=-\tilde{N}} \quad \tilde{H}_K = \sum_r h_r h_{r+\tilde{N}}. \tag{2.23}
\]

Note that
\[
|m_0(\xi)|^2 = \sum_{K=-\tilde{N}}^{\tilde{N}} \tilde{H}_K e^{-iK\xi}. \tag{2.24}
\]

The matrices \( H \) and \( \tilde{H} \) are well studied in Cohen and Daubechies (1992) and Curran and McDarby (2003).

It can be easily checked that both \( T_0 \) and \( M_\beta \) have same eigenvalues. If \( \nu_\beta \) is an eigenvector of \( M_\beta \) then the discrete Fourier transform \( e_{\gamma T_0} \beta_\beta \) is the corresponding eigenvector of \( T_0 \). Similar conclusions hold for \( \tilde{T}_0 \). Therefore, instead of analysing eigensystems of \( T_0 \) and \( \tilde{T}_0 \), we analyse eigensystems of \( M_\beta \) and \( \tilde{M}_\beta \).

Checking condition (b) of Theorem 2.4 is now much easier. This means, now we only have to check
a whether 1 is the non-degenerate eigenvalue of both \( M_\beta \) and \( \tilde{M}_\beta \)
b the discrete Fourier transforms \( e_{\gamma T_0} \beta_\beta \) and \( e_{\gamma \tilde{T}_0} \beta_\beta \) of corresponding eigenvectors \( \nu_\beta \) and \( \nu_{\tilde{\beta}} \) are strictly positive trigonometric polynomials.
3 Qualitative properties of wavelets

Clearly, we have several biorthogonal wavelets. It is difficult to choose which one is the best for all purpose. The choice of choosing depends upon the kind of application and on the qualitative properties of the wavelets. In this section, we made several references to qualitative properties of biorthogonal wavelets like vanishing moments, polynomial exactness and regularity.

3.1 Vanishing moments

**Definition:** A function \( f \in L^2(\mathbb{R}) \) is said to have vanishing moments of order \( L \) if
\[
\int_{-\infty}^{\infty} x^j f(x) \, dx = 0, \quad \text{for} \quad j = 0, 1, 2, \ldots, L - 1.
\]

**Theorem 2:** Let \( \psi \in L^2(\mathbb{R}) \) be such that
\[
\hat{\psi}(\xi) = \left(1 - e^{-i\xi}\right)^L u(\xi), \quad u(0) \neq 0.
\]

Then \( \psi \) has vanishing moment of order \( L \).

3.2 Polynomial exactness

Let \( m_0(\xi) \) be the transfer function of compactly supported refinable function \( \phi \). If \( m_0(\xi) \) has the factorised form
\[
m_0(\xi) = \left(1 + e^{-i\xi}\right)^L p(\xi),
\]

where \( p(\xi) \) is also a trigonometric polynomial such that \( p(0) = 1 \), then \( \phi \) satisfies the Strang-Fix condition of order \( L \). Moreover, \( \phi \) has polynomial exactness of order \( L \), i.e., \( \Pi_N \subset V_j \), for all \( j \), where \( \Pi_N \) is the space of polynomial of order \( N \) (Carlos et al., 1999).

3.3 Regularity

The approximation properties of wavelets are directly linked to regularity of wavelets which are as much regular as the scaling functions. The regularity of scaling functions are measured in terms of Sobolev spaces.

**Definition (Cohen, 2003):** For \( s \geq 0 \), the Sobolev space \( H^s \) is the set of all \( L^2 \) functions such that
\[
\left\| f \right\|_{H^s} := \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^{s/2} \left| \hat{f}(\xi) \right| d\xi
\]
is finite.
Definition (Cohen, 2003): For \( 1 \leq q \leq \infty \), the \( L^q \)-Fourier smoothness exponent of \( f \) is the quantity
\[
S_q(f) = \sup \left\{ s, (1 + |\cdot|)^s \hat{f} \in L^q \right\}.
\]
In case \( q = 2 \), this is exactly the classical Sobolev exponent.

Definition (Cohen, 2003): Let \( u(\xi) \) be a trigonometric polynomial and \( T_u \) be the transition operator associated with the polynomial \( u(\xi) \). Let \( M_u \) be the Lawton matrix corresponding to \( u(\xi) \). Then the spectral value of \( T_u \) is defined to be the largest eigenvalue of \( M_u \).

Following theorem determines Sobolev regularity of \( \phi \) and \( \tilde{\phi} \).

Theorem 4 (Cohen, 2003; Curran and McDarby, 2003): Let \( s_2(\phi) \) denotes the regularity exponent of \( \phi \) in the Sobolev space \( H^s \). Let, we have the factorisation
\[
m_0(\xi) = \left( \frac{1 + e^{-i\xi}}{2} \right)^{\ell} g(\xi)
\]
and \( T_g \) be the transition operator associated with the function \( g(\xi) \). Let \( \rho \) be the spectral radius of the operator \( T_g \). Then
\[
L - \frac{\log \rho}{2\log 2} \leq s_2(\phi) \leq L - \frac{1}{2}.
\]

4 Case studies

In this section, we construct biorthogonal Riesz bases of wavelets for the case \( K = 2 \). The construction for other cases follows similarly. For general \( K \), the construction follows from two different algorithms which are described at the end of this section.

When \( K = 2 \), the possible cases for the pair \( (l, \tilde{l}) \), \( l + \tilde{l} = K \) are (2, 0), (1, 1) and (0, 2). Moreover, in this case \( P(x) = 1 + 2x \). Therefore,
\[
P\left( \sin^2 \frac{\xi}{2} \right) = 2 - \cos \xi = -0.5e^{-i\xi} + 2 - 0.5e^{i\xi}.
\]
It has a factorisation
\[
P\left( \sin^2 \frac{\xi}{2} \right) = p_0(\cos \xi) \tilde{p}_0(\cos \xi)
\]
where
\[
p_0(\cos \xi) = -0.5e^{-i\xi} + 2 - 0.5e^{i\xi} \quad \text{and} \quad \tilde{p}_0(\cos \xi) = 1.
\]
Note that \( \cos^2 \frac{\xi}{2} \) in exponential form is equals to \( 0.25e^{-i\xi} + 0.5 + 0.25e^{i\xi} \).

\[
l = 2, \tilde{l} = 0
\]
(A1)
The possible cases of transfer function are

\[ m_0(\xi) = \left( \frac{1 + e^{-i\xi}}{2} \right)^2 \left( 0.25e^{-i\xi} + 0.5 + 0.25e^{i\xi} \right)^2 p_0(\cos \xi) \]

\[ = -0.03125e^{-3i\xi} + 0.28125e^{i\xi} - 0.5 + 0.28125e^{-i\xi} - 0.03125e^{3i\xi}, \]

and

\[ \tilde{m}_0(\xi) = 1. \]

\[ m_0(\xi) = \left( \frac{1 + e^{-i\xi}}{2} \right)^2 \left( 0.25e^{-i\xi} + 0.5 + 0.25e^{i\xi} \right)^2 p_0(\cos \xi) \]

\[ = -0.015625e^{-4i\xi} - 0.015625e^{-2i\xi} + 0.140625e^{-i\xi} + 0.390625e^{i\xi} \]

\[ + 0.390625e^{2i\xi} + 0.140625e^{3i\xi} - 0.015625e^{4i\xi} - 0.015625e^{2i\xi}, \]

In case (a) \( \tilde{T}_0f_0 = 2\tilde{f}_0 \), for any \( \tilde{f}_0 \) of the form (2.18). Hence, condition (b) of theorem 1 is not satisfied. Therefore, this case does not lead to a Riesz basis.

In case (b), both the transition operator \( T_0 \) and \( \tilde{T}_0 \) have unique eigenvalues 1. The corresponding eigenfunctions are \( f_0 \) and \( \tilde{f}_0 \), respectively, where

\[ f_0 = 6.78813 \times 10^{-11} e^{-6i\xi} + 1.38885 \times 10^{-7} e^{-5i\xi} \]

\[ + 0.000941807 e^{-4i\xi} + 0.000984409 e^{-3i\xi} \]

\[ - 0.0433276 e^{-2i\xi} + 0.287618 e^{-i\xi} + 0.911481 \]

\[ + 0.287618 e^{i\xi} - 0.0433276 e^{2i\xi} + 0.000984409 e^{3i\xi} \]

\[ + 0.000941807 e^{4i\xi} + 1.38885 \times 10^{-7} e^{5i\xi} + 6.78813 \times 10^{-11} e^{6i\xi} \]

and \( \tilde{f}_0 = 1 \). It is easy to observe that \( f_0 \) and \( \tilde{f}_0 \) are strictly positive polynomials. This leads to a biorthogonal Riesz wavelet bases pair \( (\psi, \tilde{\psi}) \) of \( L^2(\mathbb{R}) \). More precisely, we denote the pair by \( (\psi^{2,0,0,\tilde{b}}, \tilde{\psi}^{2,0,0,\tilde{b}}) \). They are shown in Figures 1(b) and 2(b), respectively.

Observe that we have the factorisation

\[ m_0(\xi) = \left( \frac{1 + e^{-i\xi}}{2} \right)^3 u(\xi), \]

where

\[ u(\xi) = e^{3i\xi} (0.5 - 2e^{-i\xi} + 0.5e^{-2i\xi}). \]

The spectrum of the transition operator \( T_0 \) corresponds to \( u(\xi) \) is 1.55923. Therefore, Sobolev regularities of \( \phi^{2,0,0,\tilde{b}} \) and \( \psi^{2,0,0,\tilde{b}} \) are greater than equal to 5 – 1.55923 = 3.44077. Also observe that \( \tilde{m}_0 \) is the transfer function of B-spline of order 1. Hence, Sobolev regularities of \( \tilde{\phi}^{2,0,0,\tilde{b}} \) and \( \tilde{\psi}^{2,0,0,\tilde{b}} \) are 0.5

\[ l = 1, \tilde{l} = 1 \]
The possible cases of transfer function are

\( m_0(\xi) = -0.125e^{-2\xi} + 0.25e^{-\xi} + 0.75 + 0.25e^{\xi} - 0.125e^{2\xi} \)
\( \hat{m}_0(\xi) = 0.25e^{-\xi} + 0.5 + 0.25e^{\xi} \)
\( m_0(\xi) = \left(\frac{1 + e^{-\xi}}{2}\right)(-0.125e^{-2\xi} + 0.25e^{-\xi} + 0.75 + 0.25e^{\xi} - 0.125e^{2\xi}) \)
\( = -0.0625e^{-3\xi} - 0.0625e^{-2\xi} + 0.5e^{-\xi} + 0.5 + 0.0625e^{\xi} - 0.0625e^{2\xi} \)
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\[
\hat{m}_0(\xi) = \left( \frac{1+e^{-i\xi}}{2} \right)(0.25e^{-i\xi} + 0.5 + 0.25e^{i\xi})
= 0.375 + 0.125e^{-2i\xi} + 0.375e^{-i\xi} + 0.125e^{i\xi}
\]

In case (a), both the transition operator \( T_0 \) and \( \hat{T}_0 \) have unique eigenvalues 1. The corresponding eigenfunctions are \( f_0 \) and \( \hat{f}_0 \), respectively.

\[
f_0(\xi) = 0.00146987e^{-i\xi} + 0.0529135e^{i-\xi} - 0.295444e^{-i\xi} + 0.90544
- 0.295444e^{i\xi} + 0.0529153e^{2i\xi} + 0.00146987e^{3i\xi},
\]

and

\[
\hat{f}_0 = 0.235702e^{-i\xi} + 0.942809 + 0.235702e^{i\xi}.
\]

Both are always strictly positive. Therefore, in this case we get a biorthogonal Riesz wavelet pair. We denote the pair by \( (\psi^{1,1,a}, \phi^{1,1,a}) \). They are shown in Figure 3(b) and 4(b), respectively.

Moreover, in case (a), \( \hat{m}_0 \) is the transfer function of B-spline of order 2 and therefore \( u = 1 \). The transition operator \( T_u \) associated with \( u \) has spectrum 2. Therefore \( s_2(\tilde{\phi}^{1,1,a}) = s_2(\tilde{\psi}^{1,1,a}) = 1.5 \). Moreover, \( m_0(\xi) \) has the factorisation

\[
m_0(\xi) = \left( \frac{1+e^{-i\xi}}{2} \right)^2 u(\xi),
\]

where

\[
u(\xi) = -0.5(1 - 3.73205e^{-i\xi})(1 \times 0.267949e^{-i\xi})
= -0.5(1 - 4e^{-i\xi} + e^{-2i\xi}).
\]

**Figure 3** (a) The scaling function \( \phi^{1,1,a} \) (b) The wavelet \( \psi^{1,1,a} \) (see online version for colours)
The spectrum of the transition operator $T_0$ is 2.08496. Therefore, the Sobolev regularity exponent of $\phi_{1,1,1}$ and $\psi_{1,1,1}$ is

$$s_z(\phi_{1,1,1}) = s_z(\psi_{1,1,1}) \geq 0.08496.$$ 

In case (b), both the transition operator $T_0$ and $\tilde{T}_0$ have eigenvalues 1. They have unique eigenfunctions $f_0$ and $\tilde{f}_0$, respectively corresponding to eigenvalues 1. In particular $f_0$ is the polynomial

$$f_0(\xi) = 1.52065 \times 10^{-12} e^{-14\xi} + 1.97685 \times 10^{-10} e^{-\xi} - 0.01281 e^{-2\xi} - 0.0256199 e^{-4\xi} + 0.999179 - 0.0256199 e^{4\xi} - 0.01281 e^{2\xi} + 1.97685 \times 10^{-10} e^{12\xi} + 1.52065 \times 10^{-12} e^{44\xi}$$

which is always strictly positive. Similarly,

$$\tilde{f}_0 = -0.0132337 e^{-14\xi} - 0.344077 e^{-\xi} - 0.873426 - 0.344077 e^{4\xi} - 0.0132337 e^{2\xi}$$

which is also always strictly positive. Therefore, in this case we get a biorthogonal Riesz wavelet pair. We denote the pair by $(\psi_{1,1,1}^l, \psi_{1,1,1}^{\tilde{l}})$. They are shown in Figures 5 and 6.

$$l = 0, \tilde{l} = 2$$

(A3)
Figure 5  (a) The scaling function $\phi^{1,1,b}$ (b) The wavelet $\psi^{1,1,b}$ (see online version for colours)

Figure 6  (a) The scaling function $\tilde{\phi}^{1,1,b}$ (b) The wavelet $\tilde{\psi}^{1,1,b}$ (see online version for colours)
The possible cases of transfer function are

\[ a \quad m_0(\xi) = -0.5e^{-i\xi} + 2 - 0.5e^{i\xi}, \]
\[ \tilde{m}_0(\xi) = 0.0625e^{-i2\xi} + 0.25e^{-i\xi} + 0.375 + 0.25e^{i\xi} + 0.0625e^{i2\xi} \]

\[ b \quad m_0(\xi) = \left(\frac{1 + e^{-i\xi}}{2}\right)(-0.5e^{-i\xi} + 2 - 0.5e^{i\xi}), \]
\[ \tilde{m}_0(\xi) = \left(\frac{1 + e^{-i\xi}}{2}\right)(0.0625e^{-i2\xi} + 0.25e^{-i\xi} + 0.375 + 0.25e^{i\xi} + 0.0625e^{i2\xi}). \]

In case (a), 1 is not an eigenvalue of the transition operator \( T_0 \). Therefore, this case does not lead to a biorthogonal Riesz wavelet pair. In case (b), 1 is an eigenvalue of the transition operator \( T_0 \) and the corresponding eigenfunction is

\[ f_0(\xi) = 0.0482243e^{-i2\xi} + 0.067514e^{-i\xi} - 0.289346 + 0.67514e^{i\xi} + 0.0482243e^{i2\xi} \]

which takes both positive and negative values. Therefore, this case also does not lead to a biorthogonal Riesz wavelet pair.

In case of general \( K \), we can construct the Riesz wavelet pair and study their qualitative properties once we find the transfer functions \( m_0(\xi) \) and \( \tilde{m}_0(\xi) \). The construction is done by two different algorithms which takes filters of the transfer function as input. In Algorithm 1 corresponds to case (a) where we assume that the transfer function \( m_0(\xi) \) has \( 2M + 1 \) no of filters and the transfer function \( \tilde{m}_0(\xi) \) has \( \tilde{M} \) no of filters. Applying Algorithm 1, we obtain the Riesz wavelet pair \( (\phi^{1,a}, \psi^{1,a}) \). Algorithm 2 corresponds to case (b) where we assume that the transfer function \( m_0(\xi) \) has \( 2M \) no. of filters and the transfer function \( \tilde{m}_0(\xi) \) has \( \tilde{M} \) no. of filters. Applying Algorithm 2, we obtain the Riesz wavelet pair \( (\phi^{1,b}, \psi^{1,b}) \). The dual scaling functions and dual wavelets are obtained by interchanging the role of \( m_0(\xi) \) and \( \tilde{m}_0(\xi) \) in Algorithms 1 and 2.

**Algorithm 1**  Case(a)

```plaintext
/* Filters of the transfer function */
Input (b(1), b(2), b(3) \ldots, b(2M + 1));

/* Initialisation of starting point and end point*/
xs(1) = 1 + M; xe(1) = 3M + 1;

/* Initialisation of data */
for i = 1: xe(1)
    \( p_i = 0; \)
end

\( P_{M+1} = 1; \)

/* Starting point and end point for kth iteration*/
xs(k) = 2 xs(k - 1) + M + 1;
xe(k) = 2 xe(k - 1) + M + 1;
```
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/* Data after \(k\)th iteration */
/* Even points */
for \(i = xs(k - 1): xe(k - 1) + M\)
\[ P_{2i}^{k+1} = 2\left(b(2)P_{4i}^{k} + b(4)P_{8i}^{k} + \ldots + b(2M)P_{2^{k+1}i}^{k}\right); \]
End
/* Odd points */
for \(i = xs(k - 1): xe(k - 1) + M\)
\[ P_{2i+1}^{k+1} = 2\left(b(1)P_{1i}^{k} + b(3)P_{3i}^{k} + \ldots + b(2M + 1)P_{2^{k+1}i}^{k}\right); \]
end
/* Parametrisation after \(ns\) iteration */
mid = (\(x_s(ns) + xe(ns)\))/2;
for \(i = xs(ns): xe(ns)\)
x(i) = (i - mid)/power(2, ns - 1);
end
/* The scaling function and its plot */
for \(i = 1: (xe(ns) - xs(ns) + 1)\)
F(i) = \(P_{i,x_s(ns)+i}^{ns}\);
x(i) = x(i + xs(ns) - 1);
end
plot(X, F);
/* Dilation of the scaling function */
np = xs(ns) - xs(ns);
for \(i = 1: np + 1\)
F2(i) = F(i);
X2(i) = X(i)/2;
end
/* \(r\)th translation of the dilated scaling function, \(r = 0, 1, \ldots, N - 1\) */
npi = np/(2M);
for \(i = 1: \left(1 + r \cdot npi\right): \left(np + r \cdot npi\right)\)
\[ F_{2,i} = F_{2, i - r \cdot npi}; \]
end
/* Filter of the dual transfer function */
input \((bt(1), bt(2), \ldots, bt(N))\);
/* The wavelet and its plot */
for \(i = 1: \left((N - 1)npi + np + 1\right)\)
\[ psi(i) = 2\left(bt(1) \cdot F_{2,i} - bt(2) \cdot F_{2,2i} + \ldots + \left(-1\right)^{s-1} \cdot bt(N) \cdot F_{2,2^{s-1}}(i)\right); \]
\[ Xpsi(i) = (i - 1)/\text{power}(2, ns); \]
End
plot(Xpsi, psi);
Algorithm 2  Case(b)

/* Filters of the transfer function*/
Input $(b(1), b(2), b(3) \ldots \ldots, b(2M))$;

/* Initialisation of starting point and end point*/
$$x_s(1) = 1 + M; \quad x_e(1) = 3M;$$

/* Initialisation of data */
for $i = 1$: $x_e(1)$
    $P_i^0 = 0;$
end

/* Starting point and end point for $k$th iteration*/
$$x_s(k) = 2x_s(k-1) + M;$$
$$x_e(k) = 2x_e(k-1) + M;$$

/* Data after $k$th iteration */
/* Even points */
for $i = x_s(k-1): x_e(k-1) + M$
    $$P_i^{k+1} = 2(b(2)P_{i+1}^k + b(4)P_{i+2}^k + \ldots + b(2M)P_{i+M}^k);$$
end
/* Odd points */
for $i = x_s(k-1): x_e(k-1) + M$
    $$P_i^{k+1} = 2(b(1)P_i^k + b(3)P_{i+2}^k + \ldots + b(2M-1)P_{i+M-1}^k)$$
end
/* Parameterisation after ns iteration */
mid = ($x_s(ns) + x_e(ns))/2;
for $i = x_s(ns): x_e(ns)$
    $x(i) = (i - mid)/power(2, ns - 1);$
end
/* The scaling function and its plot */
for $i = 1$: ($x_e(ns) - x_s(ns) + 1)$
    $$F(i) = P_{x_s(ns)-i}^{ns};$$
    $$X(i) = x(i) + x_s(ns) - 1;$$
end
plot($X + 0.5F$);
/* Dilation of the scaling function */
$$np = x_s(ns) - x_s(ns);$$
for $i = 1$: $np + 1$
    $$F2(i) = F(i);$$
    $$X2(i) = X(i)/2;$$
end
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/* rth translation of the dilated scaling function, r = 0,1, ……, N – 1 */
np = np/(2M – 1);
for i = (1 + r* np): (np + r* np)
F2,i = F2 (i – r* np);
end
/* Filter of the dual transfer function */
input (bt(1), bt(2), ……, bt(N));
/* The wavelet and its plot */
for i = 1: (N – 1)np + np + 1
psi(i) = 2(bt(1)* F2,0(i) – bt(2)* F2,1(i)) + bt(3)* F2,2(i) – …… + (-1)^S – 1 * bt(N) F2,N–1(i);
End
plot(Xpsi, psi);

5 Conclusions

We have seen in this paper that the filter pair \{h_{n}, \tilde{h}_{n}\}_{n=N_1} is of the transfer function pair \{m_{n}, \tilde{m}_{n}\} gives two Lawton matrices M_{T} and M_{\tilde{T}}. Their eigenvalues and eigenvectors defines the eigenvalues and eigen functions of the transition operators T and \tilde{T}, respectively. When an eigenvalue is equals to 1 and its corresponding eigenfunction is strictly positive, then only the filter pair \{h_{n}, \tilde{h}_{n}\}_{n=N_1} is considered for the biorthogonal wavelet pair (\psi, \tilde{\psi}). This pair is constructed by taking the filter pair \{h_{n}, \tilde{h}_{n}\}_{n=N_1} as input to our algorithm.

These algorithms can be extended to bivariate cases for construction of separable bivariate biorthogonal Riesz basis of wavelets (Behera and Jahan, 2013; Triebel, 2008; Azmi et al., 2015; Hwang and Lee, 2011).

References

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