Regional boundary controllability of semi-linear parabolic systems with state constraints

Touria Karite

TSI Team, MACS Laboratory,
Department of Mathematics and Informatics,
Institute of Sciences, Moulay Ismail University,
Meknes, Morocco
Email: touria.karite@gmail.com

Ali Boutoulout∗

TSI Team, MACS Laboratory,
Department of Mathematics and Informatics,
Institute of Sciences, Moulay Ismail University,
Meknes, Morocco
Email: boutouloutali@yahoo.fr
∗Corresponding author

Abstract: This work focuses on the controllability of semi-linear parabolic systems with state constraints. Sub-differential techniques are used to compute the control $u$ that steers the system $(S)$ from the initial state $y_0$ to a final one between two prescribed functions, only on a boundary subregion $\Gamma$ of the system evolution domain $\Omega$.

Keywords: boundary; constrained controllability; distributed systems; optimal control; parabolic systems; regional controllability; sub-differential.

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Biographical notes: Touria Karite is a PhD student in the Institute of Sciences, Moulay Ismail University. She got her Msc degree in Theoretical System and Informatics from the Institute of Sciences in Meknes. Her studies are focused on modeling, analysis and control of systems and optimization.

Ali Boutoulout is a full professor in Institute of Sciences, Moulay Ismail University. He obtained his State Doctorate in Regional System Analysis (2000) from the same university. He has published many papers in the area of system analysis and control. He is the head of the research at MACS Laboratory (Modeling, Analysis and Control Systems) and the director of System Theory Master at the Department of Mathematics & Informatics, Institute of Sciences, Meknes.

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1 Introduction

Control theory is one of many other branches of engineering science and applied mathematics. In the classical engineering world, everything from stereos and computers to chemical manufacturing and aircraft utilizes control theory. The development of automatic control is strongly connected to the industrial revolution and the development of modern technology. When new sources of power were discovered the need to control them immediately arose. When new production techniques were developed there were needs to keep them operating smoothly with high quality.

Many scientific and engineering problems can be modeled by partial differential equations, integral equations, or coupled ordinary and partial differential equations that can be described as differential equations in infinite dimensional spaces using semigroups (Pazy, 1983). So, the study of controllability results of such problems in finite or infinite dimensional spaces is important.

There are two main roots of control theory: regulation and trajectory optimization. An ancient real life example that use regulation mechanism was Drebbel’s Athanor (1572–1633). It was designed to combine thermal and mechanical effects in order to keep the temperature of an oven at a constant temperature. Later in the seventeenth century, Christiaan Huygens (1629–1695) invented a flywheel device for speed control of windmills. The idea was also the basis of the centrifugal governor used by James Whatt (1736–1819) (Aströma and Kumar, 2014). This successful device regulated the speed of a steam engine. It was used in all steam engines during the industrial revolution, and it became the first mass-produced control mechanism in existence.

To backtrack and follow the other historical root of control, that is, trajectory optimization, one needs to mention the beautiful brachistochrone problem, posed by Johann Bernoulli in 1696, after the discovery of differential calculus. The idea was to find the path between two given points A and B along which a body fallen just under its own weight moves in the shortest possible time.

The problem of trajectory transfer is the question of determining the paths of a dynamical system that steers the system from a given initial state to a prescribed terminal one. The regional controllability, considered here, is when the final state is in a subregion $\omega$ of the geometric domain $\Omega$. It could be an internal one (El Jai et al., 1995) or a part $\Gamma$ of the boundary $\partial \Omega$ (Zerrik et al., 2000).

Controllability of partial differential equations (PDE) with state constraints has been intensively studied since the eighties, starting with the works of Bonnans and Casas (1984, 1988, 1989) and those of Klamka (1966, 2004) who did add the concept of delay to the problem of constrained controllability. More specifically, problems with mixed control and state constraints, or with ‘pure’ state constraints, for parabolic systems have been studied in Casas (1997) and Mordukhovich (2010) with uncertainty conditions.

Various real problems can be formulated within the concept of constrained controllability, because the mathematical models are obtained from measurements or from approximation techniques and they are very often affected by perturbations. These disturbances can be modeled and represented by some constraints depending on the studied problem. The rest of the paper is organized as follows. In the next section,
we present our problem and we give some definitions and propositions. In Section 3, we
use sub-differential techniques to compute the optimal control steering the system from a
given initial state to the desired one.

2 Preliminaries and problem statement

Let \( \Omega \) be a regular bounded open set of \( \mathbb{R}^n \) \( (n \geq 1) \) with boundary \( \partial \Omega \). For a given \( T > 0 \),
let’s consider \( Q = \Omega \times [0, T] \) and \( \Sigma = \partial \Omega \times [0, T] \) and let us consider a parabolic system
excited by controls which may be applied via various types of actuators given by the
following equation:

\[
\partial_t y(x, t) - A y(x, t) = N y(x, t) + B u(t) \quad \text{in } Q
\]

with the initial condition:

\[
y(x, 0) = y_0(x) \quad \text{in } \Omega
\]

and Neumann boundary condition:

\[
\frac{\partial}{\partial \nu} A y(\xi, t) = 0 \quad \text{on } \Sigma
\]

The second-order operator \( A \) is an infinitesimal generator of a \( C_0 \) semi-group \( \{S(t)\}_{t \geq 0} \) on
\( L^2(\Omega) \) (Engel and Nagel, 2006) and \( N : L^2(0, T; L^2(\Omega)) \rightarrow L^2(\Omega) \) a non linear operator
which satisfies a Lipschitz condition in \( y \) (Pazy, 1983; Zeidler, 1995). \( \frac{\partial}{\partial \nu} y(\xi, t) \) indicates
the conormal derivative on the boundary \( \Sigma \) associated with the operator \( A \) and the unit
outward normal vector \( \nu \). \( B \in L(\mathbb{R}^m, L^2(\Omega)) \), \( y_0 \in L^2(\Omega) \) and \( u \in U = L^2(0, T; \mathbb{R}^m) \)
(where \( m \) is the number of actuators).

We denote by \( y_u(\cdot) \) the solution of (1)-(2)-(3) when it’s excited by a control \( u \), \( y_u(T) \in H^1(\Omega) \) (Lions and Magenes, 1968) and we consider:

- \( \Gamma \) a nonempty subregion of \( \partial \Omega \).
- \( \gamma_0 : H^1(\Omega) \rightarrow H^{1/2}(\partial \Omega) \) the trace operator of order zero which is linear, continuous
  and surjective.
- The restriction operator

\[
\chi_r : H^{1/2}(\partial \Omega) \rightarrow H^{1/2}(\Gamma)
\]

\[
y \mapsto \chi_r y = y|_r.
\]

Let’s consider \( H : U \rightarrow H^1(\Omega) \) defined by:

\[
\forall u \in U, \quad H u = \int_0^T S(T - s) B u(s) ds,
\]
we define also the following operator:

$$G_r : L^2(0, T; H^1(\Omega)) \rightarrow H^1(\Omega)$$

$$y(\cdot) \rightarrow \int_0^T S(T - \tau)N y(\tau) d\tau.$$  \hspace{1cm} (4)

and $\alpha(\cdot), \beta(\cdot)$ be two given real functions in $H^{1/2}(\Gamma)$ such that $\alpha(\cdot) \leq \beta(\cdot)$ on $\Gamma$, and set:

$$[\alpha(\cdot), \beta(\cdot)] = \left\{ y \in H^{1/2}(\Gamma) \mid \alpha(\cdot) \leq y(\cdot) \leq \beta(\cdot) \text{ on } \Gamma \right\}.$$  

**Definition 2.1:** We say that (1)-(2)-(3) is \$[\alpha(\cdot), \beta(\cdot)]\$-controllable on $\Gamma$ if:

$$\exists u \in \mathcal{U} \text{ such that } \alpha(\cdot) \leq \chi_r(\cdot) \leq \beta(\cdot).$$

It is clear that the system (1)-(2)-(3) is $[\alpha(\cdot), \beta(\cdot)]$-controllable on $\Gamma$ if:

$$[\alpha(\cdot), \beta(\cdot)] = \{ \chi_r \gamma_0 S(T) y_0 \} \cap (\text{Im } \chi_r \gamma_0 G_r + \text{Im } \chi_r \gamma_0 H) \neq \emptyset.$$

**Remark 1:**

1. The previous definition shows that we are interested in the transfer of the system (1)-(2)-(3) to a state just between $\alpha(\cdot)$ and $\beta(\cdot)$ on $\Gamma$.
2. If $\alpha = \beta$ we retrieve the concept of regional exact controllability. So, for $\alpha \neq \beta$ the $[\alpha(\cdot), \beta(\cdot)]$-controllability constitutes an extension of regional controllability.
3. The system which is controllable on $\Gamma$ is $[\alpha(\cdot), \beta(\cdot)]$-controllable on $\Gamma$.

A characterization of the $[\alpha(\cdot), \beta(\cdot)]$-controllability on $\Gamma$ is given by the following proposition:

**Proposition 1:** The system (1)-(2)-(3) is $[\alpha(\cdot), \beta(\cdot)]$-controllable on $\Gamma$ if and only if:

$$(\text{Ker } \chi_r + \text{Im } \gamma_0 G_r + \text{Im } \gamma_0 H) \cap [\alpha(\cdot), \beta(\cdot)] \neq \emptyset.$$  

**Proof:** We suppose that the system (1)-(2)-(3) is $[\alpha(\cdot), \beta(\cdot)]$-controllable on $\Gamma$ which is equivalent to say that

$$(\text{Im } \chi_r \gamma_0 G_r + \text{Im } \chi_r \gamma_0 H) \cap [\alpha(\cdot), \beta(\cdot)] \neq \emptyset.$$  

So there exists $z \in [\alpha(\cdot), \beta(\cdot)]$, $y(\cdot) \in L^2(0, T; H^1(\Omega))$ and $u \in \mathcal{U}$ such that

$$\chi_r \gamma_0 G_r y(\cdot) + \chi_r \gamma_0 H u = \chi_r \gamma_0 z,$$

which gives $\chi_r (z - \gamma_0 G_r y(\cdot) - \gamma_0 H u) = 0$. Let's consider

$$z_i = z - \gamma_0 G_r y(\cdot) - \gamma_0 H u,$$

and

$$z_2 = \gamma_0 G_r y(\cdot) \text{ and } z_3 = \gamma_0 H u.$$  

Then: $z = z_1 + z_2 + z_3$, where $z_1 \in \text{Ker } \chi_r$, $z_2 \in \text{Im } \gamma_0 G_r$ and $z_3 \in \text{Im } \gamma_0 H$. This proves that

$$z \in (\text{Ker } \chi_r + \text{Im } \gamma_0 G_r + \text{Im } \gamma_0 H).$$

Thus, we have

$$(\text{Ker } \chi_r + \text{Im } \gamma_0 G_r + \text{Im } \gamma_0 H) \cap [\alpha(\cdot), \beta(\cdot)] \neq \emptyset.$$
Conversely, we suppose that \((\text{Ker } \chi_{\gamma} + \text{Im } \gamma_{\alpha} G_{\tau} + \text{Im } \gamma_{\alpha} H) \cap [\alpha(\cdot), \beta(\cdot)] \neq \emptyset\) which means that there exists \(z \in [\alpha(\cdot), \beta(\cdot)]\) such that \(z \in \text{Ker } \chi_{\gamma} + \text{Im } \gamma_{\alpha} G_{\tau} + \text{Im } \gamma_{\alpha} H\). Then, we verify that \(\chi_{\gamma}(z) = \chi_{\gamma}(z_{1} + z_{2} + z_{3}) = \chi_{\gamma}(z_{1} + z_{2} + z_{3}) = \chi_{\gamma}(\gamma_{\alpha} G_{\tau} y(\cdot) + \chi_{\gamma}(\gamma_{\alpha} H u). It follows that

\[
\chi_{\gamma}(z) = \chi_{\gamma}(z_{1} + z_{2} + z_{3}) = \chi_{\gamma}(\gamma_{\alpha} G_{\tau} y(\cdot) + \chi_{\gamma}(\gamma_{\alpha} H u). \quad \Box
\]

We can also characterize the notion of controllability with output constraints by using the concept of strategic actuators. Hence, we recall that an actuator is conventionally defined by a couple \((D, f)\), where \(D\) is a nonempty closed part of \(\Omega\), and it represent the geometric support of the actuator. And \(f \in L^{2}(D)\) define the spatial distribution of the action on the support \(D\).

**Definition 2.2:** The actuator \((D, f)\) is said to be \([\alpha(\cdot), \beta(\cdot)]\)-strategic on \(\Gamma\) if the excited system is \([\alpha(\cdot), \beta(\cdot)]\)-controllable on \(\Gamma\).

In the case of a pointwise actuator (internal or boundary) \(D = \{b\}\) and \(f = \delta(b - \cdot)\), where \(\delta\) is the Dirac mass concentrated in \(b\), and the actuator is then denoted by \((b, \delta_{b})\). For definitions and properties of strategic actuators we refer to El Jai and Pritchard (1988) and Zerrik et al. (2000).

### 3 Sub-differential approach

This section is consecrated to characterize the optimal control solution of the following problem:

\[
\left\{ \begin{array}{l}
\inf \frac{1}{2} \|u\|^{2} \\
u \in U_{ad},
\end{array}\right.
\]

(5)

where \(U_{ad} = \{u \in U \mid \chi_{\gamma_{m}} y_{m}(T) \in [\alpha(\cdot), \beta(\cdot)]\}\), using an approach based on the sub-differential techniques (Penot, 1978; Kusraev and Kutateladze, 1995; Aubin and Wilson, 2002).

**Proposition 2:** If the system (1)-(2)-(3) is \([\alpha(\cdot), \beta(\cdot)]\)-controllable on \(\Gamma\), then the problem (5) has a unique solution \(u^{*} \in U_{ad}\) characterized by:

\[
\forall u \in U_{ad} \quad \langle u^{*}, u - u \rangle \leq 0.
\]

**Proof:** We have \(U_{ad} \neq \emptyset\) and the mapping \(u \mapsto \frac{1}{2} \|u\|^{2}\) is strictly convex, coercive, proper and lower semi-continuous in \(U\). So we verify that \(U_{ad}\) is a closed convex subset of \(U\). Indeed,

\[
\text{for} \quad (u, v) \in U^{2} \text{ and } t \in [0, 1], \text{ we have } \alpha(\cdot) \leq \chi_{\gamma_{m}} y_{m+(1-t)v}(T) \leq \beta(\cdot).
\]

Thus, \(U_{ad}\) is convex.
In order to prove that it is a closed subset of \( \mathcal{U} \), we consider a sequence \((u_n)\) in \( \mathcal{U}_{ad} \) such that \( u_n \longrightarrow u \) strongly in \( \mathcal{U} \). We have \( \chi_{\gamma_0} \gamma_0 H u_n \longrightarrow \chi_{\gamma_0} \gamma_0 H u \) strongly in \( H^1(\Omega) \), however \( \chi_{\gamma_0} \gamma_0 y_n(T) \in [\alpha(\cdot), \beta(\cdot)] \) which is closed, then \( \chi_{\gamma_0} \gamma_0 y_u(T) \in [\alpha(\cdot), \beta(\cdot)] \). This means that \( u \in \mathcal{U}_{ad} \). Hence \( \mathcal{U}_{ad} \) is closed and finally (5) admits a unique solution. \( \Box \)

**Notations:**

Let us consider the following:

- \( \Gamma_0(U) \) the set of functions \( f : U \rightarrow \tilde{\mathbb{R}} = ]-\infty, +\infty[ \) which are proper, lower semi-continuous (l.s.c) and convex on \( U \).

- For \( f \in \Gamma_0(U) \), the polar function \( f^* \) of \( f \) is given by:
  \[
  \forall u \in U \quad f^*(v^*) = \sup_{u \in \text{dom}(f)} \{ (v^*, u) - f(u) \},
  \]
  where \( \text{dom}(f) = \{ u \in U \mid f(u) < \infty \} \).

- For \( v_0 \in \text{dom}(f) \), the sub-differential of \( f \) at \( v_0 \) is given by the set:
  \[
  \partial f(v_0) = \{ u^* \in U \mid f(u) \geq f(v_0) + \langle u^*, u - v_0 \rangle, \quad \forall u \in U \}.
  \]

- For every \( u \in U \), we denote \( \sigma(u) = \frac{1}{2} \| u \|^2 \), the self-polar function defined in \( U \).

- For \( K \) a nonempty subset of \( U \)
  \[
  \Psi_K(u) = \begin{cases} 0 & \text{if } u \in K \\ +\infty & \text{otherwise,} \end{cases}
  \]
  denotes the indicator functional of \( K \).

With these notations, (5) is equivalent to the problem

\[
\inf_{u \in \mathcal{U}} \left( \sigma(u) + \Psi_{\mathcal{U}_{ad}}(u) \right).
\]  

We set

\[
a(\cdot) = \alpha(\cdot) - \chi_{\gamma_0} \gamma_0 [S(T)y_0 - G_r y],
\]

\[
b(\cdot) = \beta(\cdot) - \chi_{\gamma_0} \gamma_0 [S(T)y_0 - G_r y].
\]

Then \( \mathcal{U}_{ad} = \{ u \in \mathcal{U} \mid \chi_{\gamma_0} \gamma_0 H u \in [a(\cdot), b(\cdot)] \} \).

**Theorem 3:** If the system (1)-(2)-(3) is \([\alpha(\cdot), \beta(\cdot)]\)-controllable on \( \Gamma \) then \( u^* \) is the solution of (6) if and only if

\[
u^* \in \mathcal{U}_{ad} \quad \text{and} \quad \Psi_{\mathcal{U}_{ad}}(-u^*) = -\| u^* \|^2.
\]
Proof: We have $u^*$ is solution of (6) if and only if $0 \in \partial (\sigma + \Psi_{\text{ad}}) (u^*)$, where $\sigma(u^*) = \frac{1}{2} \| u^* \|_2^2$ is the self-polar function and $\mathcal{U}_{\text{ad}} = (\chi_{\text{ad}} \gamma_0 H)\dagger (a(.), b(.))$.

Since $\sigma \in \Gamma_u(U), \Psi_{\text{ad}} \in \Gamma_u(U)$ and $\text{dom} \sigma \cap \text{dom} \Psi_{\text{ad}} \neq \emptyset$ then:

$$\partial (\sigma + \Psi_{\text{ad}})(u^*) = \partial \sigma (u^*) + \partial \Psi_{\text{ad}} (u^*).$$

Then $u^*$ is solution of (6) if and only if $0 \in \left( \partial \sigma (u^*) + \partial \Psi_{\text{ad}} (u^*) \right)$. As $\sigma$ is Fréchet-differentiable, we have $\partial \sigma (u^*) = \{ \nabla \sigma(u^*) \} = \{ u^* \}$. So, $u^*$ is solution of (6) if and only if $-u^* \in \partial \Psi_{\text{ad}} (u^*)$ which is equivalent to

$$\Psi_{\text{ad}}(u^*) + \Psi_{\text{ad}}^\dagger (-u^*) = \langle -u^*, u^* \rangle = -\| u^* \|^2.$$

Thus $u^* \in \mathcal{U}_{\text{ad}}$ and $\Psi_{\text{ad}}^\dagger(-u^*) = -\| u^* \|^2$. □

We have the following characterization:

**Theorem 4:** Assume that the system (1)-(2)-(3) is $[\alpha(.), \beta(\cdot)]$-controllable on $\Gamma$, then $u^*$ is solution of (6) if and only if:

$$\min \{ \langle (\chi_{\text{ad}} \gamma_0 H)^\dagger a(.), u^* \rangle, \langle (\chi_{\text{ad}} \gamma_0 H)^\dagger b(.), u^* \rangle \} = \| u^* \|^2,$$

where $(\chi_{\text{ad}} \gamma_0 H)^\dagger = (\chi_{\text{ad}} \gamma_0 H)^\dagger (\chi_{\text{ad}} \gamma_0 H)^\dagger$ is the pseudo-inverse of $\chi_{\text{ad}} \gamma_0 H$.

Proof: We have $\mathcal{U}_{\text{ad}} = (\chi_{\text{ad}} \gamma_0 H)^\dagger [a(.), b(.)]$ and from Theorem (3), $u^*$ is a solution of (6) if and only if $u^* \in \mathcal{U}_{\text{ad}}$ and $\Psi_{\text{ad}}^\dagger(-u^*) = -\| u^* \|^2$.

Then, for all $v^* \in \mathcal{U}_{\text{ad}}$

$$\Psi_{\text{ad}}^\dagger(v^*) = \sup_{v \in \mathcal{U}_{\text{ad}}} \langle v^*, v \rangle = \sup_{v \in (\chi_{\text{ad}} \gamma_0 H)^\dagger [a(.), b(.)]} \langle v^*, v \rangle = \sup_{y \in [a(.), b(.)]} \langle v^*, (\chi_{\text{ad}} \gamma_0 H)^\dagger y \rangle.$$

The extremal points of $[a(.), b(.)]$ are $a(.)$ and $b(.)$. Moreover, the set $[a(.), b(.)]$ is bounded, closed and convex then weakly compact in the Hilbert space $H^{1/2}(\Gamma)$.

According to the Krein–Milman theorem (Chen and Cho, 2004), we obtain $[a(.), b(.)] = \overline{\text{co}}[a(.), b(.)]$ for the weak topology, where $\overline{\text{co}}[a(.), b(.)]$ is the closure of the convex hull of its extremal points. It follows that:

$$\forall v^* \in \mathcal{U}_{\text{ad}} \quad \Psi_{\text{ad}}^\dagger(v^*) = \sup_{y \in \overline{\text{co}}[a(.), b(.)]} \langle v^*, (\chi_{\text{ad}} \gamma_0 H)^\dagger y \rangle.$$

The mapping $z \mapsto (v^*, (\chi_{\text{ad}} \gamma_0 H)^\dagger z)$ is linear and continuous on $H^{1/2}(\Gamma)$, then weakly continuous, hence

$$\forall v^* \in \mathcal{U}_{\text{ad}} \quad \Psi_{\text{ad}}^\dagger(v^*) = \sup_{y \in [a(.), b(.)]} \langle v^*, (\chi_{\text{ad}} \gamma_0 H)^\dagger y \rangle,$$

which yields to the desired result. □
Remark 2:

1. If \( \alpha(\cdot) = \beta(\cdot) = \{z_d\} \), then we have \( a(\cdot) = b(\cdot) = z_d - \chi_{\gamma_0} S(T)y_0 - \chi_{\gamma_0} G_T y(\cdot) \)
   and \( U_{ad} = (\chi_{\gamma_0} H)^{\dagger} (z_d - \chi_{\gamma_0} S(T)y_0 - \chi_{\gamma_0} G_T y(\cdot)) \). It follows that:
   \( u^*(t) = (\chi_{\gamma_0} H)^{\dagger} (z_d - \chi_{\gamma_0} S(T)y_0 - \chi_{\gamma_0} G_T y(\cdot)) \), which coincides with the results shown in Zerrik et al. (2000).

2. Let \( B(z_d, \epsilon) \) be the ball of center \( z_d \) and radius \( \epsilon \).
   If \( U_{ad} = \{ u \in U \mid \chi_{\gamma_0} y_0(T) \in B(z_d, \epsilon) \} \), the result of Theorem 4 remains true.
   In this case \( u^* \) is solution of the equation:
   \[
   \|u^*\|^2 + \epsilon \|((\chi_{\gamma_0} H)^{\dagger} u^*)\| = \langle u^*, (\chi_{\gamma_0} H)^{\dagger} (z_d - \chi_{\gamma_0} S(T)y_0 - \chi_{\gamma_0} G_T y(\cdot)) \rangle.
   \]
   Indeed, we have \( U_{ad} = (\chi_{\gamma_0} H)^{\dagger} (B(z_d, \epsilon) - \chi_{\gamma_0} S(T)y_0 - \chi_{\gamma_0} G_T y(\cdot)) \).
   Moreover, \( \forall u^* \in U_{ad} \)
   \[
   \Psi_{U_{ad}}(u^*) = \sup_{z \in B(z_d, \epsilon)} \langle u^*, (\chi_{\gamma_0} H)^{\dagger} (z_d - \chi_{\gamma_0} S(T)y_0 - \chi_{\gamma_0} G_T y(\cdot)) \rangle
   \]
   \[
   = \sup_{\omega \in B(0, 1)} \langle u^*, (\chi_{\gamma_0} H)^{\dagger} (z_d - \chi_{\gamma_0} S(T)y_0 - \chi_{\gamma_0} G_T y(\cdot) + \epsilon \omega) \rangle
   \]
   \[
   = \sup_{\omega \in B(0, 1)} \langle u^*, \omega \rangle + \langle u^*, (\chi_{\gamma_0} H)^{\dagger} (z_d - \chi_{\gamma_0} S(T)y_0 - \chi_{\gamma_0} G_T y(\cdot)) \rangle
   \]
   \[
   = \langle u^*, (\chi_{\gamma_0} H)^{\dagger} (z_d - \chi_{\gamma_0} S(T)y_0 - \chi_{\gamma_0} G_T y(\cdot)) \rangle
   \]
   \[
   + \epsilon \|((\chi_{\gamma_0} H)^{\dagger} u^*)\|.
   \]
   And \( \Psi_{U_{ad}}^* = -\|u^*\|^2 \) implies the result.

3. The previous approach gives a characterization of the optimal control by using the sub-differential techniques. Although, this approach is hard to be implemented numerically.

4 Conclusion

We developed an extension of the regional controllability to a situation encountered in many real problems where we must bring the state of a system between two prescribed functions on a part of the boundary. Our results are an extension to the ones in Boutoulout et al. (2015) and Zerrik et al. (2009). They can be extended to fractional order systems. Other questions are still open and they may be the subject of future works, e.g. the problem
of constrained gradient controllability, and the problem of constrained controllability for nonlinear systems.

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