# Adaptive online successive constant rebalanced portfolio based on moving window

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**Abstract:** In the non-stationary financial market, considering that earlier observations may have little or no relevance to the current investment decision making, we design two kinds of adaptive online portfolio strategies only based on recent historical data. Firstly, we design an adaptive online portfolio strategy by linearly combining the last portfolio and the best constant rebalanced portfolio corresponding to the recent historical data, which we call moving window. We determine the length of the moving window by adaptive learning. More precisely, we consider the strategies that always adopt the best constant rebalanced portfolio corresponding to the moving window of different fixed lengths as different experts, and at the beginning of the current period, we choose the length of moving window the same as the expert achieving maximum current cumulative return. Furthermore, we determine the length of moving window by only using the recent historical data to adaptively learn, and design another adaptive online portfolio strategy. We present numerical analysis by using real stock data from the US and Chinese markets, and the results illustrate that our strategies perform well, compared with some benchmark strategies and existing online portfolio strategies.

**Keywords:** online portfolio selection; investment strategy; moving window; adaptive algorithm.

**Reference** to this paper should be made as follows: He, J., Yang, X., Lin, H. and Zhang, Y. (2020) 'Adaptive online successive constant rebalanced portfolio based on moving window', *Int. J. Industrial and Systems Engineering*, Vol. 34, No. 1, pp.107–123.

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## 1 Introduction

As the old saying goes, 'do not put all your eggs in one basket', investors often allocate their wealth among a variety of assets to reduce the risk of investment. Asset allocation theory is also known as portfolio theory, which focuses on how to scientifically determine the proportion of each asset and has been concerned by many scholars (Markowitz, 1952; Hao, 2015). These models assume that the asset prices follow certain probability distributions, and the parameters are known. However, in the investment practice, it is difficult to find suitable probability distributions to describe asset prices; even if found, it is difficult to accurately estimate the parameters, which leads to weak practicality of these models.

To overcome the shortcoming of the portfolio models with probability distributions, Cover (1991) proposed the online portfolio selection model. It is a dynamic portfolio model that divides the whole investment process into many periods and at the beginning of each period, the portfolio is determined only based on historical data. The dynamic portfolio strategy under the distribution-free assumption is an online strategy, as opposed to the offline strategy, which is determined in hindsight. The implication of 'online' is that investors only know the past and current asset prices, and know nothing about asset prices in the future, so they could obtain the latest historical data to calculate the next portfolio at the beginning of each period and then repeat this operation. Thus, it is a dynamic process. In recent years, some research methods from computer sciences have been borrowed by online portfolio theory, which designed a number of well-performing strategies (Huang et al., 2016; Lin et al., 2017; Zhang and Yang, 2017).

Although existing online portfolio strategies have been shown to achieve good performance, they often use all historical data to calculate the portfolios in each period, and usually do not adjust the relative parameters in the model throughout the whole investment process. However, the financial market, especially the emerging one, is non-stationary as it is affected by many factors. The historical data long ago may be little or not relevant to current investment decisions. Meanwhile, for an online portfolio strategy, the values of different parameters should not be fixed throughout the whole investment process. To tackle the limitations of existing strategies, we design adaptive online portfolio strategies only based on moving window. Firstly, we determine the length of moving window by adaptive learning and use a weighted method to smoothly handle vast adjustment in the portfolio. An online portfolio strategy WASCRP is proposed. Considering it still uses all the historical data indirectly, we further determine the length of moving window only by using the recent historical data to adaptively learn. Another online portfolio strategy MWASCRP is proposed. The result of numerical analysis verifies that the proposed strategies have good competitive performance.

The rest of this paper is organised as follows. Section 2 introduces some preliminary works about the online portfolio selection problem and benchmarks. Section 3 discusses related existing works. Section 4 presents our proposed strategies WASCRP and MWASCRP. Section 5 presents and analyses the experiment results of our strategies on real markets. Section 6 concludes this paper.

## 2 Preliminaries

### *2.1 Online portfolio selection*

In this section, we define some related notations and formulate the online portfolio selection problem. We assume that an investor allocates his/her wealth among $m$ assets for $n$ trading periods. The price relative vector of $m$ assets on the $t^{\text{th}}$ period is specified by a vector $\mathbf{x}_t = (x_{t,1}, \dots, x_{t,m})^\top \in \mathbf{R}_+^m$, where $x_{t,i}$ represents the proportion of the closing price of the $t$-th period to that of the $(t-1)^{\text{th}}$ period of the asset $i$. A sequence of price relative vectors for $n$ trading periods $\{\mathbf{x}_1, ..., \mathbf{x}_n\}$ is abbreviated as $\mathbf{x}_{1:n}$. Denote the portfolio vector of the $t^{\text{th}}$ period as $\mathbf{b}_t = (b_{t,1}, ..., b_{t,m})^\top$, where $b_{t,i}$ represents the proportion of wealth invested in the $t^{\text{th}}$ period of the asset $i$. All possible portfolios are represented by a set $\Delta_m$, where $\Delta_m = \{\mathbf{b} = (b_1, ..., b_m)^\top : b_i \geq 0, \sum_{i=1}^m b_i = 1\}$. The investment portfolios for $n$ periods constitute a sequence $\{\mathbf{b}_1, ..., \mathbf{b}_n\}$, which is called an online portfolio strategy and abbreviated as $\mathbf{b}_{1:n}$.

We just use the historical price relative data to construct the portfolio of each period, that is, $\mathbf{b}_{t+1} = \mathbf{b}_{t+1}(\mathbf{x}_{1:t}), t = 1, \dots, n-1$. In particular, we adopt a uniform initial portfolio, that is, $\mathbf{b}_1 = (1/m, \dots, 1/m)$. On the $t^{\text{th}}$ trading period, if the investor adopts investment portfolio $\mathbf{b}_t$, when the price relative vector $\mathbf{x}_t$ occurs, the multiplier of wealth growth is represented as $\mathbf{b}_t^\top \mathbf{x}_t = \sum_{i=1}^m b_{t,i} x_{t,i}$. Then, the final cumulative wealth after $n$ periods is

$$S_n = S_0 \prod_{t=1}^{n} \mathbf{b}_t^\top \mathbf{x}_t = S_0 \prod_{t=1}^{n} \sum_{i=1}^{m} b_{t,i} x_{t,i}, \tag{1}$$

where $S_0$ denotes the initial wealth. Without loss of generality, we set $S_0 = 1$.

The key of the online portfolio selection problem is to calculate portfolio $\mathbf{b}_t$ for each period, and to make the final cumulative wealth as much as possible. The process of selecting the portfolio is repeated in each period until the end. The general online portfolio selection framework is shown in Algorithm 1.

**Algorithm 1** Online portfolio selection framework

**Input:** $\mathbf{x}_{1:n}$ : Price relative sequence.
**Output:** $S_n$ : Final cumulative wealth.
1: Initialise $S_0 = 1, \mathbf{b}_1 = (1/m, ..., 1/m)^\top$.
2: **for** $t = 1, ..., n$ **do**
3: Receive price relative vector: $\mathbf{x}_t = (x_{t,1}, \ldots, x_{t,m})$.
4: Update the cumulative return until the $t^{\text{th}}$ period: $S_t = S_{t-1} \times (\mathbf{b}_t^\top \mathbf{x}_t)$.
5: Calculate the next portfolio: $\mathbf{b}_{t+1}$.
6: **end for**

## 2.2 Benchmarks

Buy-and-hold (BAH) is the simplest and passive baseline strategy that makes an initial investment and does not rebalance in subsequent periods. Now, readers are encouraged to know two kinds of special BAH strategies, i.e., best and market. The best strategy is that the investor puts all initial wealth on the single asset which has the best performance among all the assets he can invest during the whole trading periods. Distinctly, this special BAH strategy cannot be determined until the end of the trading period. The Market strategy is another special BAH with initial portfolio $\mathbf{b}_1 = (1/m, \ldots, 1/m)$, which is usually adopted as market strategy to produce market index.

If the same portfolio $\mathbf{b} \in \Delta_m$ is used for each period, this strategy is called constant rebalanced portfolios (CRP), whose final cumulative wealth is denoted as $S_n(CRP(\mathbf{b})) = \prod_{t=1}^{n} \mathbf{b}^\top \mathbf{x}_t$. In the situation of a fixed price relative sequence, we can determine the best CRP (BCRP) strategy as the one achieving the maximum wealth. We denote this portfolio by $\mathbf{b}^*$, and it is given by

$$\mathbf{b}^* = \arg\max_{\mathbf{b} \in \Delta_m} S_n(\mathbf{b}), \tag{2}$$

and the maximum cumulative wealth is

$$S_n^* = \max_{\mathbf{b} \in \Delta_m} S_n(\mathbf{b}). \tag{3}$$

Note that BCRP is a hindsight strategy, which can only be calculated with all price information. Cover (1991) proved that the BCRP strategy has a series of good properties, which indicates it has better performance than the best and market strategies. Thus, BCRP is usually identified as a benchmark to judge the performance of online portfolio strategies.

## 3 Existing works

Over the past 60 years, portfolio theory has obtained a lot of research results. During the investment period, investors often have to adjust their asset positions, and therefore many scholars have studied the mean-variance dynamic portfolio model (Li and Ng, 2000; Hao, 2015). In order to make the investment decision activities much closer to the reality, some scholars have studied the models that consider factors such as transaction cost (Albeverio et al., 2001; Yang et al., 2018). Many scholars have studied the portfolio selection model in a fuzzy environment (Dastkhan et al., 2013; Kazemi et al., 2014). Existing works of online portfolio selection are summarised as follows.

Cover (1991) proposed an online portfolio model that does not make any hypothesis about the price of securities. On the basis of above, Cover chose BCRP as the benchmark, constructed an online portfolio strategy comparable to this benchmark strategy, and considered the strategy as universal portfolio (UP). Cover's UP strategy uses known asset prices to weight average all CRP experts to determine the next portfolio. The UP strategy could not guarantee good returns for all sequences of price of securities. To this end, Singer (1997) researched the online portfolio strategy based on market changes to improve the return on investment strategy. Subsequently, Helmbold et al. (1998) applied multiplicative updates, and proposed the exponentiated gradient (EG) update that uses the current price information of the securities to determine the next investment portfolio. Based on the linear learning function, Zhang et al. (2012) proposed a class of universal strategies by improving EG which can update the portfolio based on current performance. Gaivoronski and Stella (2003) proposed successive constant rebalanced portfolio (SCRP) strategy by using BCRP corresponding to the past sequences of securities price as the portfolio of the next period, and gave further empirical analysis based on the data from New York Stock Exchange. O'Sullivan and Edelman (2015) proposed the adaptive universal portfolio (AUP), which retains much of the qualitative nature of Cover's UP while enhancing early performance, so it do not need to take a long time to produce significant growth. Zhang and Yang (2017) proposed that one WAAS strategy aggregate experts' strategies that only invest in a single stock and the other WAAC strategy aggregates a series of experts' CRP, and proved that the latter is a universal strategy. Moreover, Hazan and Kale (2015) combined the worst-case model with the geometric Brownian model to create a universal strategy whose regret bound has been greatly improved by probing into the information in the Brownian motion's assumptions. Using the ideas derived from signal processing and statistics, Tsagarisa et al. (2012) designed two online portfolio strategies, i.e., robust exponential least squares (R-EWRLS) strategy and regularised online minimum variance (O-VAR) strategy, both of which outperform the benchmark allocation techniques in terms of computational demand and financial performance.

Online portfolio is a dynamic portfolio, and repeated adjustments will have a bad impact on generating much more transaction costs. Blum and Kalai (1997) studied an online portfolio strategy with fixed-ratio transaction cost and proposed a solution that can be rapidly carried out in real-world application. Albeverio et al. (2001) constructed an online portfolio strategy with transaction costs by the cross rate method and proved that the final cumulative wealth of the strategy is comparable to that of the optimal securities. Given that investors may obtain side information which is beneficial to the investment, many scholars began to study online portfolio strategies with side information (Cover and Ordentlich, 1996; Fagiuoli et al., 2007; Bean and Singer, 2012), which achieve the same wealth as best state constant rebalanced portfolio (BSCRP) determined in hindsight.

Empirical evidence indicates that the current best performing stocks may not also perform well in the next trading period, especially in the short term. This trading principle is known as 'mean reversion' (Jegadeesh, 1990). Based on this principle, many scholars designed a series of online portfolio strategies. Borodin et al. (2004) put forward the Anticor strategy by calculating the correlation between the logarithmic returns of two pairs of stocks in two adjacent historical windows and using the law of mean reversion to adjust the portfolio. Although the Anticor strategy was not proved to be a universal strategy, the empirical results shown that the strategy not only surpasses UP and EG, but also BCRP. Li et al. (2012) used the passive aggressive learning technology from machine learning to construct the PAMR strategy. Passively, the strategy possesses the current poor performance strategy; aggressively, it adjusts the current better performing strategy to effectively find out the mean reversion properties of the financial market. Li et al. (2013) proposed confidence weighted average mean reversion (CWMR) strategy by confidence weighted (CW) online learning algorithm to further exploit the second order portfolio information and the mean reversion trading. Li et al. (2015) predicted the stock prices by using simple moving average and exponential moving average, and constructed two kinds of online moving average revision (OLMAR) strategies, which achieve significantly better results than most existing mean reversion algorithms. Li et al. (2014) reviewed in detail the progress of online portfolio research. Huang et al. (2016) exploited the reversion rule by using robust-median estimators and designed robust median reversion (RMR) strategy, while Lin et al. (2017) exploited mean reversion principle from a meta learning perspective and proposed two kinds of boosting moving average reversion (BMAR) strategies, and they all vastly overcame the noise and outliers in real-world markets. Yang et al. (2018) designed mean reversion strategy with transaction cost (MRTC), which can efficiently trade-off between high cumulative wealth and low transaction cost by adaptively adjusting the turnover.

## 4 Proposed strategies

### *4.1 WASCRP strategy*

The financial market is non-stationary and tends to fluctuate intensely, so the historical price relatives long ago may be little or not relevant to current investment decisions, and may even have a negative impact. Therefore, we only use the recent historical price relatives of fixed length when determining a portfolio in the next period.

To deal with such a problem, a simple idea is to take BCRP corresponding to the recent historical data of fixed length as the next portfolio. We denote the length of the historical price relatives by $r$ $(r \in \mathrm{N}_+)$, and call the sequence consisting of the $r$ historical price relatives moving window. More precisely, the portfolio at the beginning of the $(t+1)^{\text{th}}$ period is

$$\mathbf{b}_{t+1} = \mathbf{b}_t^*(r) = \underset{\mathbf{b} \in \Delta_m}{\arg\max} \prod_{\tau=t'}^{t} \mathbf{b}^\top \mathbf{x}_\tau, t' = \max\{t-r+1,\ 1\}. \tag{4}$$

In particular, the portfolio vector on the first period is always set as $\mathbf{b}_1 = (1/m, ..., 1/m)^\top$.

At this point, a new problem once again plagued us, that is, how many periods of historical price relatives should be chosen is appropriate. The idea of solving this problem is clear, that is, to use a certain method to determine the length of historical price relatives at the beginning of each period. In fact, what we can do is only to estimate the range of parameter, but it's hard to find an appropriate parameter at the very start. In addition, the value of parameter should not always be imprisoned by a fixed constant on account of non-stationary financial market. To solve this problem, we use adaptive learning method to determine the value of parameter $r$ in response to market fluctuation. Adaptive learning method is that consider the strategies that always adopt BCRP corresponding to the moving window of different fixed lengths as different experts and choose the length of moving window the same as the expert achieving maximum current cumulative return at the beginning of each period. More precisely, we firstly choose a finite number of different values of parameter $r$. If the values of parameter $r$ are different, equation (4) represents different strategies which are denoted as $\mathbf{b}_{1:n}(r)$, and we consider them as the experts, which are composed of a learning set of $\Theta \in \mathbf{Z}_{+}$. After that, we select the parameter $r$ from $\Theta$ which gains maximum wealth by the historical price relatives as the parameter value in the next period, and then compute the portfolio.

At the beginning of the $(t+1)^{\text{th}}$ period, we firstly calculate the cumulative returns of all experts $S_t(\mathbf{b}_{1:t}(r))$ up to the $t^{\text{th}}$ period, and then select the parameter $r_{t+1}$ corresponding to the expert with the maximum of cumulative return, that is,

$$r_{t+1} = r_t^* = \underset{r\in\ \Theta}{\arg\max}\, S_t(\mathbf{b}_{1:t}(r)). \tag{5}$$

We calculate BCRP corresponding to the recent $r_{t+1}$ price relatives, i.e.,

$$\mathbf{b}_t^*(r_{t+1}) = \underset{\mathbf{b}\in\Delta_m}{\arg\max} \prod_{\tau=t'}^{t} \mathbf{b}^\top \mathbf{x}_\tau, t' = \max\{t - r_{t+1} + 1,\ 1\}. \tag{6}$$

Some abnormal data may bring about substantial change, so we consider smoothing and it may be achieved by making linear combination between the last portfolio and the new one. Concretely, at the beginning of the $(t+1)^{\text{th}}$ period, we take the portfolio as a linear combination between the last portfolio $\mathbf{b}_t$ and the portfolio $\mathbf{b}_t^*(r_{t+1})$ in equation (6)

$$\mathbf{b}_{t+1} = \gamma \mathbf{b}_t + (1-\gamma)\mathbf{b}_t^*(r_{t+1}), \quad t = 1, \ldots, n-1, \tag{7}$$

where $\gamma \in [0,1]$ is the weighted average weight. We refer this online portfolio strategy as weighted adaptive successive constant rebalanced portfolio (WASCRP). Its pseudocode is presented in Algorithm 2.

**Algorithm 2** Weighted adaptive successive constant rebalanced portfolio: WASCRP

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**Input:** $\mathbf{x}_{1:n}$ : Price relative sequence; $\Theta$ : Learning set of moving window; $\gamma$ : Weighted average weight.

**Output:** $S_n$ : Final cumulative wealth.

1: Initialise $\mathbf{b}_1 = (1/m, ..., 1/m)^\top, S_0 = 1$.

2: **for** $t = 1, ..., n$ **do**

3: Receive price relative vector: $\mathbf{x}_t = (x_{t,1}, \ldots, x_{t,m})^\top$.

4: Calculate the cumulative return: $S_t = S_{t-1} \times (\mathbf{b}_t^\top \mathbf{x}_t)$.

5: Select the value of parameter $r$: $r_{t+1} = \arg\max_{r \in \Theta} S_t(\mathbf{b}_{1:t}(r))$.

6: Calculate BCRP corresponding to the recent $r_{t+1}$ price relatives: $\mathbf{b}_t^*(r_{t+1}) = \arg\max_{\mathbf{b} \in \Delta_m} \prod_{\tau=t'}^{t} \mathbf{b}^\top \mathbf{x}_\tau, t' = \max\{t - r_{t+1} + 1,\ 1\}$.

7: Update the portfolio for the $(t+1)^{\text{th}}$ period: $\mathbf{b}_{t+1} = \gamma \mathbf{b}_t + (1-\gamma)\mathbf{b}_t^*(r_{t+1})$.

8: **end for**

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### 4.2 MWASCRP strategy

While the moving window length $r$ becomes flexible and optional, there still exists some problems. It is a pity that WASCRP mentioned in the above section seems to use all of historical price relatives to determine the value of parameter $r$ at the beginning of each period, which may affect the performance of WASCRP to some extent. To solve this problem, we should adopt a learning time window $w$ for the experts to learn, where parameter $w$ represents the length of the historical price relatives. It can be better for choosing the parameter $r$ that is the most suitable in the recent periods.

At the beginning of the $(t+1)^{\text{th}}$ period , we firstly calculate the cumulative returns of all experts $S_t(\mathbf{b}_{t'':t}(r))$ from the $t''^{\text{th}}$ period to the $t^{\text{th}}$ period ($t'' = \max\{1, t - w + 1\}$), and then select the parameter $r_{t+1}(w)$ corresponding to the expert with the maximum cumulative return during the recent $w$ periods, that is,

$$r_{t+1}(w) = \arg\max_{r \in \Theta} S_{t'':t}(\mathbf{b}_{1:t}(r)), t'' = \max\{1, t - w + 1\}. \tag{8}$$

We calculate BCRP corresponding to the recent $r_{t+1}(w)$ price relatives, i.e.,

$$\mathbf{b}_t^*(r_{t+1}(w)) = \arg\max_{\mathbf{b} \in \Delta_m} \prod_{\tau=t'}^{t} \mathbf{b}^\top \mathbf{x}_\tau, t' = \max\{t - r_{t+1}(w) + 1,\ 1\}. \tag{9}$$

At the beginning of the $(t+1)^{\text{th}}$ trading period, we take the portfolio as a linear combination between the last portfolio $\mathbf{b}_t$ and the portfolio $\mathbf{b}_t^*(r_{t+1}(w))$ in equation (9)

$$\mathbf{b}_{t+1} = \gamma \mathbf{b}_t + (1-\gamma)\mathbf{b}_t^*(r_{t+1}(w)), \quad t = 1, \ldots, n-1, \tag{10}$$

where $\gamma \in [0, 1]$ is the weighted average weight. We refer this online portfolio strategy as moving window-based weighted adaptive successive constant rebalanced portfolio (MWASCRP). Its pseudocode is presented in Algorithm 3.

**Algorithm 3** Moving window-based WASCRP: MWASCRP

**Input:** $\mathbf{x}_{1:n}$ : Price relative sequence; $\Theta$ : Learning set of moving window; $w$ : The learning time window; $\gamma$ : Weighted average weight.
**Output:** $S_n$ : Final cumulative wealth.

Initialise $\mathbf{b}_1 = (1/m, ..., 1/m)^\top, S_0 = 1$;
**for** $t = 1, ..., n$ **do**
Receive price relative vector: $\mathbf{x}_t = (x_{t,1}, \ldots, x_{t,m})^\top$.
Calculate the cumulative return: $S_t = S_{t-1} \times (\mathbf{b}_t^\top \mathbf{x}_t)$.
Select the value of parameter $r$: $r_{t+1}(w) = \arg\max_{r \in \Theta} S_{t'':t}(\mathbf{b}_{1:t,r})$, $t'' = \max\{1, t - w + 1\}$.
Calculate BCRP corresponding to the recent $r_{t+1}(w)$ price relatives: $\mathbf{b}_t^*(r_{t+1}(w)) = \arg\max_{\mathbf{b} \in \Delta_m} \prod_{\tau=t'}^{t} \mathbf{b}^\top \mathbf{x}_\tau, t' = \max\{t - r_{t+1}(w) + 1,\ 1\}$.
Update the portfolio for the $(t+1)^{\text{th}}$ period: $\mathbf{b}_{t+1} = \gamma \mathbf{b}_t + (1 - \gamma)\mathbf{b}_t^*(r_{t+1}(w))$.
**end for**

## 5 Numerical experiments

Now we examine the efficacy of our proposed strategies by performing extensive numerical experiments, including the setup of our stock combinations, comparison strategies, results analysis and sensitivity analysis. The diverse data for numerical experiments is obtained from the US and Chinese markets.

### *5.1 Stock combinations*

At first, we introduce two datasets we employ in the following experiments, i.e., NYSE and S50I, which were collected from major financial markets. The dataset NYSE is a standard dataset pioneered by Cover (1991) and has been used by many researchers (Gaivoronski and Stella, 2000; Borodin et al., 2004; Li et al., 2012, 2015). It contains 5,651 daily price relatives in New York Stock Exchange for a 22-year period from 3rd July 1962 to 31 December 1984. The dataset of S50I was collected by us from shares of the Shanghai 50 index, which consists of 3,034 daily price relatives in the scope of 2 January 2004 to 30 June 2016. S50I is based on a scientific and objective way and our selection of the Shanghai stock market is in accordance with a large scale, good liquidity of most representative samples. Furthermore, we also choose the time span that covers both the Chinese bull market and bear market. We have the following data processing:

1 Select sequences of price relatives from 2 January 2004 to 30 June 2016, a total of 3,034 periods as the sample.

2 After removing those suspended and unlisted stock data, offer backward adjusted prices to the rest stocks, and the suspended and unlisted stock data is given a value of 1.

The brief information of these two datasets is listed in Table 1.

We construct six combinations of stocks from the datasets of NYSE and S50I, where two contain three stocks, two contain four stocks, and the rest two contain five stocks. The brief names of stock combinations are listed in Table 2.

**Table 1** Information of datasets

| *Dataset* | *Market* | *Region* | *Time frame* | *Periods* |
|---|---|---|---|---|
| NYSE | Stock | US | 3 July 1962–31 December 1984 | 5,651 |
| S50I | Stock | CHN | 2 January 2004–30 June 2016 | 3,034 |

**Table 2** Names of stock combinations

| *Names of combinations* | *Names of stocks* |
|---|---|
| Comb. 1 | IBM-Schlum-Dow |
| Comb. 2 | Gulf-HP-Morris-Schlum |
| Comb. 3 | Arco-Gulf-Jnj-Mei-Schlum |
| Comb. 4 | 600000-600016-600196 |
| Comb. 5 | 600016-600030-600406-600519 |
| Comb. 6 | 600028-600030-600518-600837-600887 |

## *5.2 Comparison strategies*

We conduct experiments to compare the performances of two proposed strategies, WASCRP and MWASCRP, with some benchmarks and existing online strategies. The parameters of existing strategies are set according to their original studies as listed below.

1. Market: Uniform BAH.
2. Best: The best stock in the dataset which gains the maximum cumulative wealth.
3. BCRP: Best CRP in hindsight.
4. UP: Cover's UP implemented according to Kalai and Vempala (2003), where the parameters are set as $\delta_0 = 0.004$, $\delta = 0.005$, $m = 100$ and $S = 500$.
5. EG: Exponential gradient algorithm with the best learning rate $\eta = 0.05$ according to Helmbold et al. (1998).
6. WAAC: Online portfolio selection strategy based on combining experts' advice by Zhang and Yang (2017).
7. PAMR: Passive aggressive mean reversion algorithm with $\epsilon = 0.5$ as suggested by Li et al. (2012).
8. OLMAR: Online moving average reversion with $\alpha = 0.5$ and $\epsilon = 10$ by Li et al. (2015).
9. SCRP: Successive CRP by Gaivoronski and Stella (2000).
10. SVCRP: Successive variable CRP by Gaivoronski and Stella (2000), where the parameter is set as $r = 300$.

11 WSCRP: Weighted successive CRP by Gaivoronski and Stella (2000), where the parameter is set as $\gamma = 0.95$.

There are some parameters need to be set for our strategies, such as the learning set of moving window lengths $\Theta$, the learning time window $w$, the weighted average weight $\gamma$ and so on. For these two strategies, we set $\Theta = \{250, 300, 350\}$, $\gamma = 0.95$ and $w = 300$ in all experiments except parameter sensitivity analysis.

### *5.3 Result analysis*

We measure the performances of online strategies by final cumulative wealth. We show the final cumulative wealths of strategies mentioned above on the six combinations in Table 3. In Table 3, M/B represents the ratio of final cumulative wealth of MWASCRP to that of BCRP; while, M/W represents the ratio of final cumulative wealth of MWASCRP to that of WASCRP. As can be seen, the performances of WASCRP and MWASCRP exceed all other online strategies on almost six combinations except Comb. 3. Although the final cumulative wealth of WASCRP is a little bit less than that of SVCRP, MWASCRP still achieves more final cumulative wealth than SVCRP on Comb. 3. The reason for encouraging results is that on the one hand, using adaptive learning method to determine the moving window and update portfolio in each period can effectively improve the performances of proposed strategies; on the other hand, smoothing method can effectively avoid the substantial change caused by abnormal data. We can see that the values of M/B are more than 1 on the six combinations, of which the maximum is 3.66. This indicates that MWASCRP performs better than the benchmark strategy BCRP constructed with the perfect knowledge of the future. In addition, the values of M/W are also more than 1, which further shows that the use of recent historical data improves the performances of online strategies.

Next, we illustrate the effect of adaptive learning on the performances of WASCRP and MWASCRP. We list the final cumulative wealths of SVCRP with $r = 250$, 300, 350 and their average on the six combinations in Table 4. We also list the final cumulative wealths of WASCRP and MWASCRP on the six combinations in Table 4, where the parameters are set the same as mentioned above, i.e., $\Theta = \{250, 300, 350\}$, $w = 300$ and $\gamma = 0.95$. As can be seen from Table 4, MWASCRP performs better than SVCRP for almost all combinations except Comb. 5. But the final cumulative wealth of MWASCRP is still larger than that of the average of SVCRP on Comb. 5. The above analysis further illustrates that it is effective to make online strategies achieve better performance via adaptive learning.

Then, we compare daily cumulative returns of online strategies on different combinations. We present the daily cumulative returns of BCRP, SCRP, WASCRP and MWASCRP on Comb. 3 and Comb. 6 in Figure 1. At the beginning of the whole investment periods, the daily cumulative returns of these four strategies are not significantly different in the first 1,500 periods. While the advantages of WASCRP and MWASCRP gradually emerge at about the 2,000th period on Comb. 3 and at about the 1,500th period on Comb. 6. Then, the discrepancy becomes larger between the periods of 4,500 and 5,500 on Comb. 3, and between periods of 2,000 and 3,000 on Comb. 6. We find that the daily cumulative returns of WASCRP and MWASCRP steadily exceed that of BCRP and SCRP. Meanwhile, the trends of our strategies are basically consistent and MWASCRP is continuously higher than WASCRP. All above further indicates that adaptive learning greatly enhances the performances of our strategies.

Finally, we judge whether the results are generated by simple luck or not. To this end, we list the $t$-test results of MWASCRP in Table 5. It clearly shows that the observed excess return is impossible to obtain by simple luck on these six combinations. According to $p$-value, the chances of MWASCRP are at most 4.46%, which is achieved on Comb. 5. In a word, the statistical results show that MWASCRP is a promising and reliable portfolio selection strategy to achieve high return with high confidence.

**Table 3** Comparison of the final cumulative wealths of different strategies on the six combinations

| *Strategy* | *Comb. 1* | *Comb. 2* | *Comb. 3* | *Comb. 4* | *Comb. 5* | *Comb. 6* |
|---|---|---|---|---|---|---|
| Market | 24.30 | 40.13 | 23.06 | 15.58 | 7.82 | 6.54 |
| Best | 43.13 | 54.14 | 43.13 | 25.15 | 13.83 | 10.60 |
| BCRP | 74.08 | 69.94 | 58.10 | 30.61 | 18.27 | 15.06 |
| UP | 38.17 | 60.03 | 29.46 | 21.45 | 10.18 | 10.24 |
| EG | 42.64 | 65.68 | 30.38 | 24.23 | 10.70 | 11.38 |
| WAAC | 43.37 | 58.16 | 26.71 | 24.71 | 10.86 | 11.74 |
| PAMR | 32.84 | 7.10 | 6.90 | 0.20 | 0.25 | 0.11 |
| OLMAR | 2.69 | 37.95 | 69.21 | 0.53 | 0.22 | 0.20 |
| SCRP | 26.77 | 17.79 | 18.22 | 13.30 | 7.05 | 3.30 |
| WSCRP | 29.51 | 19.56 | 18.72 | 13.24 | 5.90 | 3.44 |
| SVCRP | 45.87 | 69.21 | *136.62* | 37.17 | 14.01 | 25.10 |
| WASCRP | *134.98* | *118.13* | 103.74 | *73.06* | *21.37* | *34.21* |
| MWASCRP | *181.06* | *163.99* | *144.97* | *89.69* | *23.41* | *55.06* |
| M/B | 2.44 | 2.34 | 2.50 | 2.93 | 1.28 | 3.66 |
| M/W | 1.34 | 1.39 | 1.40 | 1.23 | 1.10 | 1.61 |

Notes: The top two final cumulative wealths in each column, except that results of strategies in hindsight, are highlighted in ital.

**Table 4** Comparison of the final cumulative wealths of SVCRP, WASCRP and MWASCRP on the six combinations ($\Theta = \{250, 300, 350\}$)

| *Strategy* | *Parameter* | *Comb. 1* | *Comb. 2* | *Comb. 3* | *Comb. 4* | *Comb. 5* | *Comb. 6* |
|---|---|---|---|---|---|---|---|
| SVCRP | 250 | 53.41 | 53.87 | 34.60 | 67.90 | 29.56 | 21.84 |
| | 300 | 45.87 | 69.21 | 136.62 | 37.17 | 14.01 | 25.10 |
| | 350 | 89.98 | 116.58 | 95.15 | 50.10 | 22.27 | 33.39 |
| | Average | 63.09 | 79.89 | 88.79 | 51.72 | 21.94 | 20.69 |
| WASCRP | $\Theta$ | 134.98 | 118.13 | 103.74 | 73.06 | 21.37 | 34.21 |
| MWASCRP | $\Theta$ | 181.06 | 163.99 | 144.97 | 89.69 | 23.41 | 55.06 |

## *5.4 Sensitivity analysis*

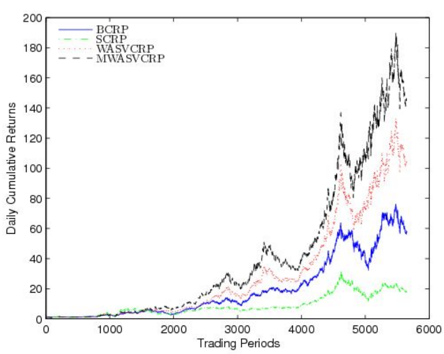
### *5.4.1 Final cumulative wealths with varying values of learning time window $w$*

In the strategy MWASCRP, the parameter $w$ is always set as 300 in the above experiments. Now we analyse the sensitivity of its performance. We show the final cumulative wealths of MWASCRP with $w = \{200, 225, ..., 400\}$ on Comb. 3 and Comb. 6 in Figure 2. To facilitate the comparison, we also list the final cumulative wealths of BCRP, SCRP and WASCRP. From Figure 2, it is easy to see that the performance of MWASCRP is better than that of SCRP, while it also exceeds BCRP and WASCRP in most cases. It shows that the performance of MWASCRP can be greatly improved with an appropriate parameter $w$, such as $w = 300$. Furthermore, it indicates that the stock price data long ago interferes with the current investment decision.

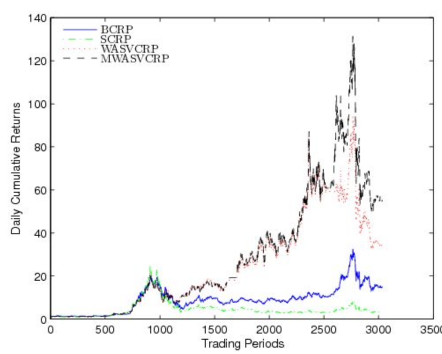
**Table 5** Statistical $t$-test of MWASCRP

| *Statistics* | *Comb. 1* | *Comb. 2* | *Comb. 3* | *Comb. 4* | *Comb. 5* | *Comb. 6* |
|---|---|---|---|---|---|---|
| Size | 5,651 | 5,651 | 5,651 | 3,034 | 3,034 | 3,034 |
| MER (strategy) | 0.0011 | 0.0011 | 0.0011 | 0.0019 | 0.0014 | 0.0017 |
| MER (market) | 0.0007 | 0.0008 | 0.0006 | 0.0012 | 0.0009 | 0.0009 |
| $\alpha$ | 0.0005 | 0.0003 | 0.0004 | 0.0007 | 0.0005 | 0.0009 |
| $\beta$ | 0.8203 | 1.0173 | 1.1126 | 1.0316 | 1.0193 | 0.9915 |
| $t$-statistics | 2.6847 | 2.1120 | 2.3132 | 2.5259 | 1.7000 | 2.6492 |
| $p$-value | 0.0036 | 0.0174 | 0.0104 | 0.0058 | 0.0446 | 0.0041 |

**Figure 1** Daily cumulative returns of BCRP, SCRP, WASCRP and MWASCRP on, (a) Comb. 3 (b) Comb. 6 (see online version for colours)




(a)

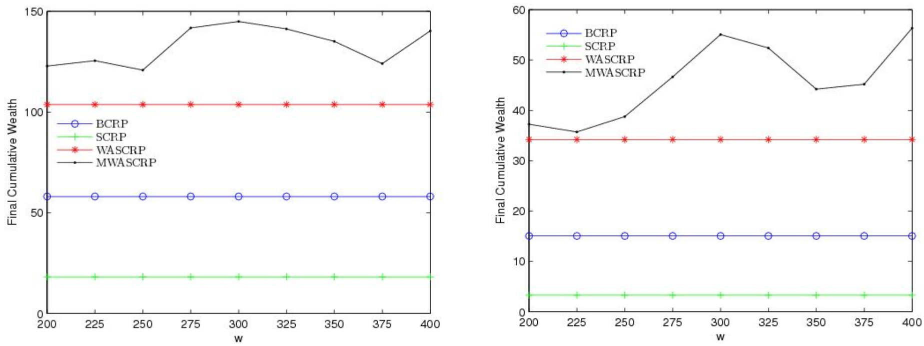



(b)

### 5.4.2 *Final cumulative wealths with varying weighted average weight $\gamma$*

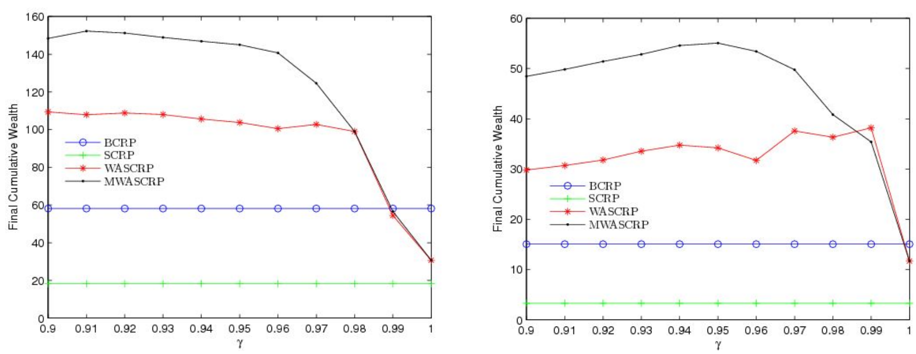
Our proposed strategies WASCRP and MWASCRP avoid substantial adjustments through weighted average weight $\gamma$. We present the final cumulative wealths of WASCRP and MWASCRP with $\gamma = \{0.90, 0.91, ..., 1\}$ on Comb. 3 and Comb. 6 in Figure 3. For the convenience of comparison, we also list the final cumulative wealths of BCRP and SCRP. From Figure 3, the final cumulative wealths of WASCRP and MWASCRP are stable and none of the values are absolutely optimal on Comb. 3 and Comb. 6 when $\gamma$ varies between 0.90 and 0.97. The cumulative wealths suddenly drop with a large amount when $\gamma$ changes from 0.97 to 1. When $\gamma$=1, the performances of WASCRP and MWASCRP are not well, because the portfolio updates $\mathbf{b}_{t+1} = \mathbf{b}_t$ which is initialised to uniform portfolio. In addition, MWASCRP can be more stable than WASCRP in most cases. In general, the final cumulative wealths are not very sensitive to the parameter $\gamma$, so we can easily choose an appropriate value for it.

**Figure 2** Final cumulative wealths with varying values of learning time window $w$, (a) Comb. 3 (b) Comb. 6 (see online version for colours)




(a) (b)

**Figure 3** Final cumulative wealths with varying weighted average weight $\gamma$, (a) Comb. 3 (b) Comb. 6 (see online version for colours)

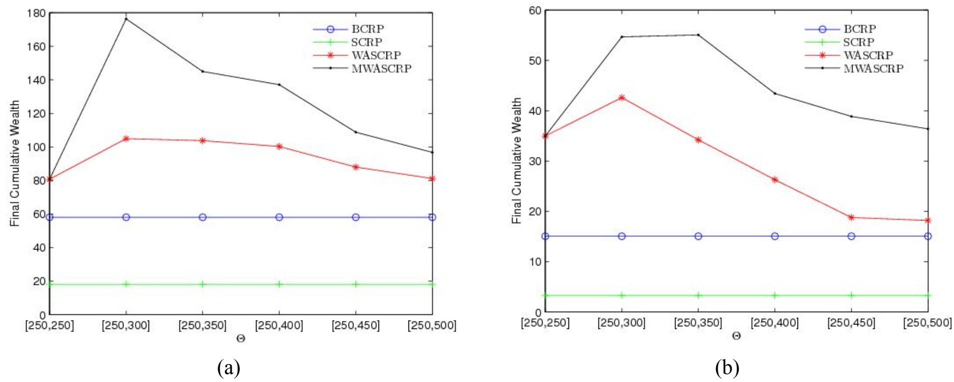



(a) (b)

### *5.4.3 Final cumulative wealths with varying learning sets of moving window lengths* $\Theta$

The WASCRP and MWASCRP can adaptively choose parameters from the learning set of moving window lengths $\Theta$. We present the final cumulative wealths of WASCRP and MWASCRP with $\Theta = \{[250, 250 + 50j]; j = 0, 1, ..., 5\}$ on Comb. 3 and Comb. 6 in Figure 4. For the convenience of comparison, we also list the final cumulative wealths of BCRP and SCRP. It is easy to find that the appropriate expansion of learning set can effectively improve the performances of WASCRP and MWASCRP on Comb. 3 and Comb. 6. The final cumulative wealths of MWASCRP and WASCRP almost reach their maximum with $\Theta = \{250, 300\}$, and then they start declining on Comb. 3 and Comb. 6. However, the final cumulative wealths of them are still higher than that of BCRP and SCRP. All above indicates that adaptive learning further enhances the performance of the strategy. In addition, MWASCRP surpasses WASCRP in almost all the learning sets, which also indicates that the stock price data long ago interferes with the current investment decision.

**Figure 4** Final cumulative wealths with varying learning sets of moving window lengths Θ, (a) Comb. 3 (b) Comb. 6 (see online version for colours)




## 6 Conclusions

In this paper, according to the objective phenomenon that the financial market is non-stationary and earlier observations may have little or no relevance to the current investment decision making, we present two kinds of novel adaptive online portfolio selection strategies based on moving window. The proposed strategies by using adaptive learning and smoothing methods are able to overcome the limitations of the existing online strategies mainly due to fixed value of moving window and vast adjustments of portfolio. The experiment results tell us that the proposed strategies perform better than the existing strategies in different markets. It indicates that, on the one hand, adaptive learning can improve the performances of online strategies, and on the other hand, the smoothing method effectively avoids vast adjustments of portfolio. Further, we adopt a learning time window for adaptive learning. The experiment results show that the performance of MWASCRP is better than that of WASCRP in most cases. It illustrates that the historical data long ago is not conducive to, even interferes with, the current investment decision.

Although achieving encouraging empirical results, there are some limitations of our proposed strategies. Firstly, WASCRP and MWASCRP run longer than some existing online portfolio strategies, such as UP, EG and OLMAR, because they need to calculate BCRP to construct the portfolio in each period. In addition, our strategies need adaptive learning, but it is a pity that both of them have a little poor computational efficiency. Nevertheless, this problem does not affect the practical application owing to advanced computer technology nowadays. Finally, we only show that our strategies achieve attractive final cumulative wealths by numerical experiments, but we have not proved that they are universal. It will be interesting to explore theoretical analysis and investigate if it is possible to bound the worst-case performance of proposed strategies.

## Acknowledgements

This work was supported by the National Natural Science Foundation of China (No. 71301029), the Training Plan for the Excellent Youth Teachers in Higher Education of Guangdong Province (No. Yq2013060), Guangdong Province Universities and Colleges Pearl River Scholar Funded Scheme (2016), Philosophy and Social Science Project of Guangdong Province (17ZS0103).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