Ride, eigenvalue and stability analysis of three-wheel vehicle using Lagrangian dynamics

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Abstract: This research paper studies the ride behaviour of coupled vertical-lateral 9 degree of freedom model of a three wheel vehicle formulated using Lagrangian dynamics. The mathematical model is validated by simulating PSD vertical and lateral acceleration of three-wheel vehicle sprung mass and comparing the same obtained from actual testing results. The natural frequencies, i.e., eigenvalues of main rigid bodies, i.e., sprung mass, front wheel steering arm and rear unsprung mass of the system are determined through MATLAB simulations. Critical speed of the vehicle is also determined in order to investigate the dynamic stability, and parameters which influence the critical speed are analysed.

Keywords: three-wheel vehicle; eigenvalue; stability; critical speed; Lagrangian dynamics.


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1 Introduction

The continuously increasing fuel costs have created a huge pressure on developing more fuel-efficient passenger vehicles. A light four-wheel passenger vehicle is designed for four to seven passengers along with their luggage. This requirement can only be met with sufficient fuel efficiency for these vehicles. Two-wheel vehicles, i.e., motorcycles, scooters, etc. are sufficiently fuel-efficient but only for a single person with limited luggage space. Moreover in extreme climates, these two-wheel vehicles are not convenient to drive. Considering these demands a three-wheel vehicle in India is a unique solution to these requirements and has becomes the popular means of public transport.
The three-wheel vehicle has one wheel in front with steering similar to those of motorcycles and scooters and the two wheels in rear which serve as driving wheels along with a differential and a suspension similar to automobiles.

Three-wheel vehicles are classified as non-guided three-track ground vehicles and termed as following type: auto-rickshaws, pedal rickshaws, delivery vans, pick-up vans, auto trailers, Dakota sun the next generation of solar powered racing vehicles, three-wheel mobile robots, i.e., automated guided vehicles (AGV) and three-wheeled road rollers, etc. The dynamic behaviour of three-wheel vehicle is characterised with longitudinal, i.e., acceleration and braking modes, ride, i.e., bounce and pitch modes and handling, i.e., lateral, yaw and roll modes. For a three-wheel vehicle lateral, yaw, roll and steering motions govern the handling along with safety and bounce and pitch motions govern the vehicle ride.

Ride quality is usually interpreted as the capability of the vehicle suspension to maintain the motion within the range of human comfort and/or within the range necessary to ensure that there is no lading damage. The ride quality of a vehicle depends on displacement, acceleration, rate of change of acceleration and other factors such as noise, dust humidity and temperature, in the environment.

Passenger comfort related to vehicle journey has many aspects. It may be experienced as uncomfortable, in particular if the passenger has to carry luggage or if the passenger is old or disabled. Other important aspects of passenger comfort include seating comfort, personal space, temperature, ventilation, etc.

Another important aspect of ride comfort is structure borne and air borne noise. Structure borne noise has no sharp boundary to mechanical vibrations, in particular in the frequency range of 20–80 Hz, where vibrations are commonly both sensible and audible.

Finally, motions and vibrations of the vehicle are most important aspects in passenger comfort. Motions and vibrations may be transient or stationary. In road vehicles vibrations are very often more or less transient. This comfort aspect, in this context, is designated as motion related comfort or more commonly: ride comfort.


The lateral dynamics of a road vehicle is concerned with its response to steering commands, road surface irregularities and to environmental inputs affecting the direction of motion of vehicle. The lateral dynamic behaviour and handling characteristics of road vehicles are very important to road safety and accident avoidance capabilities of vehicle rider system. The design of a vehicle suspension is generally a compromise between competing design requirements in order to provide a comfortable ride and good vehicle handling.

In the past analysis of vehicles, lateral and vertical motions are considered as uncoupled. However, there is coupling between these two motions and vertical inputs significantly affect the lateral and yaw responses of the vehicle.

The dynamics of two and four wheel vehicles is well studied in the past. The dynamics of three wheel vehicle is not studied in the same ratio and only a few publications are available that also focus on the stability and handling aspects: Chiang and Lee (1990), Gupta et al. (1995), Lim and Renfroe (1993), Tan and Huston (1984a, 1984b) and Rambabu (1995).

The detailed ride analysis of three-wheel vehicle is carried by Ramji and Goel (2001) and Ramji (2004) in his doctoral thesis. Raman et al. (1995) have analysed lateral dynamics of the three-wheel vehicle and determined the solution for improving the
stability with modified vehicle geometry and suspension properties. Comparisons about lateral stability and handling have also been studied between two and three wheel by Gupta et al. (1995) and between three and four-wheel vehicles by Chiang and Lee (1990) and Chang and Dhing (1994) using theoretical as well as experimental analysis. Goel and Gupta (1995) investigated the lateral dynamics and handling characteristics of Indian three-wheel motor vehicle considering the motion in longitudinal direction as well. Vertical dynamics of three-wheel vehicle is not sufficiently studied in the past as compared to its safety and handling aspects. A chapter was presented on three wheel vehicle dynamics by Korff (1990).

2 Mathematical modelling

The mathematical model of three-wheel vehicle shown in Figure 1 is formulated using Lagrangian dynamics with following main assumptions.

- the vehicle is symmetric along longitudinal plane
- the vehicle is travelling at constant speed such that the longitudinal DoF is not a rigid body motion
- the road surface is considered to be rigid
- all masses are assumed to be rigid
- the contact between tyre and road is intact
- wind drag forces are considered and are assumed to be steady.

**Figure 1** Three-wheel vehicle model
A coupled vertical-lateral three wheel vehicle model (Figure 1) has been formulated using Lagrangian dynamics with 9 DoF considered, i.e., 5 DoF of sprung mass (vertical lateral, roll, pitch and yaw), 2 DoF for front wheel steering arm (vertical and wobble motion of steering arm) and 1 DoF each for rear unsprung mass (vertical). The mass of front tyre is included in front steering mass and total rear tyre mass is the same as rear unsprung mass.

3 Ride analysis

The equations of motions for the three-wheel vehicle systems are formulated using Lagrangian dynamics in the following form

\[
[M]\{\ddot{y}_i\} + [C]\{\dot{y}_i\} + [K]\{y_i\} = [F_i(\omega)]
\]

\[(1)\]

\([M]\), \([K]\) and \([C]\) are the mass, stiffness and damping matrices respectively for the vehicle. \([F_i(\omega)]\) is force matrix for displacement excitations at the wheel/road contact points.

Equation (1) may also be written as

\[
([M](-\omega^2)+[C](i\omega)+[K])y_i = e^{i\omega t} = [F_i(\omega)]q_i e^{i\omega t}
\]

\[(2)\]

The above equations may further be written as

\[
[D_1]H_i(\omega) = F_i(\omega)
\]

\[(3)\]

where \([D_1]\) is the dynamic stiffness matrix, and \(H_i(\omega) = (y_i/q_i)\) is the complex frequency response function for \(i^{th}\) input.

For a linear system subjected to random inputs, using input-output relationships for spectral densities, the auto and cross-spectral density matrix of the response may be written as

\[
[S_{yy}(\omega)] = [H_i(\omega)][S_i(\omega)]H_i(\omega)^T
\]

\[(4)\]

The complex frequency response functions \([H_i(\omega)]\) can also be defined as the ratio of the response rate to unit harmonic input at a given point. The superscript \(T\) denotes transpose of matrix.

It may be noted here that above equation is used independently for vertical and lateral irregularities of the track. The mean square acceleration response (MSAR) at the car body mass centre expressed as \((\text{m/sec}^2)^2/\text{Hz}\), which is nothing but PSD of acceleration may be written as

\[
MSAR = (2\pi)^4[H_i(\omega)][S_i(\omega)]H_i(\omega)^T
\]

\[(5)\]

Root mean square acceleration response (RMSAR) at a centre frequency \(f_c\), power spectral density function is integrated over a one third-octave band is expressed as

\[
RMSAR = sqrt\left( \int_{0.89f_c}^{1.12f_c} [S_{yy}(f)](f)^4 df \right)
\]

\[(6)\]
The root mean square acceleration values of sprung mass centre at a series of centre frequencies within the range of interest are obtained independently in vertical and lateral directions and ride comfort is evaluated and compared with the specified ISO 2631-1 (1997) standards. In the present analysis principal frequency weightings with multiplication factors specified in ISO 2631-1 (1997) standards are applied to RMS acceleration values to obtain frequency-weighted acceleration for evaluation of passenger comfort in the sitting position.

4 Representation of road surface roughness

The irregularities on road surface are considered random and represented by power spectral density functions. For the straight road track auto and cross-power spectral density functions of vertical and lateral irregularities are represented by following expression.

\[ S(\Omega) = C_{\Omega}\Omega^{-N} \]  

(7)

\( C_{\Omega} \) is an empirical constant and \( N \) characterises the rate at which amplitude decreases with frequency.

Table 1 Parameters value for three-wheel vehicle

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Parameter value</th>
<th>Parameter</th>
<th>Parameter value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_v )</td>
<td>532 kg</td>
<td>( c_{fs} )</td>
<td>3,500 N-s/m</td>
</tr>
<tr>
<td>( m_{ij} )</td>
<td>12.5 kg</td>
<td>( k_{ij} )</td>
<td>238 kN/m</td>
</tr>
<tr>
<td>( m_s )</td>
<td>493 kg</td>
<td>( c_{ij} )</td>
<td>557 N-s/m</td>
</tr>
<tr>
<td>( m_R )</td>
<td>8.5 kg</td>
<td>( k_{rs} )</td>
<td>49,800</td>
</tr>
<tr>
<td>( m_{rt} )</td>
<td>9 kg</td>
<td>( c_{rs} )</td>
<td>2,200 N-s/m</td>
</tr>
<tr>
<td>( I^v_s )</td>
<td>182 kg-m²</td>
<td>( k_s )</td>
<td>250 kN/m</td>
</tr>
<tr>
<td>( I^v )</td>
<td>170 kg-m²</td>
<td>( c_{st} )</td>
<td>436 N-s/m</td>
</tr>
<tr>
<td>( I^t )</td>
<td>163 kg-m²</td>
<td>( x )</td>
<td>2 m</td>
</tr>
<tr>
<td>( I^v_f )</td>
<td>1.9 kg-m²</td>
<td>( y )</td>
<td>1.15 m</td>
</tr>
<tr>
<td>( I^v_{rt} )</td>
<td>1.07 kg-m²</td>
<td>( r_w )</td>
<td>0.205 m</td>
</tr>
<tr>
<td>( I^t_f )</td>
<td>1.3 kg-m²</td>
<td>( \alpha )</td>
<td>20°</td>
</tr>
<tr>
<td>( k_{rt} )</td>
<td>32,700 N/m</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5 Ride comfort

The three-wheel vehicle considered for the present analysis is Bajaj independent suspension three-wheel vehicle. The values of the parameters of the same are obtained from Bajaj Tempo Ltd, Pune (India) and are listed from Table 1. The weighted RMS accelerations of sprung mass of three-wheel vehicle obtained by mathematical modelling
in vertical and lateral directions are shown in Figures 2 and 3 respectively. The three-wheel vehicle is considered to be moving at a constant speed of 40 km/hr on a straight track. The overall weighted vertical RMS acceleration and lateral RMS acceleration of the three-wheel vehicle for the frequency range from 1 to 80 Hz obtained from the present analysis are 2.63 m/s² and 0.81 m/s² respectively. The result of weighted vertical RMS acceleration response (Figure 2) indicates that the response lies well within the one hour ISO-2631 comfort boundary except for frequency range from 2.5 to 4.1 Hz and 5.5 to 6.1 Hz. The result of weighted lateral RMS acceleration response (Figure 3) indicates that the response lies well within the four-hour ISO comfort boundary except for frequency from 2.8 to 3.1 Hz. For 2.5 hrs and 1 hr comfort lateral ride lies in comfort range.

**Figure 2**  Weighted vertical RMS acceleration of three-wheel vehicle (see online version for colours)

![Weighted vertical RMS acceleration of three-wheel vehicle](image)

**Figure 3**  Weighted lateral RMS acceleration of three-wheel vehicle (see online version for colours)

![Weighted lateral RMS acceleration of three-wheel vehicle](image)
6 Validation of mathematical model

The mathematical model is validated by simulating PSD vertical and lateral acceleration of three-wheel vehicle sprung mass and comparing the same obtained from actual testing results. PSD of vertical and lateral acceleration obtained from simulations are shown in Figures 4 and 5, respectively. The acceleration records of actual testing are obtained in time domain which is converted into frequency domain with fast Fourier transformation (FFT). The data acquisition is completed in 1 km straight specimen run. The acceleration is recorded with the accelerometer which is a strain gauge transducer, which measures low frequency (up to 240 Hz) vibrations of low magnitude (up to 50 m/s²). The accelerometer is kept at the middle of back seat of the vehicle. Accelerometer has variable sensitivity which is function of amplifier gain of dynamic strain amplifier. An integral signal conditioner conditions the signal of the sensing element to a user-friendly voltage. In this testing was used. The transducer is set for the rated output of 553 µV/g, sensitivity of 1.43 V/g, strain of 110.6 µ and safe excitation of 5 V. PSD of vertical and lateral acceleration of three wheel vehicle obtained from actual testing are shown in Figures 6 and 7, respectively. The results from simulations and results from actual testing are in good agreement which validate the present mathematical model. The actual testing and simulated results slightly vary due to following reasons:

- In modelling the acceleration is determined at the c.g. of the sprung mass and in actual testing the acceleration is measured at the middle of back seat of the vehicle where exactly the c.g. of the sprung mass may not be located.

- It is possible that the track where the random inputs are measured and track where acceleration measurements are carried out may not have been exactly same. It is also possible that due to the time gap between these two measurements the track profile may have been changed.

Figure 4  PSD of vertical acceleration of three-wheel vehicle (simulations)
Figure 5  PSD of lateral acceleration of three-wheel vehicle (simulations)

![Graph showing PSD of lateral acceleration of three-wheel vehicle (simulations)](image)

Figure 6  PSD of vertical acceleration of three-wheel vehicle (actual testing)

![Graph showing PSD of vertical acceleration of three-wheel vehicle (actual testing)](image)

Figure 7  PSD of lateral acceleration of three-wheel vehicle (actual testing)

![Graph showing PSD of lateral acceleration of three-wheel vehicle (actual testing)](image)
7 Eigenvalue analysis

The final equations of motion of three-wheel vehicle [equation (1)] may also be expressed in the following form

\[ [M]\{\ddot{y}\} + [C]\{\dot{y}\} + [K]\{y\} = 0 \]  \hspace{1cm} (8)

The linear nine second-order differential equations of motion is reduced to a set of 18 first-order equations in the state space form by the introduction of following variables.

\[
x_1 = \dot{y}_1 = \dot{x}_2 \\
x_2 = y_1 \\
x_3 = \dot{y}_2 = \dot{x}_4 \\
x_4 = y_2 \\
\text{and so on}
\]  \hspace{1cm} (9)

The equations of motions in the state space form are represented as

\[
[A](\dot{X}) = [B](X)
\]  \hspace{1cm} (10)

\[
(\dot{X}) = A^{-1}[B](X)
\]  \hspace{1cm} (11)

where \([A]\) and \([B]\) are square, non-symmetric matrices of order \(18 \times 18\). \(\dot{X}\) is \([X_1, X_2, X_3, \ldots, X_{18}]\) and \(X\) is \([X_1, X_2, X_3, \ldots, X_{18}]\)

<table>
<thead>
<tr>
<th>S.N.</th>
<th>Real part ((\alpha'')) (decay rate in 1/sec)</th>
<th>Imaginary part ((\beta'')) (damped frequency rad/sec)</th>
<th>Damping rate (\zeta = \frac{-\alpha}{\sqrt{\alpha'^2 + \beta'^2}})</th>
<th>Damped natural frequency in Hz</th>
<th>Natural (undamped) frequency in Hz</th>
<th>Vibration mode description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-43.28</td>
<td>63.42</td>
<td>0.563</td>
<td>10.104</td>
<td>12.22</td>
<td>Rear left wheel vertical</td>
</tr>
<tr>
<td>2</td>
<td>-38.63</td>
<td>69.33</td>
<td>0.487</td>
<td>11.037</td>
<td>12.63</td>
<td>Rear right wheel vertical</td>
</tr>
<tr>
<td>3</td>
<td>-33.11</td>
<td>42.51</td>
<td>0.614</td>
<td>6.77</td>
<td>8.579</td>
<td>Front wheel vertical</td>
</tr>
<tr>
<td>4</td>
<td>-26.94</td>
<td>38.56</td>
<td>0.563</td>
<td>6.29</td>
<td>7.62</td>
<td>Steering arm wobble</td>
</tr>
<tr>
<td>5</td>
<td>-11.85</td>
<td>32.63</td>
<td>0.34</td>
<td>5.198</td>
<td>5.527</td>
<td>Sprung mass roll</td>
</tr>
<tr>
<td>6</td>
<td>-8.11</td>
<td>28.47</td>
<td>0.274</td>
<td>4.53</td>
<td>4.71</td>
<td>Sprung mass pitch</td>
</tr>
<tr>
<td>7</td>
<td>-9.62</td>
<td>30.11</td>
<td>0.304</td>
<td>4.79</td>
<td>5.03</td>
<td>Sprung mass yaw</td>
</tr>
<tr>
<td>8</td>
<td>-7.53</td>
<td>20.24</td>
<td>0.348</td>
<td>3.22</td>
<td>3.44</td>
<td>Sprung mass vertical</td>
</tr>
<tr>
<td>9</td>
<td>-3.11</td>
<td>18.56</td>
<td>0.165</td>
<td>2.95</td>
<td>2.99</td>
<td>Sprung mass lateral</td>
</tr>
</tbody>
</table>
The eigenvalues of the matrix \([A][B]\) provide an insight into system’s relative stability, damping ratios and natural frequencies. The three-wheel vehicle is considered to be moving at a constant speed of 40 km/hr over a straight track. The 9 DoF vehicle model will yield as many natural frequencies and modes of vibration. However, due to the effects of coupling in a multi degree of freedom system where one vibration is likely to influence others, the shape of the response curves may be remarkably changed. It is possible that some peaks are attenuated, or shifted to a different frequency or may even sometimes disappear. The general motion would be superposition of all these modes. Since damping is present in the system, some modes are complex and occur in conjugate pairs. The real part of the eigenvalue, which are negative gives the decay rate (related to damping) while the imaginary part gives the damped natural frequency. The eigenvalue for a particular mode was identified on the basis of varying the parameters, which are likely to influence it most, i.e., mass, mass moment of inertia, stiffness, damping coefficient. From the eigenvalue analysis it is observed that the sprung mass damped natural frequencies in vertical, lateral, roll, pitch and yaw mode is 3.22 Hz, 2.95 Hz, 5.198 Hz, 4.53 Hz and 4.79 Hz respectively. The eigenvalues of rigid bodies considered in 9 DoF coupled vertical-lateral three-wheel vehicle system are listed in Table 2.

**Figure 8** Locus of eigenvalue for steering wobble mode
8 Stability analysis

The stability of three-wheel vehicle is investigated using the eigenvalues obtained in the present analysis. The rigid body is found unstable if the real part of the eigenvalue becomes positive. The parameter altered for the analysis of stability is the speed or in other words the critical speed of the three-wheel vehicle is investigated from the eigenvalue analysis. The vehicle is found stable at the speed of 40 km/hr as all the rigid bodies have negative real part in their eigenvalues. On increasing speed instability is first obtained in steering arm wobble mode at the speed of 79 km/hr which can be termed is critical speed. Locus of eigenvalue for steering wobble mode is shown in Figure 8. This instability can be controlled with proper amount of front suspension vertical damping with yaw increased mass moment of inertia. However control in steering arm wobble mode stability demands significant increment in parameter values from present existing and are not recommended from design perspective of three-wheel vehicle. This will also demand in increased power requirement and affect the fuel economy. Therefore it is always recommended to keep the three-wheel vehicle below this critical speed.

9 Conclusions

Present study investigates the ride behaviour of the three-wheel vehicle and it is found that the vehicle vertical ride lies in discomfort zone for frequency range from 2.5 to 4.1 Hz and from 5.5 to 6.1 Hz as per one hour ISO-2631 comfort specifications. The vehicle lateral ride lies in discomfort zone for frequency range from 2.8 to 3.1 Hz for 4 hours ISO comfort specifications. For 2.5 hrs and 1 hr comfort lateral ride lies in comfort range. PSD vertical and lateral acceleration obtained from simulations and the same obtained from actual testing results compare well which validates the mathematical model. The equations of motion are further utilised to determine eigenvalues of main rigid bodies of the system. These eigenvalues are helpful to the vehicle designer as eigenvalues provide an insight into system’s relative stability, damping ratios and natural frequencies. From the present eigenvalue analysis it is observed that the sprung mass damped natural frequencies in vertical, lateral, roll, pitch and yaw mode is 3.22 Hz, 2.95 Hz, 5.198 Hz, 4.53 Hz and 4.79 Hz respectively. From the stability analysis the critical speed observed is 79 km/hr and instability is analysed in steering arm wobble mode. It is not recommended to change the concerned vehicle parameters to improve this critical speed as this would demand increased power requirement and affect the fuel economy.

References


Nomenclature

\[ m_v \] Total mass of three wheel vehicle.

\[ m_s \] Sprung mass of three wheeler.

\[ m_{fs} \] Front steering mass of three wheel vehicle.

\[ m_f \] Mass of front wheel.

\[ m_{rt} \] Mass of single rear wheel.

\[ I_{x,y,z}^{s} \] Roll, pitch and yaw mass moment of inertia of sprung mass about its c.g.

\[ I_{x}^{s} \] Roll, pitch and yaw mass moment of inertia of front steering mass about its c.g.

\[ k_{fs} \] Front suspension vertical stiffness.

\[ c_{fs} \] Front suspension vertical damping coefficient.

\[ k_f \] Front tyre vertical stiffness.

\[ c_f \] Front tyre vertical damping coefficient.

\[ k_{ru} \] Rear suspension vertical stiffness (½ part).

\[ c_{ru} \] Rear suspension vertical damping coeff. (½ part).

\[ k_r \] Rear tyre vertical stiffness.

\[ c_r \] Rear tyre vertical damping coefficient.
Rear tyre vertical damping coefficient.

Wheelbase of three wheel vehicle.

Wheel gauge of three wheel vehicle.

Radius of both front and rear wheel of three wheel vehicle.

Steering axis inclination angle.