Single controller for synchronisation of coupled neural networks with distributed time-varying delays

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Abstract: This paper deals with global synchronisation in arrays of delayed chaotic neural networks with nonlinear hybrid coupling. By constructing a new Lyapunov-Krasovskii functional, a novel synchronisation criterion is presented in terms of matrix inequalities based on Chen's integral inequalities and reciprocal convex technique. These established conditions are heavily dependent on the bounds of both time-delay and its derivative. Through employing Matlab Toolbox and adjusting some matrix parameters in the derived results, the design and applications of the generalised networks can be realised. The effectiveness and applicability of the proposed methods is demonstrated by a numerical example with simulations.

Keywords: synchronisation; matrix inequality; hybrid coupled neural networks; reciprocal convex technique.

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1 Introduction

Over the past few decades, synchronisation of various chaotic systems has drawn considerable attention since the pioneering works of Pecora and Carroll was published (Pecora and Carroll, 1990). Presently, it is widely known that many advantages of having synchronisation or chaos synchronisation can be found in various engineering fields such as secure communication, image processing, harmonic oscillation generation, and soon. Also, the existence of synchronisation in language emergence and development results can help think out the common vocabulary and agents synchronisation in organisation management can make their work effectiveness better. In recent years, the issue on synchronisation in chaotic systems has been extensively studied owing to the potential applications in various engineering areas (Wang et al., 2010; Zhang et al., 2015; Wang et al., 2015a; Liu et al., 2010; Chen et al., 2004; Yang and Cao, 2009; Liang et al., 2008; Zhang et al., 2013).

When the coupled networks cannot achieve synchronisation only relying on their internal structure, it is necessary to append an external controller into the associated coupled networks, and it is so-called controlled synchronisation. The controlled synchronisation of coupled neural networks has drawn increasing attention. Cao et al. (2007) and Li and Bohnerb (2010) investigated the synchronisation of coupled stochastic neural networks with time delays by use of adaptive feedback controller. By proposing an appropriate augmented LKF, appending several new free-weighting matrices and utilising the Jensen inequality, Wang et al. (2014) established an LMI-based synchronisation criterion for nonlinear coupled static neural networks with time-varying delay. Yang et al. (2011) studied the global exponential synchronisation for a class of switched delayed neural networks via impulsive control method. By appending several new free-weighting matrices and utilising the Jensen inequality, Zhang et al. (2012) derived two LMI-based synchronisation criteria for neutral-type neural networks with mixed time delays and hybrid nonlinear coupling strengths. Zhou et al. (2008) and Song et al. (2012) researched the synchronisation in an array of linearly coupled delayed neural networks by use of pinning control. Wang et al. (2015b) tackled the global exponential synchronisation of coupled neural networks with mixed time-varying delays and stochastic perturbations via single pinning impulsive control. By use of Kronecker product techniques and the Jensen inequality, Huang et al. (2015) obtained two LMI-based synchronisation criteria for static neural networks with hybrid couplings and constant time delays. By designing a Luenberger-type observer, Shi et al. (2017) studies the problem of exponential passive filtering for a class of stochastic neutral-type neural networks with both mixed time delays and semi-Markovian jump parameters. By utilising asynchronous control strategy, Li et al. (2017) discussed the problem of passivity-based asynchronous sliding mode control for uncertain singular Markovian jump systems with nonlinear perturbations and time-delay.

However, the synchronisation problem for neural networks with mixed time delays and coupling strengths has drawn very little research effort primarily due to the mathematical complexity. Therefore it is the motivation of our current investigation to shorten such a gap by doing a study on the synchronisation problem for hybrid coupled neural networks with time-varying discrete delay and continuous distributed delays.

As well known, in the field of synchronisation analysis, gaining tight bounds of integral terms of quadratic functions plays a key role in reducing the conservatism. In the last decade, the Jensen inequality (Gu et al., 2000) has been intensively utilised to get these bounds. Recently, Seuret and Gouaisbaut (2013) established a Wirtinger-based integral inequality, which contains the Jensen one as a special case and leads to tighter bounds of integral terms of quadratic functions than the Jensen one; Park et al. (2015) derived some auxiliary function-based integral inequalities, which encompass the Wirtinger-based one and obtains tighter bounds of integral terms of quadratic functions than the Wirtinger-based one. In order to further reduce conservatism from using these inequalities, Chen et al. (2016) proposed two general integral inequalities from which the Jensen inequality, the Wirtinger-based one and the auxiliary function-based ones can be obtained.

This paper discusses the synchronisation issue for a class of hybrid neural networks with coupling strengths and mixed time-delays. The main contributions of this paper can be listed as follows:

- Only one controller is used for the synchronisation of coupled neural networks.
- By applying a simple variation of the reciprocal convex approach in Park et al. (2011), tighter upper bounds of some reciprocal convex combinations are derived with less conservative approximation. For detail please refer to Remark 2.
- Proper combination of Chen's inequalities (see Lemma 3) with the reciprocal convex combination technique ensures that we derive less conservative synchronisation criteria.

Notations: Throughout this paper, we denote W^T, W^{-1}, $\lambda_M(W)$ the transpose, the inverse, and the largest eigenvalue of a square matrix W respectively. $W > 0(< 0)$ expresses a positive (negative) definite symmetric matrix, I expresses the identity matrix with compatible dimension, $0_{m\times n}$ expresses the $m \times n$ zero matrix, the symbol "*" expresses a block that is readily inferred by symmetry. The shorthand $\mathrm{col}\{M_1, M_2, ..., M_k\}$ expresses a column matrix with the matrices $M_1, M_2, ..., M_k$. $\mathrm{sym}(A)$ is defined as $A + A^T$, $\mathrm{diag}\{\cdot\}$ denotes a diagonal or block-diagonal matrix. $||\cdot||$ denotes the Euclidean norm; matrices, if not explicitly stated, are assumed to have appropriate dimensions.

2 Problem description and preliminaries

In this paper, we consider the following general model of an array of coupled neural networks with discrete and continuous distributed time-varying delays:

$$\begin{aligned}\dot{z}_i(t) =& -Cz_i(t) + A\tilde{f}(z_i(t)) + B\tilde{f}(z_i(t-\tau(t))) + \sigma_1\sum_{j=1}^{N} q_{1ij}\Phi_1 z_j(t)\\ &+ \sigma_2\sum_{j=1}^{N} q_{2ij}\Phi_2 z_j(t-\tau(t))\\ &+ \sigma_3\sum_{j=1}^{N} q_{3ij}\Phi_3\int_{t-\tau(t)}^{t} z_j(v)\mathrm{d}v + I(t), \quad i \in \mathcal{N} = \{1,2,...,N\},\end{aligned} \tag{1}$$

where $z_i(t) = (z_{i1}(t), z_{i2}(t), ..., z_{im}(t))^T \in \mathbb{R}^m$ is the state vector of the ith node of the coupled networks at moment t, positive integer m is the number of neurons. $C = \text{diag}\{c_1, c_2, ..., c_m\}$ is a positive diagonal matrix with $c_j (j = 1, 2, ..., m)$ being the rate with which the $j-$th neuron will reset its potential to the resting state in isolation, A, B represent the connection weight matrix, the discretely delayed connection weight matrix, respectively. $\tilde{f}(z_i(t)) = \left(\tilde{f}_1(z_{i1}(t)), \tilde{f}_2(z_{i2}(t)), ..., \tilde{f}_m(z_{im}(t))\right)^T \in \mathbb{R}^m$ is the neural activation function. Bounded function $\tau(t)$ is the unknown time-varying delay with $0 \le \tau(t) \le \bar{\tau}, \dot{\tau}(t) \le \tau'$, where $\bar{\tau}$ is a positive scalar. Positive constants $\sigma_1, \sigma_2, \sigma_3$ represent the coupling strengths of non-delayed coupling, discretely time-delayed coupling and distributively time-delayed coupling respectively. $I(t)$ denotes an external input vector, $\Phi_k = \text{diag}\{\phi_{k1}, \phi_{k2}, ..., \phi_{km}\} > 0 (k = 1, 2, 3)$ denotes inner coupling matrix between coupled nodes. Symmetric matrix $Q_k = (q_{kij})_{N\times N} (k = 1, 2, 3)$ is outer coupling matrix of the whole network. Throughout this paper, we assume that the network is connected, which implies that the corresponding Laplacian matrix is irreducible.

The initial value of networks (1) is $z_i(s) = \varphi_i(s)$ with $\varphi_i(s)$ being a continuous function from $[-\bar{\tau}, 0]$ to $\mathbb{R}^m$.

Throughout this paper, we do the following assumptions.

Assumption 1 (Liu et al., 2006)**:** There are constants $\lambda_j^- < \lambda_j^+$ such that

$$\lambda_j^- \le \frac{\tilde{f}_j(p) - \tilde{f}_j(q)}{p - q} \le \lambda_j^+, \ \ j = 1, 2, ..., m$$

for any $p \ne q, \ p, q \in \mathbb{R}$.

For simplicity, we set $\Lambda_1 = \text{diag}\left\{\lambda_1^+, \lambda_2^+, \ldots, \lambda_m^+\right\}, \ \Lambda_2 = \text{diag}\left\{\lambda_1^-, \lambda_2^-, \ldots, \lambda_m^-\right\}$, $\Lambda = \text{diag}\left\{\lambda_1, \lambda_2, \ldots, \lambda_m\right\}$, where $\lambda_j = \max\{|\lambda_j^-|, |\lambda_j^+|\}, \ j = 1, 2, ..., m.$

Assumption 2 : Q_k is the configuration matrix that is irreducible and satisfies the following diffusive conditions:

$$q_{kij} = q_{kji} \ge 0, \ \ q_{kii} = -\sum_{j=1, j\ne i}^{N} q_{kij}, \quad i \ne j, \ i, j \in \mathcal{N}, \ k = 1, 2, 3,$$

where $q_{kij} > 0$, if there is a connection between $i-$th node and $j-$th node, and otherwise, $q_{kij} = 0$.

The purpose of this paper is to synchronise all the states of coupled neural network (1) to the following manifold $z_i(t) \to \varsigma(t) (i \in \mathcal{N})$ by designing a controller $u_i(t) \in \mathbb{R}^m$ into each individual node, where $\varsigma(t) \in \mathbb{R}^m$ is defined as

$$\dot{\varsigma}(t) = -C\varsigma(t) + A\tilde{f}(\varsigma(t)) + B\tilde{f}(\varsigma(t - \tau(t))) + I(t), \tag{2}$$

and $\varsigma(t)$ can be any one of desired states: equilibrium point, a nontrivial periodic orbit, or even a chaotic orbit.

With the controllers $u_i(t) (i \in \mathcal{N})$, the controlled coupled neural networks (1) are written as:

$$\begin{aligned}\dot{z}_i(t) =& -Cz_i(t) + A\tilde{f}(z_i(t)) + B\tilde{f}(z_i(t-\tau(t))) + I(t) + \sigma_1\sum_{j=1}^{N} q_{1ij}\Phi_1 z_j(t)\\ &+\sigma_2\sum_{j=1}^{N} q_{2ij}\Phi_2 z_j(t-\tau(t)) + \sigma_3\sum_{j=1}^{N} q_{3ij}\Phi_3\int_{t-\tau(t)}^{t} z_j(\upsilon)\mathrm{d}\upsilon + u_i(t), \quad i\in\mathcal{N}.\end{aligned} \tag{3}$$

Let $e_i(t) = z_i(t) - \varsigma(t)$, from Assumption 2 one can easily verify that

$$\begin{aligned}\sum_{j=1}^{N} q_{1ij}\Phi_1 z_j(t) &= \sum_{j=1}^{N} q_{1ij}\Phi_1[e_j(t)+\varsigma(t)] = \sum_{j=1}^{N} q_{1ij}\Phi_1 e_j(t) + \sum_{j=1}^{N} q_{1ij}\Phi_1\varsigma(t)\\ &= \sum_{j=1}^{N} q_{1ij}\Phi_1 e_j(t).\end{aligned}$$

Similarly one has

$$\begin{aligned}\sum_{j=1}^{N} q_{2ij}\Phi_2 z_j(t-\tau(t)) &= \sum_{j=1}^{N} q_{2ij}\Phi_2 e_j(t-\tau(t)),\\ \sum_{j=1}^{N} q_{3ij}\Phi_3\int_{t-\tau(t)}^{t} z_j(\upsilon)\mathrm{d}\upsilon &= \sum_{j=1}^{N} q_{3ij}\Phi_3\int_{t-\tau(t)}^{t} e_j(\upsilon)\mathrm{d}\upsilon.\end{aligned}$$

By subtracting (2) from (3), one gets the following error dynamical system:

$$\begin{aligned}\dot{e}_i(t) =& -Ce_i(t) + Af(e_i(t)) + Bf(e_i(t-\tau(t))) + \sigma_1\sum_{j=1}^{N} q_{1ij}\Phi_1 e_j(t)\\ &+\sigma_2\sum_{j=1}^{N} q_{2ij}\Phi_2 e_j(t-\tau(t)) + \sigma_3\sum_{j=1}^{N} q_{3ij}\Phi_3\int_{t-\tau(t)}^{t} e_j(\upsilon)\mathrm{d}\upsilon + u_i(t), \quad i\in\mathcal{N},\end{aligned} \tag{4}$$

where $f(e_i(\cdot)) = \tilde{f}(z_i(\cdot)) - \tilde{f}(\varsigma(\cdot)),\ i\in\mathcal{N}$.

Therefore, it follows from Assumption 1 that $f_j(0)=0$ and

$$\lambda_j^- \le \frac{f_j(p)-f_j(q)}{p-q} \le \lambda_j^+,\ \ \lambda_j^- \le \frac{f_j(r)}{r} \le \lambda_j^+,\ \ p\neq q,\ r\neq 0,\ p,q,r\in\mathbb{R},\ \ j\in\mathcal{N}, \tag{5}$$

In order to achieve the goal, we utilise the following controllers on the nodes

$$u_i(t) = \begin{cases} -\sigma_1 q_1\Phi_1 e_i(t) - \sigma_2 q_2\Phi_2 e_i(t-\tau(t)), & i=1,\\ 0, & i=2,3,...,m, \end{cases} \tag{6}$$

where q_1, q_2 are positive constants to be determined.

Under the controllers defined in equation (6), the error dynamical system (4) is transformed into

$$\dot{e}_i(t) = -Ce_i(t) + Af(e_i(t)) + Bf(e_i(t-\tau(t))) + \sigma_1 \sum_{j=1}^{N} \bar{g}_{1ij}\Phi_1 e_j(t) + \sigma_2 \sum_{j=1}^{N} \bar{g}_{2ij}\Phi_2 e_j(t-\tau(t)) + \sigma_3 \sum_{j=1}^{N} q_{3ij}\Phi_3 \int_{t-\tau(t)}^{t} e_j(\upsilon)\mathrm{d}\upsilon, \quad i \in \mathcal{N}, \quad (7)$$

where matrix $\bar{Q}_k = (\bar{g}_{kij})_{N\times N}$ is defined as $\bar{g}_{k11} = g_{k11} - q_k$, and, $\bar{g}_{kij} = g_{kij}$, for $i, j = 1, 2, ..., N, i + j > 2; k = 1, 2$.

Obviously, the synchronisation problem of system (3) is equivalent to the stabilisation problem of the error system (7) at the origin.

In order to obtain the results, we need the following lemmas.

Lemma 1 (Chen et al., 2007)**:** *If $L = (l_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ is an irreducible matrix with* $\mathrm{Rank}(L) = n - 1$ *such that $l_{ij} = l_{ji} \geq 0$, for $i \neq j$, and $\sum_{j=1}^{n} l_{ij} = 0$, for $i = 1, 2, ..., n$. Then for any constant $l > 0$, any eigenvalue of the matrix*

$$\bar{L} = \begin{pmatrix} l_{11} - l & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & \cdots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix}$$

is negative.

Lemma 2 (Park et al., 2011)**:** *If $f_1, f_2, ..., f_N : \mathbb{R}^m \to \mathbb{R}$ have positive values in an open subset D of $\mathbb{R}^m$. Then, the reciprocally convex combination of f_i over D satisfies the following equality:*

$$\min_{\left\{\delta_i | \delta_i > 0,\ \sum_i \delta_i = 1\right\}} \sum_i \frac{1}{\delta_i} f_i(t) = \sum_i f_i(t) + \max_{q_{i,j}(t)} \sum_{i\neq j} q_{i,j}(t),$$

$$\text{subject to}: \quad \left\{ q_{i,j} : \mathbb{R}^m \to \mathbb{R},\ q_{j,i}(t) = q_{i,j}(t),\ \begin{pmatrix} f_i(t) & q_{i,j}(t) \\ q_{j,i}(t) & f_j(t) \end{pmatrix} \geq 0 \right\}.$$

Lemma 3 (Chen et al., 2016)**:** *For any given matrix $U > 0$ and continuous vector function $\mu : [a, b] \to \mathbb{R}^{m}$, the following inequalities are true:*

$$(b-a)\int_a^b \mu(\upsilon)^T U \mu(\upsilon)\mathrm{d}\upsilon \geq \Theta_1^T U \Theta_1 + 3(\Theta_1 - \Theta_2)^T U (\Theta_1 - \Theta_2) + 5\bar{\Theta}_1^T U \bar{\Theta}_1 + 7\bar{\Theta}_2^T U \bar{\Theta}_2,$$

$$2\int_a^b \int_u^b \mu(\upsilon)^T U \mu(\upsilon)\mathrm{d}\upsilon\mathrm{d}u \geq \Theta_2^T U \Theta_2 + 8(\Theta_2 - \Theta_3)^T U (\Theta_2 - \Theta_3) + 3\bar{\Theta}_3^T U \bar{\Theta}_3,$$

$$2\int_a^b \int_a^u \mu(\upsilon)^T U \mu(\upsilon)\mathrm{d}\upsilon\mathrm{d}u \geq (2\Theta_1 - \Theta_2)^T U (2\Theta_1 - \Theta_2) + 8\bar{\Theta}_4^T U \bar{\Theta}_4 + 3\bar{\Theta}_5^T U \bar{\Theta}_5,$$

where $\Theta_1 = \int_a^b \mu(\upsilon)\mathrm{d}\upsilon$, $\Theta_2 = \frac{2}{b-a}\int_a^b \int_\psi^b \mu(\upsilon)\mathrm{d}\upsilon\mathrm{d}\psi$, $\bar{\Theta}_1 = \Theta_1 - 3\Theta_2 + 2\Theta_3$, $\bar{\Theta}_2 = \Theta_1 - 6\Theta_2 + 10\Theta_3 - 5\Theta_3$, $\bar{\Theta}_3 = 3\Theta_2 - 8\Theta_3 + 5\Theta_4$, $\bar{\Theta}_4 = \Theta_1 - 2\Theta_2 + \Theta_3$, $\bar{\Theta}_5 = 2\Theta_1 - 9\Theta_2 + 12\Theta_3 - 5\Theta_4$ *with* $\Theta_3 = \frac{6}{(b-a)^2}\int_a^b \int_\psi^b \int_\upsilon^b \mu(\upsilon)\mathrm{d}\upsilon\mathrm{d}\upsilon\mathrm{d}\psi$, $\Theta_4 = \frac{24}{(b-a)^3}\int_a^b \int_\psi^b \int_\upsilon^b \int_u^b \mu(\upsilon)\mathrm{d}\upsilon\mathrm{d}u\mathrm{d}\upsilon\mathrm{d}\psi$.

3 Main result

Before proposing our main result, for notational simplicity, we denote $e_{it} = e_i(t), e_{i\tau} = e_i(t-\tau(t)), e_{i\bar{\tau}} = e_i(t-\bar{\tau}),\ \dot{e}_{it} = \dot{e}_i(t)$. Now we define a new vector as

$$\delta_i(t) = \mathrm{col}\left\{e_{it},\ e_{i\tau},\ e_{i\bar{\tau}},\ \dot{e}_{it},\ v_{it},\ w_{it},\ x_{it},\ \varrho_{it}, \omega_{it},\ \chi_{it},\ \ f(e_{it}),\ f(e_{i\tau}),\ f(e_{i\bar{\tau}})\right\},$$

where

$$\begin{aligned}
v_{it} =& \frac{1}{\tau(t)}\int_{t-\tau(t)}^{t} e_{i\upsilon}\mathrm{d}\upsilon, \quad w_{it} = \frac{1}{\bar{\tau}-\tau(t)}\int_{t-\bar{\tau}}^{t-\tau(t)} e_{i\upsilon}\mathrm{d}\upsilon,\\
x_{it} =& \frac{2}{\tau^2(t)}\int_{t-\tau(t)}^{t}\int_{\psi}^{t} e_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi, \quad \varrho_{it} = \frac{2}{[\bar{\tau}-\tau(t)]^2}\int_{t-\bar{\tau}}^{t-\tau(t)}\int_{\psi}^{t-\tau(t)} e_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi,\\
\omega_{it} =& \frac{6}{\tau^3(t)}\int_{t-\tau(t)}^{t}\int_{\psi}^{t}\int_{\rho}^{t} e_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\rho\mathrm{d}\psi,\\
\chi_{it} =& \frac{6}{[\bar{\tau}-\tau(t)]^3}\int_{t-\bar{\tau}}^{t-\tau(t)}\int_{\psi}^{t-\tau(t)}\int_{\rho}^{t-\tau(t)} e_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\rho\mathrm{d}\psi.
\end{aligned}$$

From the mean-value theorem for integral, it follows that

$$\begin{aligned}
\lim_{\tau(t)\to 0^+} v_{it} = \lim_{\tau(t)\to 0^+} \omega_{it} = \lim_{\tau(t)\to 0^+} x_{it} = e_{it}, \quad & \lim_{\tau(t)\to\bar{\tau}^-} w_{it} = \lim_{\tau(t)\to\bar{\tau}^-} \varrho_{it}\\
&= \lim_{\tau(t)\to\bar{\tau}^-} \chi_{it} = e_{i\bar{\tau}}.
\end{aligned}$$

Therefore $v_{it}, w_{it}, x_{it}, \varrho_{it}, \omega_{it}, \chi_{it}$ are well defined after we set

$$v_{it}|_{\tau(t)=0} = x_{it}|_{\tau(t)=0} = \omega_{it}|_{\tau(t)=0} = e_{it}, \quad w_{it}|_{\tau(t)=\bar{\tau}} = \varrho_{it}|_{\tau(t)=\bar{\tau}} = \chi_{it}|_{\tau(t)=\bar{\tau}} = e_{i\bar{\tau}}. \tag{8}$$

Denote

$$\begin{aligned}
\mathcal{R}_2 =& \mathrm{diag}\{R_2+R_4, 3(R_2+R_4), 5(R_2+R_4), 7R_2\},\\
\mathcal{S}_1 =& \mathrm{diag}\{S_1, 3S_1, 5S_1, 7S_1\}, \ \ \mathcal{S}_2 = \mathrm{diag}\{S_2, 3S_2, 5S_2, 7S_2\},\\
\Upsilon =& \nu_2^T\Lambda D_4\Lambda\nu_2 - \nu_5^T D_4\nu_5 + \begin{bmatrix}\nu_2\\ \nu_5-\Lambda_2\nu_2\end{bmatrix}^T(\mathcal{Q}-\mathcal{W})\begin{bmatrix}\nu_2\\ \nu_5-\Lambda_2\nu_2\end{bmatrix} + \begin{bmatrix}\nu_1\\ \nu_3\end{bmatrix}^T\mathcal{U}\begin{bmatrix}\nu_1\\ \nu_3\end{bmatrix}\\
&- \begin{bmatrix}\nu_1\\ \nu_4\end{bmatrix}^T\mathcal{Y}\begin{bmatrix}\nu_1\\ \nu_4\end{bmatrix} + \mathrm{sym}\left\{\begin{bmatrix}\nu_1\\ \nu_3\end{bmatrix}^T\mathcal{U}\begin{bmatrix}0\\ \nu_2-\nu_3\end{bmatrix} - \begin{bmatrix}\nu_1\\ \nu_4\end{bmatrix}^T\mathcal{Y}\begin{bmatrix}0\\ \nu_2-\nu_4\end{bmatrix}\right\},\\
\Omega_0 =& \mathrm{sym}\left\{\iota_4^T\left[D_1(\Lambda_1\iota_1-\iota_{11}) + D_2(\iota_{11}-\Lambda_2\iota_1) + D_3(\iota_{11}+\Lambda\iota_1)\right]\right\}\\
&+ \tau'\begin{bmatrix}\iota_1\\ \iota_5\end{bmatrix}^T\mathcal{U}\begin{bmatrix}\iota_1\\ \iota_5\end{bmatrix} + \mathrm{sym}\left\{\begin{bmatrix}\iota_1\\ \iota_5\end{bmatrix}^T\mathcal{U}\begin{bmatrix}0\\ \iota_1-(1-\tau')\iota_2-\tau'\iota_5\end{bmatrix}\right\}\\
&- \tau'\begin{bmatrix}\iota_1\\ \iota_6\end{bmatrix}^T\mathcal{Y}\begin{bmatrix}\iota_1\\ \iota_6\end{bmatrix} + \mathrm{sym}\left\{\begin{bmatrix}\iota_1\\ \iota_6\end{bmatrix}^T\mathcal{Y}\begin{bmatrix}0\\ -\iota_3+(1-\tau')\iota_2+\tau'\iota_6\end{bmatrix}\right\}\\
&+ \iota_1^T\Lambda D_4\Lambda\iota_1 - \iota_{11}^T D_4\iota_{11} - (1-\tau')(\iota_2^T\Lambda D_4\Lambda\iota_2 - \iota_{12}^T D_4\iota_{12})\\
&+ \frac{1}{2}\bar{\tau}^2\iota_4^T(S_1+S_2)\iota_4
\end{aligned}$$

$$
\begin{aligned}
&+\mathrm{sym}\left\{\begin{bmatrix}\iota_1\\0\end{bmatrix}^T\mathcal{P}\begin{bmatrix}\iota_4\\\iota_1-\iota_3\end{bmatrix}\right\}-(1-\tau')\begin{bmatrix}\iota_2\\\iota_{12}-\Lambda_2\iota_2\end{bmatrix}^T(\mathcal{Q}-\mathcal{W})\begin{bmatrix}\iota_2\\\iota_{12}-\Lambda_2\iota_2\end{bmatrix}\\
&+\begin{bmatrix}\iota_1\\\iota_{11}-\Lambda_2\iota_1\end{bmatrix}^T\mathcal{Q}\begin{bmatrix}\iota_1\\\iota_{11}-\Lambda_2\iota_1\end{bmatrix}-\begin{bmatrix}\iota_3\\\iota_{13}-\Lambda_2\iota_3\end{bmatrix}^T\mathcal{W}\begin{bmatrix}\iota_3\\\iota_{13}-\Lambda_2\iota_3\end{bmatrix}\\
&+\bar{\tau}^2\begin{bmatrix}\iota_1\\\iota_4\end{bmatrix}^T\mathcal{R}_1\begin{bmatrix}\iota_1\\\iota_4\end{bmatrix}-\begin{bmatrix}\beta_1\\\beta_2\end{bmatrix}^T\begin{bmatrix}\mathcal{R}_2&\mathcal{X}\\ *&\mathcal{R}_2\end{bmatrix}\begin{bmatrix}\beta_1\\\beta_2\end{bmatrix}\\
&+\bar{\tau}\left[\iota_1^TH_1\iota_1-\iota_2^T(H_1-H_2)\iota_2-\iota_3^TH_2\iota_3\right]+\sigma_1\lambda_M(\bar{Q}_1)\mathrm{sym}\{\iota_4^TD_6\Phi_1\iota_1\}\\
&-\bar{\tau}\cdot\mathrm{sym}\{\iota_5^T(R_5+H_1)(\iota_1-\iota_2)+3(\iota_5-\iota_7)^T(R_5+H_1)(\iota_1+\iota_2-2\iota_5)\}\\
&-5\bar{\tau}\cdot\mathrm{sym}\{(\iota_5-3\iota_7+2\iota_9)^T(R_5+H_1)(\iota_1-\iota_2+6\iota_5-6\iota_7)\}\\
&-\bar{\tau}\cdot\mathrm{sym}\{\iota_6^T(R_5+H_2)(\iota_2-\iota_3)+3(\iota_6-\iota_8)^T(R_5+H_2)(\iota_2+\iota_3-2\iota_6)\}\\
&-5\bar{\tau}\cdot\mathrm{sym}\{(\iota_6-3\iota_8+2\iota_{10})^T(R_5+H_2)(\iota_2-\iota_3+6\iota_6-6\iota_8)\}\\
&-2(\iota_2-\iota_6)^TS_1(\iota_2-\iota_6)-4(\iota_2+2\iota_6-3\iota_8)^TS_1(\iota_2+2\iota_6-3\iota_8)\\
&-6(\iota_2-3\iota_6+12\iota_8-10\iota_{10})^TS_1(\iota_2-3\iota_6+12\iota_8-10\iota_{10})\\
&-2(\iota_1-\iota_5)^TS_1(\iota_1-\iota_5)-4(\iota_1+2\iota_5-3\iota_7)^TS_1(\iota_1+2\iota_5-3\iota_7)\\
&-6(\iota_1-3\iota_5+12\iota_7-10\iota_9)^TS_1(\iota_1-3\iota_5+12\iota_7-10\iota_9)\\
&-2(\iota_2-\iota_5)^TS_2(\iota_2-\iota_5)-4(\iota_2-4\iota_5+3\iota_7)^TS_2(\iota_2-4\iota_5+3\iota_7)\\
&-6(\iota_2-9\iota_5+18\iota_7-10\iota_9)^TS_2(\iota_2-9\iota_5+18\iota_7-10\iota_9)\\
&-2(\iota_3-\iota_6)^TS_2(\iota_3-\iota_6)-4(\iota_3-4\iota_6+3\iota_8)^TS_2(\iota_3-4\iota_6+3\iota_8)\\
&-6(\iota_3-9\iota_6+18\iota_8-10\iota_{10})^TS_2(\iota_3-9\iota_6+18\iota_8-10\iota_{10})\\
&+\mathrm{sym}\{(D_5\iota_1+D_6\iota_5)^T(-\iota_5-C\iota_1+A\iota_{11}+B\iota_{12})\}\\
&+\sigma_2\lambda_M(\bar{Q}_2)\mathrm{sym}\{\iota_4^TD_6\Phi_2\iota_2\}+2\sigma_1\lambda_M(\bar{Q}_1)\left(\iota_1^TD_5\Phi_1\iota_1\right)\\
&+\sigma_2\lambda_M(\bar{Q}_2)\mathrm{sym}\{\iota_1^TD_5\Phi_2\iota_2\}+\mathrm{sym}\left\{(\iota_{11}-\Lambda_2\iota_1)^TW_1(\Lambda_1\iota_1-\iota_{11})\right\}\\
&+\mathrm{sym}\left\{(\iota_{12}-\Lambda_2\iota_2)^TW_2(\Lambda_1\iota_2-\iota_{12})\right\}+\mathrm{sym}\left\{(\iota_{13}-\Lambda_2\iota_3)^TW_3(\Lambda_1\iota_3-\iota_{13})\right\}\\
&+\mathrm{sym}\left\{[(\iota_{11}-\iota_{12})-\Lambda_2(\iota_1-\iota_2)]^TW_4[\Lambda_1(\iota_1-\iota_2)-(\iota_{11}-\iota_{12})]\right\}\\
&+\mathrm{sym}\left\{[(\iota_{11}-\iota_{13})-\Lambda_2(\iota_1-\iota_3)]^TW_5[\Lambda_1(\iota_1-\iota_3)-(\iota_{11}-\iota_{13})]\right\}\\
&+\mathrm{sym}\left\{[(\iota_{12}-\iota_{13})-\Lambda_2(\iota_2-\iota_3)]^TW_6[\Lambda_1(\iota_2-\iota_3)-(\iota_{12}-\iota_{13})]\right\},
\end{aligned}
$$

$$
\begin{aligned}
\Omega_1=&\mathrm{sym}\left\{\begin{bmatrix}0\\\iota_5\end{bmatrix}^T\mathcal{P}\begin{bmatrix}\iota_4\\\iota_1-\iota_3\end{bmatrix}+\begin{bmatrix}\iota_1\\\iota_5\end{bmatrix}^T\mathcal{U}\begin{bmatrix}\iota_4\\0\end{bmatrix}\right\}-\bar{\tau}\iota_5^T(R_1+R_3)\iota_5\\
&-3\bar{\tau}(\iota_5-\iota_7)^T(R_1+R_3)(\iota_5-\iota_7)-5\bar{\tau}(\iota_5-3\iota_7+2\iota_9)^T(R_1+R_3)\times\\
&(\iota_5-3\iota_7+2\iota_9)+\sigma_3\lambda_M(Q_3)\mathrm{sym}\{\left(\iota_1^TD_5+\iota_4^TD_6\right)\Phi_3\iota_5\},\\
\Omega_2=&-\bar{\tau}\iota_6^T(R_1+R_3)\iota_6-3\bar{\tau}(\iota_6-\iota_8)^T(R_1+R_3)(\iota_6-\iota_8)\\
&-5\bar{\tau}(\iota_6-3\iota_8+2\iota_{10})^T(R_1+R_3)(\iota_6-3\iota_8+2\iota_{10})\\
&+\mathrm{sym}\left\{\begin{bmatrix}0\\\iota_6\end{bmatrix}^T\mathcal{P}\begin{bmatrix}\iota_4\\\iota_1-\iota_3\end{bmatrix}+\begin{bmatrix}\iota_1\\\iota_6\end{bmatrix}^T\mathcal{Y}\begin{bmatrix}\iota_4\\0\end{bmatrix}\right\},
\end{aligned}
$$

with $\nu_p=\left(0_{n\times(p-1)n}\ I_n\ 0_{n\times(5-p)n}\right), p=1,2,...5, \iota_q=\left(0_{n\times(q-1)n}\ I_n\ 0_{n\times(13-q)n}\right)$, $q=1,2,...,13$, and $\beta_1=\mathrm{col}\{\iota_1-\iota_2,\iota_1+\iota_2-2\iota_5,\iota_1-\iota_2+6\iota_5-6\iota_7,\iota_1+\iota_2-$

$12\iota_5 + 30\iota_7 - 20\iota_9\}$, $\beta_2 = \mathrm{col}\{\iota_2 - \iota_3, \iota_2 + \iota_3 - 2\iota_6, \iota_2 - \iota_3 + 6\iota_6 - 6\iota_8, \iota_2 + \iota_3 - 12\iota_6 + 30\iota_8 - 20\iota_{10}\}$.

Now we put forward our main result about the synchronisation of the coupled neural networks (3).

Theorem 1 (See Appendix I for a proof)**:** *Under Assumptions* 1-2*,* $\bar{\tau}, \tau'$ *are given scalars with* $\bar{\tau} > 0$*. The coupled neural networks* (3) *are globally asymptotically synchronised for* $0 \le \tau(t) \le \bar{\tau}$*,* $\dot{\tau}(t) \le \tau'$*, if there are* $2m \times 2m$ *positive definite matrices* $\mathcal{P}, \mathcal{Q}, \mathcal{U}, \mathcal{W}, \mathcal{Y}$*,* $m \times m$ *positive definite matrices* $R_j (j = 1, 2, 3, 4), S_k$*,* $m \times m$ *positive diagonal matrices* $D_l, W_l (l = 1, ..., 6)$*,* $m \times m$ *symmetric matrices* H_k*,* $m \times m$ *real matrix* R_5*, and* $4m \times 4m$ *real matrix* $\mathcal{X}$*, such that the following matrix inequalities are true*

$$\mathcal{R}_1 = \begin{bmatrix} R_1 + R_3 & R_5 \\ * & R_2 + R_4 \end{bmatrix} \ge 0, \quad \begin{bmatrix} \mathcal{R}_2 + \mathcal{S}_1 & \mathcal{X} \\ * & \mathcal{R}_2 + \mathcal{S}_2 \end{bmatrix} \ge 0, \tag{9}$$

$$\Psi_k = \begin{bmatrix} R_3 & R_5 + H_k \\ * & R_4 \end{bmatrix} \ge 0, \quad \Upsilon \ge 0, \tag{10}$$

$$\Omega_0 + \bar{\tau}\Omega_k < 0, \quad k = 1, 2. \tag{11}$$

Remark 1: When $\dot{\tau}(t)$ is unknown or $\dot{\tau}(t) \ge 1$, $\mathcal{Q}$ will no longer be helpful to improve the synchronisation condition since $-[1 - \dot{\tau}(t)]\mathcal{Q}$ may be nonnegative definite. Similarly, $\mathcal{W}, D_4$ are no longer be helpful either. By setting $D_4 = 0$ and $\mathcal{Q} = \mathcal{W} = 0$ in Theorem 1, one can get a synchronisation condition for $\dot{\tau}(t) \ge 1$ or unknown $\dot{\tau}(t)$.

Remark 2: To treat the $\frac{\varpi}{\vartheta}$-dependent terms in inequality (30) of Ji et al. (2014), inequalities (17) and (18) of Liu et al. (2012), inequality (27) of Song and Cao (2007), and the $\frac{\vartheta}{\varpi}$-dependent terms in inequalities (17) and (18) of Liu et al. (2012), the authors have utilised an approximation as $\frac{\varpi}{\vartheta} \ge \varpi$, $\frac{\vartheta}{\varpi} \ge \vartheta$, which leads to considerable conservatism. However, by use of the relations $\frac{\varpi}{\vartheta} = \frac{1}{\vartheta} - 1$ and $\frac{\vartheta}{\varpi} = \frac{1}{\varpi} - 1$, it can also be regarded as one reciprocally convex combination, which can be effectively handled by the simple variation of the reciprocal convex approach in Park et al. (2011). Similar to inequality (17) of Lee and Park (2014), in inequality (33) of this paper, the tighter upper bound is presented by employing the conditions (9) with less conservative approximation.

Remark 3: When coping with integral term $\int_{t-\bar{\tau}}^{t} \dot{e}_{iv}^T R \dot{e}_{iv} \mathrm{d}v$, the traditional method is to use Jensen's inequality (see Park et al. (2012)) or the reciprocally convex combination technique. In this paper, by combining Chen's integral inequalities (see Lemma 3) with the reciprocally convex combination technique (see Lemma 2), we derive synchronisation conditions with still less conservativeness than (Park et al., 2012).

Remark 4: When treating integral term $\int_{t-\bar{\tau}}^{t} \int_{\psi}^{t} \dot{e}_{iv}^T S \dot{e}_{iv} \mathrm{d}v \mathrm{d}\psi$, Zhang et al. (2015) utilised Jensen's inequality. In this paper, by splitting it into three parts, we derive less conservative synchronisation conditions than Zhang et al. (2015) by use of combining Chen's integral inequalities (see Lemma 3) with the reciprocally convex combination technique (see Lemma 2).

Remark 5: Since the computation time of LMIs is dependent on the maximal order of the LMIs (MOL) and the total number of the scalar decision variables (NDV), those factors are considered as the indices of calculation complexity (Li et al., 2014). In Theorem 1, the MOL is $13m$ and the NDV is $40m^2 + 16m$. However the MOL and NDV are $13m,\ 63.5m^2 + 11.5m$ respectively in Li et al. (2013). The percentage of the decreasing NDV of Theorem 1 compared with that of Li et al. (2013) is $44.27\%, 48.53\%, 50.85\%$ with the cases $m = 2, 3, 4$ respectively.

4 Illustrative example

In this section, we put forward an example to show the effectiveness of our theoretical results.

Example 1: Consider networks (1) with $m = 2, N = 3$ and the following parameters:

$$C = \begin{bmatrix} 1.7 & 0 \\ 0 & 1.4 \end{bmatrix},\ A = \begin{bmatrix} 1.6 & -1.6 \\ 2.2 & 2.1 \end{bmatrix},\ B = \begin{bmatrix} 1.9 & -1.4 \\ 0.6 & -1.0 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} -1.7 & 0.9 & 0.8 \\ * & -1.5 & 0.6 \\ * & * & -1.4 \end{bmatrix},\ Q_2 = \begin{bmatrix} -1.4 & 0.9 & 0.5 \\ * & -1.2 & 0.3 \\ * & * & -0.8 \end{bmatrix},\ Q_3 = \begin{bmatrix} -1.3 & 0.6 & 0.7 \\ * & -1.5 & 0.9 \\ * & * & -1.6 \end{bmatrix},$$
$$\Phi_1 = \begin{bmatrix} -1.7 & 0 \\ 0 & 2.1 \end{bmatrix},\ \Phi_2 = \begin{bmatrix} 1.5 & 0 \\ 0 & -1.1 \end{bmatrix},\ \Phi_3 = \begin{bmatrix} -1.7 & 0 \\ 0 & 0.6 \end{bmatrix},\ I(t) = \begin{bmatrix} 1.0 \\ -0.2 \end{bmatrix}.$$

Set coupling strengths as $\sigma_1 = \sigma_2 = \sigma_3 = 4$. The activation functions are $\tilde{f}_1(y) = \tilde{f}_2(y) = \tanh(y)$, and the time delay is $\tau(t) = 0.7 + 0.7\sin(0.5t)$. Then $\bar{\tau} = 1.4,\ \tau' = 0.35$, and Assumption 1 is satisfied with $\Lambda = \Lambda_1 = 0.5I, \Lambda_2 = 0$. By introducing the controller (6) with $q_1 = q_2 = 5$ into the neural network, one can get error network (7) with

$$\bar{Q}_1 = \begin{bmatrix} -6.7 & 0.9 & 0.8 \\ * & -1.5 & 0.6 \\ * & * & -1.4 \end{bmatrix},\ \bar{Q}_2 = \begin{bmatrix} -6.4 & 0.9 & 0.5 \\ * & -1.2 & 0.3 \\ * & * & -0.8 \end{bmatrix}.$$

Solving the LMIs (9)-(11) in Theorem 1 by utilising the Matlab LMI Control Toolbox, one can obtain feasible solutions. The decision matrices of one solution are listed as follows

$$\mathcal{P} = \begin{bmatrix} 7.6774 & -0.5316 & 0.9892 & 1.4572 \\ -0.5316 & 1.1523 & -0.1034 & 0.2004 \\ 0.9892 & -0.1034 & 1.0564 & 0.4523 \\ 1.4572 & 0.2004 & 0.4523 & 0.5350 \end{bmatrix},\ \mathcal{Q} = \begin{bmatrix} 34.7062 & 3.2531 & -12.2843 & 2.9210 \\ 3.2531 & 16.3223 & 1.2919 & 2.1110 \\ -12.2843 & 1.2919 & 7.7655 & -0.4005 \\ 2.9210 & 2.1110 & -0.4005 & 5.5738 \end{bmatrix},$$
$$\mathcal{U} = \begin{bmatrix} 4.0249 & 0.8143 & 0.1006 & 2.3430 \\ 0.8143 & 1.4177 & 0.2576 & 1.2170 \\ 0.1006 & 0.2576 & 0.8485 & 0.4700 \\ 2.3430 & 1.2170 & 0.4700 & 2.0388 \end{bmatrix},\ \mathcal{W} = \begin{bmatrix} 0.6266 & -0.2164 & -0.1134 & -0.0322 \\ -0.2164 & 0.1029 & 0.0664 & 0.0187 \\ -0.1134 & 0.0664 & 0.2138 & 0.0832 \\ -0.0322 & 0.0187 & 0.0832 & 0.0336 \end{bmatrix},$$
$$\mathcal{Y} = \begin{bmatrix} 0.1943 & -0.0049 & 0.0531 & -0.0163 \\ -0.0049 & 0.1191 & 0.0407 & 0.0443 \\ 0.0531 & 0.0407 & 0.1261 & 0.0203 \\ -0.0163 & 0.0443 & 0.0203 & 0.0506 \end{bmatrix},\ R_1 = \begin{bmatrix} 0.9847 & 0.6765 \\ 0.6765 & 0.8912 \end{bmatrix},\ R_2 = \begin{bmatrix} 0.6143 & 0.4977 \\ 0.4977 & 0.6010 \end{bmatrix},$$
$$R_3 = \begin{bmatrix} 0.3187 & 0.2683 \\ 0.2683 & 0.2518 \end{bmatrix},\ R_4 = \begin{bmatrix} 7.0012 & 1.2290 \\ 1.2290 & 0.5792 \end{bmatrix},\ R_5 = \begin{bmatrix} -0.1354 & -0.0944 \\ -0.0973 & -0.0753 \end{bmatrix},$$
$$S_1 = \begin{bmatrix} 0.4627 & -0.1832 \\ -0.1832 & 0.2408 \end{bmatrix},\ S_2 = \begin{bmatrix} 0.1557 & -0.0071 \\ -0.0071 & 0.0746 \end{bmatrix},\ D_1 = \begin{bmatrix} 0.6925 & 0 \\ 0 & 0.5933 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.4176 & 0 \\ 0 & 0.6201 \end{bmatrix},\ D_3 = \begin{bmatrix} 0.7131 & 0 \\ 0 & 0.4996 \end{bmatrix},\ D_4 = \begin{bmatrix} 0.6644 & 0 \\ 0 & 0.7125 \end{bmatrix},$$

$$D_5 = \begin{bmatrix} 0.9288 & 0 \\ 0 & 0.1640 \end{bmatrix},\ D_6 = \begin{bmatrix} 0.1990 & 0 \\ 0 & 0.3121 \end{bmatrix},\ W_1 = \begin{bmatrix} 0.7055 & 0 \\ 0 & 0.5498 \end{bmatrix},$$

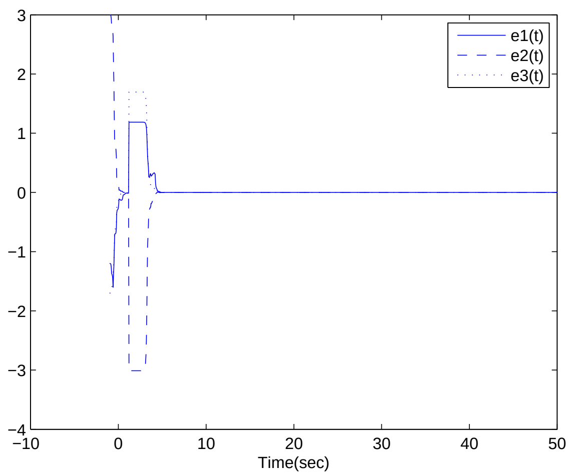
$$W_2 = \begin{bmatrix} 0.6063 & 0 \\ 0 & 0.2754 \end{bmatrix},\ W_3 = \begin{bmatrix} 0.3164 & 0 \\ 0 & 0.1267 \end{bmatrix},\ W_4 = \begin{bmatrix} 0.7446 & 0 \\ 0 & 0.1049 \end{bmatrix},$$

$$W_5 = \begin{bmatrix} 0.2387 & 0 \\ 0 & 0.1637 \end{bmatrix},\ W_6 = \begin{bmatrix} 0.6143 & 0 \\ 0 & 0.6010 \end{bmatrix},\ H_1 = \begin{bmatrix} 0.0981 & 0.0235 \\ 0.0251 & 0.0342 \end{bmatrix},$$

$$\mathcal{X} = \begin{bmatrix} -1.0415 & 0.4149 & 1.0176 & -0.2113 & 0.0011 & -0.0465 \\ 0.0549 & -0.4014 & -0.3780 & -0.2944 & -0.0412 & 0.0204 \\ -0.0667 & -0.9882 & -0.2539 & -0.7523 & -0.0553 & 0.0025 \\ 0.4571 & 0.3022 & 0.0480 & 0.4558 & -0.0128 & 0.0055 \\ -8.6295 & 3.5899 & 5.4528 & -0.3761 & -2.0343 & 0.4843 \\ -0.8173 & -0.1605 & 0.8467 & 0.3004 & 0.1288 & 1.6728 \end{bmatrix},\ H_2 = \begin{bmatrix} -0.3164 & -0.0961 \\ -0.2977 & -0.1267 \end{bmatrix}.$$

Therefore, we arrive at the conclusion that this system is globally asymptotically synchronised. The stability of the error dynamical system (7) is shown in Figure 1.

Figure 1 The error state of t–$e_1(t)$–$e_2(t)$–$e_3(t)$ (see online version for colours)



However, it is easy to verify that the conditions of Zhang et al. (2012), Feng and Cao (205) and Li et al. (2013) fail to justify the synchronisation for this example, which explains that our methods are less conservative than those in Zhang et al. (2012), Feng and Cao (205) and Li et al. (2013). This numerical example demonstrates the efficiency and less conservatism of the proposed methods over existing ones (Zhang et al., 2012; Feng and Cao, 205; Li et al., 2013).

5 Conclusion

This paper studies the global synchronisation in arrays of delayed chaotic neural networks with nonlinear hybrid coupling. By presenting one new Lyapunov-Krasovskii functional and utilising Chen's integral inequalities and reciprocal convex technique, a novel synchronisation criterion has been derived, which is heavily dependent on the bounds of both time-delay and its derivative. The established results, which are expressed in the form of linear matrix inequality, can be easily verified via Matlab LMI Toolbox. Through predetermining some matrix parameters in the proposed results, the efficiency and applicability of the derived methods are showed by a numerical example with simulations. The proposed method can be easily utilised to more complicated neural networks with impulsive effect, stochastic perturbation or reaction-diffusion terms, and these are our future research topics.

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Appendix I

Proof of Theorem 1

Inspired by Zhang et al. (2016), we consider the following LKF:

$$V(t) = \sum_{l=1}^{5} V_l(t), \tag{A1}$$

where

$$V_1(t) = 2\sum_{i=1}^{N}\sum_{j=1}^{m}\int_0^{e_{ij}(t)} \Big\{ d_{1j}\left[\lambda_j^+ s - f_j(\upsilon)\right] + d_{2j}\left[f_j(\upsilon) - \lambda_j^- s\right] + d_{3j}\left[f_j(\upsilon) + \lambda_j s\right] \Big\} \mathrm{d}\upsilon,$$

$$V_2(t) = \sum_{i=1}^{N}\Big\{ \tau(t)\eta_i(t)^T \mathcal{U}\eta_i(t) + [\bar{\tau} - \tau(t)]\kappa_i(t)^T \mathcal{Y}\kappa_i(t) + \gamma_i(t)^T \mathcal{P}\gamma_i(t) \Big\},$$

$$V_3(t) = \sum_{i=1}^{N}\Big\{ \int_{t-\tau(t)}^{t} \Big[e_{i\upsilon}^T \Lambda D_4 \Lambda e_{i\upsilon} - f(e_{i\upsilon})^T D_4 f(e_{i\upsilon}) \Big] \mathrm{d}\upsilon$$

$$+\int_{t-\tau(t)}^{t}\varepsilon_i(\upsilon)^T\mathcal{Q}\varepsilon_i(\upsilon)\mathrm{d}\upsilon+\int_{t-\bar{\tau}}^{t-\tau(t)}\varepsilon_i(\upsilon)^T\mathcal{W}\varepsilon_i(\upsilon)\mathrm{d}\upsilon\Big\},$$

$$V_4(t)=\bar{\tau}\sum_{i=1}^{N}\int_{t-\bar{\tau}}^{t}\int_{\psi}^{t}\zeta_i(\upsilon)^T\mathcal{R}_1\zeta_i(\upsilon)\mathrm{d}\upsilon\mathrm{d}\psi,$$

$$V_5(t)=\sum_{i=1}^{N}\Big\{\int_{t-\bar{\tau}}^{t}\int_{\psi}^{t}\int_{\rho}^{t}\dot{e}_{i\upsilon}^TS_1\dot{e}_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\rho\mathrm{d}\psi+\int_{t-\bar{\tau}}^{t}\int_{t-\bar{\tau}}^{\psi}\int_{\rho}^{t}\dot{e}_{i\upsilon}^TS_2\dot{e}_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\rho\mathrm{d}\psi\Big\},$$

with $\eta_i(t)=\begin{bmatrix}e_{it}\\ v_{it}\end{bmatrix}$, $\kappa_i(t)=\begin{bmatrix}e_{it}\\ w_{it}\end{bmatrix}$, $\gamma_i(t)=\begin{bmatrix}e_{it}\\ \int_{t-\bar{\tau}}^{t}e_{i\upsilon}\mathrm{d}\upsilon\end{bmatrix}$, $\varepsilon_i(t)=\begin{bmatrix}e_{it}\\ f(e_{it})-\Lambda_2e_{it}\end{bmatrix}$, $\zeta_i(t)=\begin{bmatrix}e_{it}\\ \dot{e}_{it}\end{bmatrix}$, $D_l=\mathrm{diag}\{d_{l1},d_{l2},...,d_{lm}\}$, $l=1,2,...,6$.

From Assumption 1, the following inequality is true for any positive diagonal matrix D_4

$$e_{it}^T\Lambda D_4\Lambda e_{it}-f(e_{it})^TD_4f(e_{it})\geq 0.$$

Therefore $V(t)$ defined as in (A1) is positive definite if all the matrices involved in $V(t)$ satisfy the requirement of Theorem 1.

Taking the derivative of $V(t)$ along the trajectory of the coupled neural networks (6) yields

$$\dot{V}(t)=\sum_{l=1}^{5}\dot{V}_l(t),\tag{A2}$$

where

$$\begin{aligned}\dot{V}_1(t)=&\sum_{i=1}^{N}2\dot{e}_{it}^T\{D_1[\Lambda_1e_{it}-f(e_{it})]+D_2[f(e_{it})-\Lambda_2e_{it}]+D_3[f(e_{it})+\Lambda e_{it}]\}\\ =&\sum_{i=1}^{N}\delta_i(t)^T\mathrm{sym}\{\iota_4^T[D_1(\Lambda_1\iota_1-\iota_{11})+D_2(\iota_{11}-\Lambda_2\iota_1)+D_3(\iota_{11}+\Lambda\iota_1)]\}\delta_i(t),\end{aligned}\tag{A3}$$

$$\begin{aligned}\dot{V}_2(t)=&\sum_{i=1}^{N}\Big\{\dot{\tau}(t)\eta_i(t)^T\mathcal{U}\eta_i(t)+2\tau(t)\eta_i(t)^T\mathcal{U}\dot{\eta}_i(t)\\ &-\dot{\tau}(t)\kappa_i(t)^T\mathcal{Y}\kappa_i(t)+2[\bar{\tau}-\tau(t)]\kappa_i(t)^T\mathcal{Y}\dot{\kappa}_i(t)+2\gamma_i(t)^T\mathcal{P}\dot{\gamma}_i(t)\Big\}\\ =&\sum_{i=1}^{N}\Big\{\dot{\tau}(t)\eta_i(t)^T\mathcal{U}\eta_i(t)+2\eta_i(t)^T\mathcal{U}\begin{bmatrix}\tau(t)\dot{e}_{it}\\ e_{it}-[1-\dot{\tau}(t)]e_{i\tau}-\dot{\tau}(t)v_{it}\end{bmatrix}\\ &-\dot{\tau}(t)\kappa_i(t)^T\mathcal{Y}\kappa_i(t)+2\kappa_i(t)^T\mathcal{Y}\begin{bmatrix}[\bar{\tau}-\tau(t)]\dot{e}_{it}\\ -e_{i\bar{\tau}}+[1-\dot{\tau}(t)]e_{i\tau}+\dot{\tau}(t)w_{it}\end{bmatrix}\\ &+2\begin{bmatrix}e_{it}\\ \tau(t)v_{it}+[\bar{\tau}-\tau(t)]w_{it}\end{bmatrix}^T\mathcal{P}\begin{bmatrix}\dot{e}_{it}\\ e_{it}-e_{i\bar{\tau}}\end{bmatrix}\Big\}\\ =&\sum_{i=1}^{N}\delta_i(t)^T\Big\{\dot{\tau}(t)\begin{bmatrix}\iota_1\\ \iota_5\end{bmatrix}^T\mathcal{U}\begin{bmatrix}\iota_1\\ \iota_5\end{bmatrix}+\mathrm{sym}\Big\{\begin{bmatrix}\iota_1\\ \iota_5\end{bmatrix}^T\mathcal{U}\begin{bmatrix}\tau(t)\iota_4\\ \iota_1-[1-\dot{\tau}(t)]\iota_2-\dot{\tau}(t)\iota_5\end{bmatrix}\Big\}\end{aligned}$$

$$- \dot{\tau}(t)\begin{bmatrix}\iota_1\\ \iota_6\end{bmatrix}^T \mathcal{Y}\begin{bmatrix}\iota_1\\ \iota_6\end{bmatrix} + \mathrm{sym}\Big\{\begin{bmatrix}\iota_1\\ \iota_6\end{bmatrix}^T \mathcal{Y}\begin{bmatrix}[\bar{\tau}-\tau(t)]\iota_4\\ -\iota_3+[1-\dot{\tau}(t)]\iota_2+\dot{\tau}(t)\iota_6\end{bmatrix}\Big\}$$
$$+ \mathrm{sym}\Big\{\begin{bmatrix}\iota_1\\ \tau(t)\iota_5+[\bar{\tau}-\tau(t)]\iota_6\end{bmatrix}^T \mathcal{P}\begin{bmatrix}\iota_4\\ \iota_1-\iota_3\end{bmatrix}\Big\}\Big\}\delta_i(t), \tag{A4}$$

$$\dot{V}_3(t) = \sum_{i=1}^{N}\Big\{\Big[e_{it}^T\Lambda D_4\Lambda e_{it} - f(e_{it})^T D_4 f(e_{it})\Big]$$
$$- [1-\dot{\tau}(t)]\Big[e_{i\tau}^T\Lambda D_4\Lambda e_{i\tau} - f(e_{i\tau})^T D_4 f(e_{i\tau})\Big]$$
$$+ \varepsilon_i(t)^T\mathcal{Q}\varepsilon_i(t) - [1-\dot{\tau}(t)]\varepsilon_i(t-\tau(t))^T(\mathcal{Q}-\mathcal{W})\varepsilon_i(t-\tau(t)) - \varepsilon_i(t-\bar{\tau})^T\mathcal{W}\varepsilon_i(t-\bar{\tau})\Big\}$$
$$= \sum_{i=1}^{N}\delta_i(t)^T\Big\{(\iota_1^T\Lambda D_4\Lambda\iota_1 - \iota_{11}^T D_4\iota_{11}) - [1-\dot{\tau}(t)](\iota_2^T\Lambda D_4\Lambda\iota_2 - \iota_{12}^T D_4\iota_{12})$$
$$- [1-\dot{\tau}(t)]\begin{bmatrix}\iota_2\\ \iota_{12}-\Lambda_2\iota_2\end{bmatrix}^T(\mathcal{Q}-\mathcal{W})\begin{bmatrix}\iota_2\\ \iota_{12}-\Lambda_2\iota_2\end{bmatrix}$$
$$+ \begin{bmatrix}\iota_1\\ \iota_{11}-\Lambda_2\iota_1\end{bmatrix}^T\mathcal{Q}\begin{bmatrix}\iota_1\\ \iota_{11}-\Lambda_2\iota_1\end{bmatrix} - \begin{bmatrix}\iota_3\\ \iota_{13}-\Lambda_2\iota_3\end{bmatrix}^T\mathcal{W}\begin{bmatrix}\iota_3\\ \iota_{13}-\Lambda_2\iota_3\end{bmatrix}\Big\}\delta_i(t), \tag{A5}$$

$$\dot{V}_4(t) = \sum_{i=1}^{N}\Big\{\bar{\tau}^2\zeta_i(t)^T\mathcal{R}_1\zeta_i(t) - \bar{\tau}\int_{t-\bar{\tau}}^{t}\zeta_i(\upsilon)^T\mathcal{R}_1\zeta_i(\upsilon)\mathrm{d}\upsilon\Big\}$$
$$= \sum_{i=1}^{N}\Big\{\bar{\tau}^2\delta_i(t)^T\begin{bmatrix}\iota_1\\ \iota_4\end{bmatrix}^T\mathcal{R}_1\begin{bmatrix}\iota_1\\ \iota_4\end{bmatrix}\delta_i(t) - \bar{\tau}\int_{t-\bar{\tau}}^{t}\zeta_i(\upsilon)^T\mathcal{R}_1\zeta_i(\upsilon)\mathrm{d}\upsilon\Big\}, \tag{A6}$$

$$\dot{V}_5(t) = \sum_{i=1}^{N}\Big\{\frac{\bar{\tau}^2}{2}\dot{e}_{it}^T(S_1+S_2)\dot{e}_{it} - \int_{t-\bar{\tau}}^{t}\int_{\psi}^{t}\dot{e}_{i\upsilon}^T S_1\dot{e}_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi - \int_{t-\bar{\tau}}^{t}\int_{t-\bar{\tau}}^{\psi}\dot{e}_{i\upsilon}^T S_2\dot{e}_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi\Big\}$$
$$= \sum_{i=1}^{N}\Big\{\frac{\bar{\tau}^2}{2}\delta_i(t)^T\iota_4^T(S_1+S_2)\iota_4\delta_i(t) - \int_{t-\bar{\tau}}^{t}\int_{\psi}^{t}\dot{e}_{i\upsilon}^T S_1\dot{e}_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi$$
$$- \int_{t-\bar{\tau}}^{t}\int_{t-\bar{\tau}}^{\psi}\dot{e}_{i\upsilon}^T S_2\dot{e}_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi\Big\}. \tag{A7}$$

For $0 < \tau(t) < \bar{\tau}$, applying Lemma 3 to R_1-dependent terms gives

$$-\bar{\tau}\int_{t-\bar{\tau}}^{t}e_{i\upsilon}^T R_1 e_{i\upsilon}\mathrm{d}\upsilon = -\bar{\tau}\int_{t-\tau(t)}^{t}e_{i\upsilon}^T R_1 e_{i\upsilon}\mathrm{d}\upsilon - \bar{\tau}\int_{t-\bar{\tau}}^{t-\tau(t)}e_{i\upsilon}^T R_1 e_{i\upsilon}\mathrm{d}\upsilon$$
$$\le -\bar{\tau}\cdot\tau(t)\big\{v_{it}^T R_1 v_{it} + 3(v_{it}-x_{it})^T R_1(v_{it}-x_{it})$$
$$+ 5(v_{it}-3x_{it}+2\omega_{it})^T R_1(v_{it}-3x_{it}+2\omega_{it})\big\}$$
$$-\bar{\tau}[\bar{\tau}-\tau(t)]\big\{w_{it}^T R_1 w_{it} + 3(w_{it}-\varrho_{it})^T R_1(w_{it}-\varrho_{it})$$
$$+ 5(w_{it}-3\varrho_{it}+2\chi_{it})^T R_1(w_{it}-3\varrho_{it}+2\chi_{it})\big\}$$
$$= -\delta_i(t)^T\Big\{\tau(t)\big[\iota_5^T(\bar{\tau}R_1)\iota_5 + 3(\iota_5-\iota_7)^T(\bar{\tau}R_1)(\iota_5-\iota_7)$$
$$+ 5(\iota_5-3\iota_7+2\iota_9)^T(\bar{\tau}R_1)(\iota_5-3\iota_7+2\iota_9)\big]$$
$$+ [\bar{\tau}-\tau(t)]\big[\iota_6^T(\bar{\tau}R_1)\iota_6 + 3(\iota_6-\iota_8)^T(\bar{\tau}R_1)(\iota_6-\iota_8)$$

$$+ 5(\iota_6 - 3\iota_8 + 2\iota_{10})^T(\bar{\tau}R_1)(\iota_6 - 3\iota_8 + 2\iota_{10})\Big]\Big\}\delta_i(t). \tag{A8}$$

Setting $\vartheta = \frac{\tau(t)}{\bar{\tau}}, \varpi = 1 - \vartheta$, when $0 < \tau(t) < \bar{\tau}$, applying Lemma 3 to R_2-dependent terms derives

$$\begin{aligned}
&-\bar{\tau}\int_{t-\tau(t)}^{t} \dot{e}_{iv}^T R_2 \dot{e}_{iv} \mathrm{d}v \\
\le& -\frac{1}{\vartheta}\Big\{(e_{it} - e_{i\tau})^T R_2 (e_{it} - e_{i\tau}) \\
&+ 3(e_{it} + e_{i\tau} - 2v_{it})^T R_2 (e_{it} + e_{i\tau} - 2v_{it}) \\
&+ 5(e_{it} - e_{i\tau} + 6v_{it} - 6x_{it})^T R_2 (e_{it} - e_{i\tau} + 6v_{it} - 6x_{it})) \\
&+ 7(e_{it} + e_{i\tau} - 12v_{it} + 30x_{it} - 20\omega_{it})^T R_2 (e_{it} + e_{i\tau} - 12v_{it} + 30x_{it} - 20\omega_{it})\Big\} \\
=& -\frac{1}{\vartheta}\delta_i(t)^T \beta_1^T \mathrm{diag}\{R_2, 3R_2, 5R_2, 7R_2\}\beta_1\delta_i(t),
\end{aligned} \tag{A9}$$

$$\begin{aligned}
&-\bar{\tau}\int_{t-\bar{\tau}}^{t-\tau(t)} \dot{e}_{iv}^T R_2 \dot{e}_{iv} \mathrm{d}v \\
\le& -\frac{1}{\varpi}\Big\{(e_{i\tau} - e_{i\bar{\tau}})^T R_2 (e_{i\tau} - e_{i\bar{\tau}}) \\
&+ 3(e_{i\tau} + e_{i\bar{\tau}} - 2w_{it})^T R_2 (e_{i\tau} + e_{i\bar{\tau}} - 2w_{it}) \\
&+ 5(e_{i\tau} - e_{i\bar{\tau}} + 6w_{it} - 6\varrho_{it})^T R_2 (e_{i\tau} - e_{i\bar{\tau}} + 6w_{it} - 6\varrho_{it}) \\
&+ 7(e_{i\tau} + e_{i\bar{\tau}} - 12w_{it} + 30\varrho_{it} - 20\chi_{it})^T R_2 (e_{i\tau} + e_{i\bar{\tau}} - 12w_{it} + 30\varrho_{it} - 20\chi_{it})\Big\} \\
=& -\frac{1}{\varpi}\delta_i(t)^T \beta_2^T \mathrm{diag}\{R_2, 3R_2, 5R_2, 7R_2\}\beta_2\delta_i(t).
\end{aligned} \tag{A10}$$

Inspired by the work of Kim et al. (2010), the following zero equalities with any $m \times m$ symmetric matrices H_1, H_2 are proposed according to the integration by part:

$$\begin{aligned}
0 =& \bar{\tau}\Big\{e_{it}^T H_1 e_{it} - e_{i\tau}^T H_1 e_{i\tau} - 2\int_{t-\tau(t)}^{t} e_{iv}^T H_1 \dot{e}_{iv} \mathrm{d}v\Big\}, \\
0 =& \bar{\tau}\Big\{e_{i\tau}^T H_2 e_{i\tau} - e_{i\bar{\tau}}^T H_2 e_{i\bar{\tau}} - 2\int_{t-\bar{\tau}}^{t-\tau(t)} e_{iv}^T H_2 \dot{e}_{iv} \mathrm{d}v\Big\}.
\end{aligned}$$

Adding the above two zero equalities into the rest terms in $\dot{V}_4(t)$ yields

$$\begin{aligned}
&-\bar{\tau}\int_{t-\bar{\tau}}^{t} \zeta_i(v)^T \begin{bmatrix} R_3 & R_5 \\ * & R_4 \end{bmatrix} \zeta_i(v) \mathrm{d}v \\
=& \bar{\tau}\Big\{e_{it}^T H_1 e_{it} - e_{i\tau}^T (H_1 - H_2) e_{i\tau} - e_{i\bar{\tau}}^T H_2 e_{i\bar{\tau}} - \int_{t-\tau(t)}^{t} \zeta_i(v)^T \Psi_1 \zeta_i(v) \mathrm{d}v \\
&- \int_{t-\bar{\tau}}^{t-\tau(t)} \zeta_i(v)^T \Psi_2 \zeta_i(v) \mathrm{d}v\Big\} \\
=& \bar{\tau}\delta_i(t)^T \left[\iota_1^T H_1 \iota_1 - \iota_2^T (H_1 - H_2)\iota_2 - \iota_3^T H_2 \iota_3\right] \delta_i(t) \\
&- \bar{\tau}\int_{t-\tau(t)}^{t} \zeta_i(v)^T \Psi_1 \zeta_i(v) \mathrm{d}v - \bar{\tau}\int_{t-\bar{\tau}}^{t-\tau(t)} \zeta_i(v)^T \Psi_2 \zeta_i(v) \mathrm{d}v.
\end{aligned} \tag{A11}$$

For $0 < \tau(t) < \bar{\tau}$, applying Lemma 3 to the last two single integral terms of the above equality (A11) yields

$$
\begin{aligned}
& -\bar{\tau}\int_{t-\tau(t)}^{t}\zeta_i(\upsilon)^T\Psi_1\zeta_i(\upsilon)\mathrm{d}\upsilon \\
\leq & -\frac{\bar{\tau}}{\tau(t)}\begin{bmatrix}\tau(t)v_{it}\\ e_{it}-e_{i\tau}\end{bmatrix}^T\Psi_1\begin{bmatrix}\tau(t)v_{it}\\ e_{it}-e_{i\tau}\end{bmatrix}-\frac{3\bar{\tau}}{\tau(t)}\begin{bmatrix}\tau(t)(v_{it}-x_{it})\\ e_{it}+e_{i\tau}-2v_{it}\end{bmatrix}^T\Psi_1\begin{bmatrix}\tau(t)(v_{it}-x_{it})\\ e_{it}+e_{i\tau}-2v_{it}\end{bmatrix} \\
& -\frac{5\bar{\tau}}{\tau(t)}\begin{bmatrix}\tau(t)(v_{it}-3x_{it}+2\omega_{it})\\ e_{it}-e_{i\tau}+6v_{it}-6x_{it}\end{bmatrix}^T\Psi_1\begin{bmatrix}\tau(t)(v_{it}-3x_{it}+2\omega_{it})\\ e_{it}-e_{i\tau}+6v_{it}-6x_{it}\end{bmatrix} \\
= & -\tau(t)v_{it}^T(\bar{\tau}R_3)v_{it}-2\bar{\tau}v_{it}^T(R_5+H_1)(e_{it}-e_{i\tau})-\frac{\bar{\tau}}{\tau(t)}(e_{it}-e_{i\tau})^TR_4(e_{it}-e_{i\tau}) \\
& -3\tau(t)(v_{it}-x_{it})^T(\bar{\tau}R_3)(v_{it}-x_{it})-6\bar{\tau}(v_{it}-x_{it})^T(R_5+H_1)(e_{it}+e_{i\tau}-2v_{it}) \\
& -\frac{3\bar{\tau}}{\tau(t)}(e_{it}+e_{i\tau}-2v_{it})^TR_4(e_{it}+e_{i\tau}-2v_{it}) \\
& -5\tau(t)(v_{it}-3x_{it}+2\omega_{it})^T(\bar{\tau}R_3)(v_{it}-3x_{it}+2\omega_{it}) \\
& -10\bar{\tau}(v_{it}-3x_{it}+2\omega_{it})^T(R_5+H_1)(e_{it}-e_{i\tau}+6v_{it}-6x_{it}) \\
& -\frac{5\bar{\tau}}{\tau(t)}(e_{it}-e_{i\tau}+6v_{it}-6x_{it})^TR_4(e_{it}-e_{i\tau}+6v_{it}-6x_{it}) \\
= & -\delta_i(t)^T\Big\{\tau(t)\big[\iota_5^T(\bar{\tau}R_3)\iota_5+3(\iota_5-\iota_7)^T(\bar{\tau}R_3)(\iota_5-\iota_7) \\
& \qquad +5(\iota_5-3\iota_7+2\iota_9)^T(\bar{\tau}R_3)(\iota_5-3\iota_7+2\iota_9)\big] \\
& \quad +\bar{\tau}\cdot\mathrm{sym}\big\{\iota_5^T(R_5+H_1)(\iota_1-\iota_2)+3(\iota_5-\iota_7)^T(R_5+H_1)(\iota_1+\iota_2-2\iota_5) \\
& \quad +5(\iota_5-3\iota_7+2\iota_9)^T(R_5+H_1)(\iota_1-\iota_2+6\iota_5-6\iota_7)\big\}\Big\}\delta_i(t) \\
& -\frac{1}{\vartheta}(e_{it}-e_{i\tau})^TR_4(e_{it}-e_{i\tau})-\frac{3}{\vartheta}(e_{it}+e_{i\tau}-2v_{it})^TR_4(e_{it}+e_{i\tau}-2v_{it}) \\
& -\frac{5}{\vartheta}(e_{it}-e_{i\tau}+6v_{it}-6x_{it})^TR_4(e_{it}-e_{i\tau}+6v_{it}-6x_{it}),
\end{aligned}
\tag{A12}
$$

$$
\begin{aligned}
& -\bar{\tau}\int_{t-\bar{\tau}}^{t-\tau(t)}\zeta_i(\upsilon)^T\Psi_2\zeta_i(\upsilon)\mathrm{d}\upsilon \\
\leq & -\frac{\bar{\tau}}{\bar{\tau}-\tau(t)}\begin{bmatrix}[\bar{\tau}-\tau(t)]w_{it}\\ e_{i\tau}-e_{i\bar{\tau}}\end{bmatrix}^T\Psi_2\begin{bmatrix}[\bar{\tau}-\tau(t)]w_{it}\\ e_{i\tau}-e_{i\bar{\tau}}\end{bmatrix} \\
& -\frac{3\bar{\tau}}{\bar{\tau}-\tau(t)}\begin{bmatrix}[\bar{\tau}-\tau(t)](w_{it}-\varrho_{it})\\ e_{i\tau}+e_{i\bar{\tau}}-2w_{it}\end{bmatrix}^T\Psi_2\begin{bmatrix}[\bar{\tau}-\tau(t)](w_{it}-\varrho_{it})\\ e_{i\tau}+e_{i\bar{\tau}}-2w_{it}\end{bmatrix} \\
& -\frac{5\bar{\tau}}{\bar{\tau}-\tau(t)}\begin{bmatrix}[\bar{\tau}-\tau(t)](w_{it}-3\varrho_{it}+2\chi_{it})\\ e_{i\tau}-e_{i\bar{\tau}}+6w_{it}-6\varrho_{it}\end{bmatrix}^T\Psi_2\begin{bmatrix}[\bar{\tau}-\tau(t)](w_{it}-3\varrho_{it}+2\chi_{it})\\ e_{i\tau}-e_{i\bar{\tau}}+6w_{it}-6\varrho_{it}\end{bmatrix} \\
= & -[\bar{\tau}-\tau(t)]w_{it}^T(\bar{\tau}R_3)w_{it}-2\bar{\tau}w_{it}^T(R_5+H_2)(e_{i\tau}-e_{i\bar{\tau}}) \\
& \qquad -\frac{\bar{\tau}}{\bar{\tau}-\tau(t)}(e_{i\tau}-e_{i\bar{\tau}})^TR_4(e_{i\tau}-e_{i\bar{\tau}}) \\
& -3[\bar{\tau}-\tau(t)](w_{it}-\varrho_{it})^T(\bar{\tau}R_3)(w_{it}-\varrho_{it}) \\
& \quad -6\bar{\tau}[w_{it}-\varrho_{it}]^T(R_5+H_2)(e_{i\tau}+e_{i\bar{\tau}}-2w_{it})
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{3\bar{\tau}}{\bar{\tau}-\tau(t)}(e_{i\tau}+e_{i\bar{\tau}}-2w_{it})^T R_4(e_{i\tau}+e_{i\bar{\tau}}-2w_{it}) \\
&- 5[\bar{\tau}-\tau(t)](w_{it}-3\varrho_{it}+2\chi_{it})^T(\bar{\tau}R_3)(w_{it}-3\varrho_{it}+2\chi_{it}) \\
&- 10\bar{\tau}(w_{it}-3\varrho_{it}+2\chi_{it})^T(R_5+H_2)(e_{i\tau}-e_{i\bar{\tau}}+6w_{it}-6\varrho_{it}) \\
&- \frac{5\bar{\tau}}{\bar{\tau}-\tau(t)}(e_{i\tau}-e_{i\bar{\tau}}+6w_{it}-6\varrho_{it})^T R_4(e_{i\tau}-e_{i\bar{\tau}}+6w_{it}-6\varrho_{it}) \\
=& -\delta_i(t)^T\Big\{[\bar{\tau}-\tau(t)]\big[\iota_6^T(\bar{\tau}R_3)\iota_6+3(\iota_6-\iota_8)^T(\bar{\tau}R_3)(\iota_6-\iota_8) \\
&\qquad + 5(\iota_6-3\iota_8+2\iota_{10})^T(\bar{\tau}R_3)(\iota_6-3\iota_8+2\iota_{10})\big] \\
&\quad + \bar{\tau}\cdot\mathrm{sym}\big\{\iota_6^T(R_5+H_2)(\iota_2-\iota_3)+3(\iota_6-\iota_8)^T(R_5+H_2)(\iota_2+\iota_3-2\iota_6) \\
&\qquad + 5(\iota_6-3\iota_8+2\iota_{10})^T(R_5+H_2)(\iota_2-\iota_3+6\iota_6-6\iota_8)\big\}\Big\}\delta_i(t) \\
&- \frac{1}{\varpi}(e_{i\tau}-e_{i\bar{\tau}})^T R_4(e_{i\tau}-e_{i\bar{\tau}}) - \frac{3}{\varpi}\left[e_{i\tau}+e_{i\bar{\tau}}-2w_{it}\right]^T R_4(e_{i\tau}+e_{i\bar{\tau}}-2w_{it}) \\
&- \frac{5}{\varpi}(e_{i\tau}-e_{i\bar{\tau}}+6w_{it}-6\varrho_{it})^T R_4(e_{i\tau}-e_{i\bar{\tau}}+6w_{it}-6\varrho_{it}). \qquad \text{(A13)}
\end{aligned}
$$

Obviously the following equalities are true for any $t > 0$

$$
\begin{aligned}
\int_{t-\bar{\tau}}^{t}\int_{\psi}^{t}\dot{e}_{i\upsilon}^T S_1\dot{e}_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi &= \int_{t-\bar{\tau}}^{t-\tau(t)}\int_{\psi}^{t-\tau(t)}\dot{e}_{i\upsilon}^T S_1\dot{e}_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi \\
&+ \int_{t-\tau(t)}^{t}\int_{\psi}^{t}\dot{e}_{i\upsilon}^T S_1\dot{e}_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi + [\bar{\tau}-\tau(t)]\int_{t-\tau(t)}^{t}\dot{e}_{i\upsilon}^T S_1\dot{e}_{i\upsilon}\mathrm{d}\upsilon. \qquad \text{(A14)}
\end{aligned}
$$

When $0 < \tau(t) < \bar{\tau}$, applying Lemma 3 to S_1-dependent double integral terms yields

$$
\begin{aligned}
&\int_{t-\bar{\tau}}^{t-\tau(t)}\int_{\psi}^{t-\tau(t)}\dot{e}_{i\upsilon}^T S_1\dot{e}_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi \\
\geq& \frac{2}{[\bar{\tau}-\tau(t)]^2}\left([\bar{\tau}-\tau(t)]e_{i\tau}-\int_{t-\bar{\tau}}^{t-\tau(t)}e_{i\upsilon}\mathrm{d}\upsilon\right)^T S_1\left([\bar{\tau}-\tau(t)]e_{i\tau}-\int_{t-\bar{\tau}}^{t-\tau(t)}e_{i\upsilon}\mathrm{d}\upsilon\right) \\
&+ \frac{16}{[\bar{\tau}-\tau(t)]^2}\left(\frac{1}{2}[\bar{\tau}-\tau(t)]e_{i\tau}+\int_{t-\bar{\tau}}^{t-\tau(t)}e_{i\upsilon}\mathrm{d}\upsilon-\frac{3}{\bar{\tau}-\tau(t)}\int_{t-\bar{\tau}}^{t-\tau(t)}\int_{\psi}^{t-\tau(t)}e_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi\right)^T \\
&\quad \times S_1\left(\frac{1}{2}[\bar{\tau}-\tau(t)]e_{i\tau}+\int_{t-\bar{\tau}}^{t-\tau(t)}e_{i\upsilon}\mathrm{d}\upsilon-\frac{3}{\bar{\tau}-\tau(t)}\int_{t-\bar{\tau}}^{t-\tau(t)}\int_{\psi}^{t-\tau(t)}e_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi\right) \\
&+ \frac{54}{[\bar{\tau}-\tau(t)]^2}\left(\frac{1}{3}[\bar{\tau}-\tau(t)]e_{i\tau}-\int_{t-\bar{\tau}}^{t-\tau(t)}e_{i\upsilon}\mathrm{d}\upsilon+\frac{8}{\bar{\tau}-\tau(t)}\int_{t-\bar{\tau}}^{t-\tau(t)}\int_{\psi}^{t-\tau(t)}e_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi\right. \\
&\left.- \frac{20}{[\bar{\tau}-\tau(t)]^2}\int_{t-\bar{\tau}}^{t-\tau(t)}\int_{\psi}^{t-\tau(t)}\int_{\rho}^{t-\tau(t)}e_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\rho\mathrm{d}\psi\right)^T S_1\left(\frac{1}{3}[\bar{\tau}-\tau(t)]e_{i\tau}-\int_{t-\bar{\tau}}^{t-\tau(t)}e_{i\upsilon}\mathrm{d}\upsilon\right. \\
&\left.+ \frac{8}{\bar{\tau}-\tau(t)}\int_{t-\bar{\tau}}^{t-\tau(t)}\int_{\psi}^{t-\tau(t)}e_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi-\frac{20}{[\bar{\tau}-\tau(t)]^2}\int_{t-\bar{\tau}}^{t-\tau(t)}\int_{\psi}^{t-\tau(t)}\int_{\rho}^{t-\tau(t)}e_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\rho\mathrm{d}\psi\right) \\
=& 2(e_{i\tau}-w_{it})^T S_1(e_{i\tau}-w_{it})+4(e_{i\tau}+2w_{it}-3\varrho_{it})^T S_1(e_{i\tau}+2w_{it}-3\varrho_{it}) \\
&+ 6(e_{i\tau}-3w_{it}+12\varrho_{it}-10\chi_{it})^T S_1(e_{i\tau}-3w_{it}+12\varrho_{it}-10\chi_{it}) \\
=& \delta_i(t)^T\big\{2(\iota_2-\iota_6)^T S_1(\iota_2-\iota_6)+4(\iota_2+2\iota_6-3\iota_8)^T S_1(\iota_2+2\iota_6-3\iota_8) \\
&\quad + 6(\iota_2-3\iota_6+12\iota_8-10\iota_{10})^T S_1(\iota_2-3\iota_6+12\iota_8-10\iota_{10})\big\}\delta_i(t), \qquad \text{(A15)}
\end{aligned}
$$

$$\begin{aligned}
&\int_{t-\tau(t)}^{t}\int_{\psi}^{t}\dot{e}_{i\upsilon}^{T}S_1\dot{e}_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi\\
\geq&2(e_{it}-v_{it})^TS_1(e_{it}-v_{it})+4(e_{it}+2v_{it}-3x_{it})^TS_1(e_{it}+2v_{it}-3x_{it})\\
&+6(e_{it}-3v_{it}+12x_{it}-10\omega_{it})^TS_1(e_{it}-3v_{it}+12x_{it}-10\omega_{it})\\
=&\delta_i(t)^T\big\{2(\iota_1-\iota_5)^TS_1(\iota_1-\iota_5)+4(\iota_1+2\iota_5-3\iota_7)^TS_1(\iota_1+2\iota_5-3\iota_7)\\
&\quad+6(\iota_1-3\iota_5+12\iota_7-10\iota_9)^TS_1(\iota_1-3\iota_5+12\iota_7-10\iota_9)\big\}\delta_i(t).
\end{aligned}\tag{A16}$$

For $0<\tau(t)<\bar{\tau}$, applying Lemma 3 to S_1-dependent single integral term derives

$$\begin{aligned}
&[\bar{\tau}-\tau(t)]\int_{t-\tau(t)}^{t}\dot{e}_{i\upsilon}^{T}S_1\dot{e}_{i\upsilon}\mathrm{d}\upsilon\\
\geq&\frac{\varpi}{\vartheta}\Big\{(e_{it}-e_{i\tau})^TS_1(e_{it}-e_{i\tau})\\
&\quad+3(e_{it}+e_{i\tau}-2v_{it})^TS_1(e_{it}+e_{i\tau}-2v_{it})\\
&\quad+5(e_{it}-e_{i\tau}+6v_{it}-6x_{it})^TS_1(e_{it}-e_{i\tau}+6v_{it}-6x_{it})\\
&\quad+7(e_{it}+e_{i\tau}-12v_{it}+30x_{it}-20\omega_{it})^TS_1(e_{it}+e_{i\tau}-12v_{it}+30x_{it}-20\omega_{it})\Big\}\\
=&\frac{\varpi}{\vartheta}\delta_i(t)^T\beta_1^T\mathcal{S}_1\beta_1\delta_i(t).
\end{aligned}\tag{A17}$$

It is easy to see that the following equalities are true for any $t>0$

$$\begin{aligned}
\int_{t-\bar{\tau}}^{t}\int_{t-\bar{\tau}}^{\psi}\dot{e}_{i\upsilon}^{T}S_2\dot{e}_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi=&\int_{t-\tau(t)}^{t}\int_{t-\tau(t)}^{\psi}\dot{e}_{i\upsilon}^{T}S_2\dot{e}_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi\\
&+\int_{t-\bar{\tau}}^{t-\tau(t)}\int_{t-\bar{\tau}}^{\psi}\dot{e}_{i\upsilon}^{T}S_2\dot{e}_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi+\tau(t)\int_{t-\bar{\tau}}^{t-\tau(t)}\dot{e}_{i\upsilon}^{T}S_2\dot{e}_{i\upsilon}\mathrm{d}\upsilon.
\end{aligned}\tag{A18}$$

When $0<\tau(t)<\bar{\tau}$, applying Lemma 3 to S_2-dependent double integral terms gives

$$\begin{aligned}
&\int_{t-\tau(t)}^{t}\int_{t-\tau(t)}^{\psi}\dot{e}_{i\upsilon}^{T}S_2\dot{e}_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi\\
\geq&\frac{2}{\tau^2(t)}\left(\int_{t-\tau(t)}^{t}e_{i\upsilon}\mathrm{d}\upsilon-\tau(t)e_{i\tau}\right)^TS_2\left(\int_{t-\tau(t)}^{t}e_{i\upsilon}\mathrm{d}\upsilon-\tau(t)e_{i\tau}\right)\\
&+\frac{16}{\tau^2(t)}\left(\frac{1}{2}\tau(t)e_{i\tau}-2\int_{t-\tau(t)}^{t}e_{i\upsilon}\mathrm{d}\upsilon+\frac{3}{\tau(t)}\int_{t-\tau(t)}^{t}\int_{\psi}^{t}e_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi\right)\\
&\qquad\times S_2\left(\frac{1}{2}\tau(t)e_{i\tau}-2\int_{t-\tau(t)}^{t}e_{i\upsilon}\mathrm{d}\upsilon+\frac{3}{\tau(t)}\int_{t-\tau(t)}^{t}\int_{\psi}^{t}e_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi\right)\\
&+\frac{6}{\tau^2(t)}\Bigg(\tau(t)e_{i\tau}-9\int_{t-\tau(t)}^{t}e_{i\upsilon}\mathrm{d}\upsilon+\frac{36}{\tau(t)}\int_{t-\tau(t)}^{t}\int_{\psi}^{t}e_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi\\
&-\frac{60}{\tau^2(t)}\int_{t-\tau(t)}^{t}\int_{\psi}^{t}\int_{\rho}^{t}e_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\rho\mathrm{d}\psi\Bigg)\\
&\quad\times S_2\Bigg(\tau(t)e_{i\tau}-9\int_{t-\tau(t)}^{t}e_{i\upsilon}\mathrm{d}\upsilon+\frac{36}{\tau(t)}\int_{t-\tau(t)}^{t}\int_{\psi}^{t}e_{i\upsilon}\mathrm{d}\upsilon\mathrm{d}\psi
\end{aligned}$$

$$- \frac{60}{\tau^2(t)} \int_{t-\tau(t)}^{t} \int_{\psi}^{t} \int_{\rho}^{t} e_{i\upsilon} \mathrm{d}\upsilon \mathrm{d}\rho \mathrm{d}\psi \Big)$$

$$=2(e_{i\tau} - v_{it})^T S_2 (e_{i\tau} - v_{it}) + 4(e_{i\tau} - 4v_{it} + 3x_{it})^T S_2 (e_{i\tau} - 4v_{it} + 3x_{it})$$

$$+ 6(e_{i\tau} - 9v_{it} + 18x_{it} - 10\omega_{it})^T S_2 (e_{i\tau} - 9v_{it} + 18x_{it} - 10\omega_{it})$$

$$=\delta_i(t)^T \big\{ 2(\iota_2 - \iota_5)^T S_2 (\iota_2 - \iota_5) + 4(\iota_2 - 4\iota_5 + 3\iota_7)^T S_2 (\iota_2 - 4\iota_5 + 3\iota_7)$$

$$+ 6(\iota_2 - 9\iota_5 + 18\iota_7 - 10\iota_9)^T S_2 (\iota_2 - 9\iota_5 + 18\iota_7 - 10\iota_9) \big\} \delta_i(t), \quad \text{(A19)}$$

$$\int_{t-\bar{\tau}}^{t-\tau(t)} \int_{t-\bar{\tau}}^{\psi} \dot{e}_{i\upsilon}^T S_2 \dot{e}_{i\upsilon} \mathrm{d}\upsilon \mathrm{d}\psi$$

$$\geq 2(e_{i\bar{\tau}} - w_{it})^T S_2 (e_{i\bar{\tau}} - w_{it}) + 4(e_{i\bar{\tau}} - 4w_{it} + 3\varrho_{it})^T S_2 (e_{i\bar{\tau}} - 4w_{it} + 3\varrho_{it})$$

$$+ 6(e_{i\bar{\tau}} - 9w_{it} + 18\varrho_{it} - 10\chi_{it})^T S_2 (e_{i\bar{\tau}} - 9w_{it} + 18\varrho_{it} - 10\chi_{it})$$

$$=\delta_i(t)^T \big\{ 2(\iota_3 - \iota_6)^T S_2 (\iota_3 - \iota_6) + 4(\iota_3 - 4\iota_6 + 3\iota_8)^T S_2 (\iota_3 - 4\iota_6 + 3\iota_8)$$

$$+ 6(\iota_3 - 9\iota_6 + 18\iota_8 - 10\iota_{10})^T S_2 (\iota_3 - 9\iota_6 + 18\iota_8 - 10\iota_{10}) \big\} \delta_i(t). \quad \text{(A20)}$$

For $0 < \tau(t) < \bar{\tau}$, applying Lemma 3 to S_2-dependent single integral term yields

$$\tau(t) \int_{t-\bar{\tau}}^{t-\tau(t)} \dot{e}_{i\upsilon}^T S_2 \dot{e}_{i\upsilon} \mathrm{d}\upsilon$$

$$\geq \frac{\vartheta}{\varpi} \Big\{ (e_{i\tau} - e_{i\bar{\tau}})^T S_2 (e_{i\tau} - e_{i\bar{\tau}})$$

$$+ 3(e_{i\tau} + e_{i\bar{\tau}} - 2w_{it})^T S_2 (e_{i\tau} + e_{i\bar{\tau}} - 2w_{it})$$

$$+ 5(e_{i\tau} - e_{i\bar{\tau}} + 6w_{it} - 6\varrho_{it})^T S_2 (e_{i\tau} - e_{i\bar{\tau}} + 6w_{it} - 6\varrho_{it})$$

$$+ 7(e_{i\tau} + e_{i\bar{\tau}} - 12w_{it} + 30\varrho_{it} - 20\chi_{it})^T S_2 (e_{i\tau} + e_{i\bar{\tau}} - 12w_{it} + 30\varrho_{it} - 20\chi_{it}) \Big\}$$

$$= \frac{\vartheta}{\varpi} \delta_i(t)^T \beta_2^T \mathcal{S}_2 \beta_2 \delta_i(t). \quad \text{(A21)}$$

According to the second inequality of conditions (9), the following inequality is true for any $t > 0$

$$\delta_i(t)^T \begin{bmatrix} \sqrt{\frac{\varpi}{\vartheta}} \beta_1 \\ -\sqrt{\frac{\vartheta}{\varpi}} \beta_2 \end{bmatrix}^T \begin{bmatrix} \mathcal{R}_2 + \mathcal{S}_1 & \mathcal{X} \\ * & \mathcal{R}_2 + \mathcal{S}_2 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{\varpi}{\vartheta}} \beta_1 \\ -\sqrt{\frac{\vartheta}{\varpi}} \beta_2 \end{bmatrix} \delta_i(t) \geq 0.$$

Noting that $\vartheta + \varpi = 1$, one obtains

$$- \frac{1}{\vartheta} \delta_i(t)^T \beta_1^T \mathrm{diag}\, \{R_2, 3R_2, 5R_2, 7R_2\}\, \beta_1 \delta_i(t) \quad \text{(A22)}$$

$$- \frac{1}{\varpi} \delta_i(t)^T \beta_2^T \mathrm{diag}\{R_2, 3R_2, 5R_2, 7R_2\} \beta_2 \delta_i(t)$$

$$- \frac{1}{\vartheta} (e_{it} - e_{i\tau})^T R_4 (e_{it} - e_{i\tau}) - \frac{3}{\vartheta} (e_{it} + e_{i\tau} - 2v_{it})^T R_4 (e_{it} + e_{i\tau} - 2v_{it})$$

$$- \frac{5}{\vartheta} (e_{it} - e_{i\tau} + 6v_{it} - 6x_{it})^T R_4 (e_{it} - e_{i\tau} + 6v_{it} - 6x_{it})$$

$$- \frac{1}{\varpi} (e_{i\tau} - e_{i\bar{\tau}})^T R_4 (e_{i\tau} - e_{i\bar{\tau}})$$

$$
\begin{aligned}
&-\frac{3}{\varpi}(e_{i\tau}+e_{i\bar{\tau}}-2w_{it})^T R_4(e_{i\tau}+e_{i\bar{\tau}}-2w_{it})-\frac{5}{\varpi}(e_{it}-e_{i\tau}+6w_{it}-6\varrho_{it})^T\\
&\times R_4(e_{it}-e_{i\tau}+6w_{it}-6\varrho_{it})-\frac{\varpi}{\vartheta}\delta_i(t)^T\beta_1^T\mathcal{S}_1\beta_1\delta_i(t)-\frac{\vartheta}{\varpi}\delta_i(t)^T\beta_2^T\mathcal{S}_2\beta_2\delta_i(t)\\
&=-\delta_i(t)^T\Big\{\frac{1}{\vartheta}\beta_1^T\mathcal{R}_2\beta_1+\frac{1}{\varpi}\beta_2^T\mathcal{R}_2\beta_2+\frac{\varpi}{\vartheta}\beta_1^T\mathcal{S}_1\beta_1+\frac{\vartheta}{\varpi}\beta_2^T\mathcal{S}_2\beta_2\Big\}\delta_i(t)\\
&\le -\delta_i(t)^T\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}^T\begin{bmatrix}\mathcal{R}_2 & \mathcal{X}\\ * & \mathcal{R}_2\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}\delta_i(t). \qquad \text{(A23)}
\end{aligned}
$$

Considering networks (6), the following zero equality is true for any positive diagonal matrix D_5, D_6

$$
\begin{aligned}
0 &= 2\sum_{i=1}^{N}(D_5e_{it}+D_6\dot{e}_{it})^T\Big\{-\dot{e}_{it}-Ce_{it}+Af(e_{it})+Bf(e_{i\tau})\\
&\quad +\sigma_1\sum_{j=1}^{N}\bar{g}_{1ij}\Phi_1e_{jt}+\sigma_2\sum_{j=1}^{N}\bar{g}_{2ij}\Phi_2e_{j\tau}+\sigma_3\sum_{j=1}^{N}q_{3ij}\Phi_3\int_{t-\tau(t)}^{t}e_{js}\mathrm{d}\upsilon\Big\}\\
&=\sum_{i=1}^{N}\delta_i(t)^T\mathrm{sym}\big\{(D_5\iota_1+D_6\iota_5)^T(-\iota_5-C\iota_1+A\iota_{11}+B\iota_{12})\big\}\delta_i(t)\\
&\quad +2\sum_{i=1}^{N}(D_5e_{it}+D_6\dot{e}_{it})^T\Big\{\sigma_1\sum_{j=1}^{N}\bar{g}_{1ij}\Phi_1e_{jt}+\sigma_2\sum_{j=1}^{N}\bar{g}_{2ij}\Phi_2e_{j\tau}\\
&\quad +\sigma_3\sum_{j=1}^{N}q_{3ij}\Phi_3\int_{t-\tau(t)}^{t}e_{js}\mathrm{d}\upsilon\Big\}.
\end{aligned}
$$

Setting $\tilde{e}_k(t)=\mathrm{col}\{e_{1k}(t),e_{2k}(t),...,e_{Nk}(t)\}$, $\dot{\tilde{e}}_k(t)=\mathrm{col}\{\dot{e}_{1k}(t),\dot{e}_{2k}(t),...,\dot{e}_{Nk}(t)\}$, $k=1,2,...,m$, then one has

$$
\begin{aligned}
2\sum_{i=1}^{N}(D_6\dot{e}_{it})^T\sigma_1\sum_{j=1}^{N}\bar{g}_{1ij}\Phi_1e_{jt} &= 2\sigma_1\sum_{i,j=1}^{N}\bar{g}_{1ij}\dot{e}_{it}^TD_6\Phi_1e_{jt}\\
&=2\sigma_1\sum_{i,j=1}^{N}\bar{g}_{1ij}\sum_{k=1}^{m}d_{6k}\phi_{1k}\dot{e}_{ik}(t)e_{jk}(t)\\
&=2\sigma_1\sum_{k=1}^{m}d_{6k}\phi_{1k}\dot{\tilde{e}}_k(t)\bar{Q}_1\tilde{e}_k(t)\\
&\le 2\sigma_1\lambda_M(\bar{Q}_1)\sum_{k=1}^{m}d_{6k}\phi_{1k}\dot{\tilde{e}}_k(t)\tilde{e}_k(t)\\
&=2\sigma_1\lambda_M(\bar{Q}_1)\sum_{i=1}^{N}\dot{e}_{it}^TD_6\Phi_1e_{it}\\
&=\sigma_1\lambda_M(\bar{Q}_1)\sum_{i=1}^{N}\delta_i(t)^T\mathrm{sym}\big\{\iota_4^TD_6\Phi_1\iota_1\big\}\delta_i(t), \qquad \text{(A24)}
\end{aligned}
$$

$$
\begin{aligned}
2\sum_{i=1}^{N}(D_6\dot{e}_{it})^T\sigma_2\sum_{j=1}^{N}\bar{g}_{2ij}\Phi_2 e_{j\tau} &= 2\sigma_2\sum_{i,j=1}^{N}\bar{g}_{2ij}\dot{e}_{it}^T D_6\Phi_2 e_{j\tau} \\
&= 2\sigma_2\sum_{i,j=1}^{N}\bar{g}_{2ij}\sum_{k=1}^{m}d_{6k}\phi_{2k}\dot{e}_{ik}(t)e_{jk}(t-\tau(t)) \\
&= 2\sigma_2\sum_{k=1}^{m}d_{6k}\phi_{2k}\dot{\tilde{e}}_k(t)\bar{Q}_2\tilde{e}_k(t-\tau(t)) \\
&\le 2\sigma_2\lambda_M(\bar{Q}_2)\sum_{k=1}^{m}d_{6k}\phi_{2k}\dot{\tilde{e}}_k(t)\tilde{e}_k(t-\tau(t)) \\
&= 2\sigma_2\lambda_M(\bar{Q}_2)\sum_{i=1}^{N}\dot{e}_{it}^T D_6\Phi_2 e_{i\tau} \\
&= \sigma_2\lambda_M(\bar{Q}_2)\sum_{i=1}^{N}\delta_i(t)^T\mathrm{sym}\{\iota_4^T D_6\Phi_2\iota_2\}\delta_i(t),
\end{aligned}
\tag{A25}
$$

$$
\begin{aligned}
2\sum_{i=1}^{N}(D_6\dot{e}_{it})^T\sigma_3\sum_{j=1}^{N}g_{3ij}\Phi_3\int_{t-\tau(t)}^{t}e_{js}\mathrm{d}\upsilon &= 2\sigma_3\sum_{i,j=1}^{N}g_{3ij}\dot{e}_{it}^T D_6\Phi_3\int_{t-\tau(t)}^{t}e_{js}\mathrm{d}\upsilon \\
&= 2\sigma_3\sum_{i,j=1}^{N}g_{3ij}\sum_{k=1}^{m}d_{6k}\phi_{3k}\dot{e}_{ik}(t)\int_{t-\tau(t)}^{t}e_{jk}(\upsilon)\mathrm{d}\upsilon \\
&= 2\sigma_3\sum_{k=1}^{m}d_{6k}\phi_{3k}\dot{\tilde{e}}_k(t)Q_3\int_{t-\tau(t)}^{t}\tilde{e}_k(\upsilon)\mathrm{d}\upsilon \\
&\le 2\sigma_3\lambda_M(Q_3)\sum_{k=1}^{m}d_{6k}\phi_{3k}\dot{\tilde{e}}_k(t)\int_{t-\tau(t)}^{t}\tilde{e}_k(\upsilon)\mathrm{d}\upsilon \\
&= 2\sigma_3\lambda_M(Q_3)\sum_{i=1}^{N}\dot{e}_{it}^T D_6\Phi_3\int_{t-\tau(t)}^{t}e_{i\upsilon}\mathrm{d}\upsilon \\
&= \sigma_3\tau(t)\lambda_M(Q_3)\sum_{i=1}^{N}\delta_i(t)^T\mathrm{sym}\{\iota_4^T D_6\Phi_3\iota_5\}\delta_i(t).
\end{aligned}
\tag{A26}
$$

Similarly one has

$$
\begin{aligned}
2\sum_{i=1}^{N}(D_5 e_{it})^T\sigma_1\sum_{j=1}^{N}\bar{g}_{1ij}\Phi_1 e_{jt} &\le 2\sigma_1\lambda_M(\bar{Q}_1)\sum_{i=1}^{N}e_{it}^T D_5\Phi_1 e_{it} \\
&= 2\sigma_1\lambda_M(\bar{Q}_1)\sum_{i=1}^{N}\delta_i(t)^T\big(\iota_1^T D_5\Phi_1\iota_1\big)\delta_i(t),
\end{aligned}
\tag{A27}
$$

$$
\begin{aligned}
2\sum_{i=1}^{N}(D_5 e_{it})^T\sigma_2\sum_{j=1}^{N}\bar{g}_{2ij}\Phi_2 e_{j\tau} &\le 2\sigma_2\lambda_M(\bar{Q}_2)\sum_{i=1}^{N}e_{it}^T D_5\Phi_2 e_{i\tau} \\
&= \sigma_2\lambda_M(\bar{Q}_2)\sum_{i=1}^{N}\delta_i(t)^T\mathrm{sym}\{\iota_1^T D_5\Phi_2\iota_2\}\delta_i(t),
\end{aligned}
\tag{A28}
$$

$$2\sum_{i=1}^{N}(D_5e_{it})^T\sigma_3\sum_{j=1}^{N}g_{3ij}\Phi_3\int_{t-\tau(t)}^{t}e_{js}\mathrm{d}\upsilon \le 2\sigma_3\lambda_M(Q_3)\sum_{i=1}^{N}e_{it}^TD_5\Phi_3\int_{t-\tau(t)}^{t}e_{i\upsilon}\mathrm{d}\upsilon$$
$$= \sigma_3\tau(t)\lambda_M(Q_3)\sum_{i=1}^{N}\delta_i(t)^T\mathrm{sym}\big\{\iota_1^TD_5\Phi_3\iota_5\big\}\delta_i(t). \tag{A29}$$

From the second inequality of conditions (10), the $\dot{\tau}(t)$-dependent terms in $\dot{V}(t)$ can be combined and estimated as follows:

$$\dot{\tau}(t)\rho_i(t)^T\Upsilon\rho_i(t) \le \tau'\rho_i(t)^T\Upsilon\rho_i(t), \tag{A30}$$

where $\rho_i(t) = \mathrm{col}\left\{e_{it},\ e_{i\tau},\ v_{it},\ w_{it},\ f(e_{i\tau})\right\}$.

By taking into account the conditions (5) of the activation function, the following inequalities are true for any positive diagonal matrices $W_l(l = 1, 2, ..., 6)$:

$$0 \le 2[f(e_{it}) - \Lambda_2e_{it}]^TW_1[\Lambda_1e_{it} - f(e_{it})]$$
$$= \delta_i(t)^T\mathrm{sym}\left\{(\iota_{11} - \Lambda_2\iota_1)^TW_1(\Lambda_1\iota_1 - \iota_{11})\right\}\delta_i(t), \tag{A31}$$
$$0 \le 2[f(e_{i\tau}) - \Lambda_2e_{i\tau}]^TW_2[\Lambda_1e_{i\tau} - f(e_{i\tau})]$$
$$= \delta_i(t)^T\mathrm{sym}\left\{(\iota_{12} - \Lambda_2\iota_2)^TW_2(\Lambda_1\iota_2 - \iota_{12})\right\}\delta_i(t), \tag{A32}$$
$$0 \le 2[f(e_{i\bar{\tau}}) - \Lambda_2e_{i\bar{\tau}}]^TW_3[\Lambda_1e_{i\bar{\tau}} - f(e_{i\bar{\tau}})]$$
$$= \delta_i(t)^T\mathrm{sym}\left\{(\iota_{13} - \Lambda_2\iota_3)^TW_3(\Lambda_1\iota_3 - \iota_{13})\right\}\delta_i(t), \tag{A33}$$
$$0 \le 2\big\{[f(e_{it}) - f(e_{i\tau})] - \Lambda_2(e_{it} - e_{i\tau})\big\}^TW_4\big\{\Lambda_1(e_{it} - e_{i\tau}) - [f(e_{it}) - f(e_{i\tau})]\big\}$$
$$= \delta_i(t)^T\mathrm{sym}\left\{[(\iota_{11} - \iota_{12}) - \Lambda_2(\iota_1 - \iota_2)]^TW_4[\Lambda_1(\iota_1 - \iota_2) - (\iota_{11} - \iota_{12})]\right\}\delta_i(t), \tag{A34}$$
$$0 \le 2\big\{[f(e_{it}) - f(e_{i\bar{\tau}})] - \Lambda_2(e_{it} - e_{i\bar{\tau}})\big\}^TW_5\big\{\Lambda_1(e_{it} - e_{i\bar{\tau}}) - [f(e_{it}) - f(e_{i\bar{\tau}})]\big\}$$
$$= \delta_i(t)^T\mathrm{sym}\left\{[(\iota_{11} - \iota_{13}) - \Lambda_2(\iota_1 - \iota_3)]^TW_5[\Lambda_1(\iota_1 - \iota_3) - (\iota_{11} - \iota_{13})]\right\}\delta_i(t), \tag{A35}$$
$$0 \le 2\big\{[f(e_{i\tau}) - f(e_{i\bar{\tau}})] - \Lambda_2(e_{i\tau} - e_{i\bar{\tau}})\big\}^TW_6\big\{\Lambda_1(e_{i\tau} - e_{i\bar{\tau}}) - [f(e_{i\tau}) - f(e_{i\bar{\tau}})]\big\}$$
$$= \delta_i(t)^T\mathrm{sym}\left\{[(\iota_{12} - \iota_{13}) - \Lambda_2(\iota_2 - \iota_3)]^TW_6[\Lambda_1(\iota_2 - \iota_3) - (\iota_{12} - \iota_{13})]\right\}\delta_i(t). \tag{A36}$$

For $0 < \tau(t) < \bar{\tau}$, substituting (A3)-(A36) into (A2) gives

$$\dot{V}(t) \le \sum_{i=1}^{N}\delta_i(t)^T\big\{\Omega_0 + \tau(t)\Omega_1 + [\bar{\tau} - \tau(t)]\Omega_2\big\}\delta_i(t). \tag{A37}$$

As $\Omega_0 + \tau(t)\Omega_1 + [\bar{\tau} - \tau(t)]\Omega_2$ is a linear convex combination of $\tau(t)$, $\dot{V}(t) < 0$ is true if and only if inequalities (11) are true simultaneously for $k = 1, 2$. Therefore, if inequalities (11) hold, from inequality (A37) one has that $\dot{V}(t) < 0$ is true for any $t > 0$. This ends the proof of Theorem 1.