On inventory control with reference prices: a technical note

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Abstract: A retailer sets prices, as well as, simultaneously, selects quantities to order from the supplier. We consider a two-period setting where the second period's demand depends on the first period's ('reference') price, as well as on the second period price. We consider a linear-additive demand function as well as a novel iso-elastic multiplicative model.

Keywords: inventory control; reference price; stochastic demand.

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1 Introduction

As commonly argued in the psychological and marketing literature, we assume that customers react, positively or negatively, to the difference between current and previous ('reference') price (e.g., Briesch et al., 1997; Kalyanaram and Winer, 1995; for OM models incorporating reference prices, see Popescu and Wu, 2007; Kocabiyikoğlu and Popescu, 2011; Güler et al., 2014; Chen et al., 2016). So we have a two-period model, where in the second period the retailer set its price, but customers react to both that and to previous price. At the first period, then, when setting its price, the retailer has to also consider its effect on the next period's demand.

We first consider a second period where demand is linear with additive noise. We explore the dependence of optimal second period price on first period price, in general and in an example with uniformly distributed noise. We then formulate a novel 'reference' iso-elastic model with multiplicative noise where we focus on an example with uniformly distributed noise. We briefly then consider the first period price setting problem.

2 Linear-additive demand: second period

We assume that at the second period, demand is affected by the price in the first period, p_1, as well as by the price in the second period, p_2 (to be set). See Güler et al. (2014) for another model with reference prices. If $p_2 > p_1$, the extended linear additive model is $D_2(p_1, p_2) = a - b(p_2 - p_1) - kp_2 + \varepsilon_2$, $k > a > 0$, $b > 0$, $\varepsilon_2 \sim G_2$, $E(\varepsilon_2) = 0$. Presumably, the reference price effect, b, is smaller than the direct price effect, k. We shall assume that if $p_2 < p_1$, the same coefficient applies.

Thus,

$$P\left[D_2\left(p_1, p_2\right) \leq x\right] = G_2\left[b\left(p_2 - p_1\right) + kp_2 + x - a\right].$$

In the order quantity is q_2, the expected sales are

$$S_2\left(q_2, p_2 \,|\, p_1\right) = q_2 - \int_0^{q_2} G_2\left[b\left(p_2 - p_1\right) + kp_2 + x - a\right]dx,$$

and thus the expected profit is

$$E\pi_2 = p_2\left\{q_2 - \int_0^{q_2} G_2\left[b\left(p_2 - p_1\right) + kp_2 + x - a\right]dx\right\} - wq_2,$$

where w is the wholesale price in the second period.

That has to be maximised with respect to q_2 and p_2 for any given p_1. So the second period's optimality equations are:

$$\frac{dE\pi_2}{dp_2} = q_2 - \int_0^{q_2} G_2\left[b\left(p_2 - p_1\right) + kp_2 + x - a\right]dx - p_2\left\{(b+k)\left[G_2\left[b\left(p_2 - p_1\right) + kp_2 + q_2 - a\right] - G_2\left[b\left(p_2 - p_1\right) + kp_2 - a\right]\right]\right\} = 0$$

and

$$\frac{dE\pi_2}{dq_2} = p_2\bar{G}_2\left[b\left(p_2 - p_1\right) + kp_2 + q_2 - a\right] - w = 0.$$

From the second equation

$$q_2\left(p_2 \,|\, p_1\right) = a - b\left(p_2 - p_1\right) - kp_2 + \bar{G}_2^{-1}\left(\frac{w}{p_2}\right).$$

Note that q_2 is increasing in b iff $p_2 < p_1$, and is decreasing in k and w.

Substituting into the first equation:

$$\begin{aligned} 0 = \frac{dE\pi_2}{dp_2} &= a - 2(b+k)p_2 + bp_1 + (b+k)w \\ &- \int_0^{a - b(p_2 - p_1) - kp_2 + \bar{G}_2^{-1}\left(\frac{w}{p_2}\right)} G_2\left[b\left(p_2 - p_1\right) + kp_2 + x - a\right]dx \\ &+ p_2(b+k)G_2\left[b\left(p_2 - p_1\right) + kp_2 - a\right] \end{aligned} \qquad (*)$$

We now explore the direction of dependence of p_2^* on p_1. It can be shown that

$$\frac{dp_2^*}{dp_1}=\frac{\begin{array}{c} b\left(p_2-w\right)-(b+k)p_2G_2\left[b\left(p_2-p_1\right)+kp_2-a\right] \\ +b(b+k)p_2^2g_2\left[b\left(p_2-p_1\right)+kp_2-a\right]\end{array}}{\begin{array}{c}(b+k)\left\{-2p_2-p_2+w+p_2G_2\left[b\left(p_2-p_1\right)+kp_2-a\right]\right. \\ \left. +p_2-w+(b+k)p_2^2g_2\left[b\left(p_2-p_1\right)+kp_2-a\right]\right\}\end{array}}$$

The sign of the denominator is that of

$$-2+G_2\left[b\left(p_2-p_1\right)+kp_2-a\right]+(b+k)p_2g_2\left[b\left(p_2-p_1\right)+kp_2-a\right],$$

so denominator is positive if

$$G_2\left[b\left(p_2-p_1\right)+kp_2-a\right]+(b+k)p_2g_2\left[b\left(p_2-p_1\right)+kp_2-a\right]>2$$

[which requires that $(b + k)p_2g_2[b(p_2 - p_1) + kp_2 - a] > 1$.]

The numerator equals

$$p_2\left\{b-(b+k)\left[p_2G_2\left(b\left(p_2-p_1\right)+kp_2-a\right)-bp_2g_2\left(b\left(p_2-p_1\right)+kp_2-a\right)\right]-bw\right\},$$

so it is positive if

$$w<\frac{p_2\left\{b-(b+k)\left[p_2G_2\left(b\left(p_2-p_1\right)+kp_2-a\right)-bp_2g_2\left(b\left(p_2-p_1\right)+kp_2-a\right)\right]\right\}}{b}$$

[That requires that $G_2-bg_2>\dfrac{b}{(b+k)p_2}$]

So $\dfrac{dp_2^*}{dp_1}>0$ if either

$$G_2+(b+k)p_2g_2>2$$

and

$$w<\frac{p_2\left\{b-(b+k)\left[p_2G_2\left(b\left(p_2-p_1\right)+kp_2-a\right)-bp_2g_2\left(b\left(p_2-p_1\right)+kp_2-a\right)\right]\right\}}{b}$$

[requires that $G_2-bg_2>\dfrac{b}{(b+k)p_2}$].

If we approximate the second condition by the requirement, then combining the two conditions obtains

$$(b+k)p_2g_2+bg_2>2-\frac{b}{(b+k)p_2}$$

i.e.,

$$g_2\left[b\left(p_2-p_1\right)+kp_2-a\right]>\frac{2(b+k)^2-b}{(b+k)^2\left[(b+k)p_2+b\right]}.$$

That can be written as

$$p_1<\frac{(b+k)p_2-a-g_2^{-1}\left[\dfrac{2(b+k)^2-b}{(b+k)^2\left[(b+k)p_2+b\right]}\right]}{b}$$

Alternatively, $\frac{dp_2^*}{dp_1}>0$ if

$$G_2+(b+k)p_2g_2<2$$

and

$$w>\frac{p_2\left\{b-(b+k)\left[p_2G_2\left(b\left(p_2-p_1\right)+kp_2-a\right)-bp_2g_2\left(b\left(p_2-p_1\right)+kp_2\right)-a\right]\right\}}{b}$$

[obviously holds if $G_2-bg_2<\frac{b}{(b+k)p_2}$]

Combining the first condition with the approximation of the second, obtain

$$g_2\left[b\left(p_2-p_1\right)+kp_2-a\right]<\frac{2(b+k)^2-b}{(b+k)^2\left[(b+k)p_2+b\right]}$$

i.e.,

$$p_1>\frac{(b+k)p_2-a-g_2^{-1}\left[\dfrac{2(b+k)^2-b}{(b+k)^2\left[(b+k)p_2+b\right]}\right]}{b}.$$

2.1 Example

$$\varepsilon_2\sim U[-1,1]\Rightarrow\bar{G}_2^{-1}(\theta)=1-2\theta.$$

Then, from (*), it can be shown that

$$\begin{aligned}3\left(b^2+6bk+5k^2\right)p_2^4-4\left(1+a+bp_1\right)(b+k)p_2^3\\+\left[2a(1+a)+2bp_1(1+a)+4(b+k)w+b^2p_1^2-3\right]p_2^2\\+4w(2-a)p_2-4w^2=0.\end{aligned}$$

Can show that

$$\frac{dp_2^*}{dp_1}=\frac{p_2^2\left[2b(b+k)-1-a-b^2p_1\right]}{2\left(3b^2+5k^2+6bk\right)p_2^3-6(b+k)\left(1+a+bp_1\right)p_2^2}$$
$$+\left[2a(1+a)+2(1+a)p_1+4(b+k)w+b^2p_1^2-3\right]p_2+2(2-a)w.$$

Numerator is positive iff $p_1 < \dfrac{2b(b+k)-1-a}{b^2}$.

As for the denominator,

- If $p_2 = 0$, it is positive iff $a < 2$.
- If $p_2 = p_1$, then

  a if $\Delta < 0$, $den > 0\ \forall p_1$

  b if $\Delta > 0$, $den > 0$ if $p_1 > \dfrac{-2(1+a-3b^2-3bk)+\sqrt{\Delta}}{2b^2}$

  where

  $$\begin{aligned}\Delta = {} & 3b^4 + 2(9k-2w+3+3a)b^3 \\ & + \left(k^2-3-8a-4kw-4w+2aw+78k+6ak-2a^2\right)b^2 \\ & - 6k(1+a)b-(1+a)^2.\end{aligned}$$

- If $p_2 = 1$,

  a if $\tilde{\Delta} < 0$, $den > 0\ \forall p_1$

  b if $\tilde{\Delta} > 0$, $den > 0$ iff $p_1 > \dfrac{-2(1+a-3b^2-3bk)+\sqrt{\Delta}}{2b^2}$

  where

  $$\begin{aligned}\tilde{\Delta} = {} & 3b^4 + 2(9k-22+3+3a)b^3 \\ & + \left(k^2-3-8a-4kw-4w+2aw+78k+6ak-2a^2\right)b^2 \\ & -6k(1+a)b-(1+a)^2.\end{aligned}$$

3 Iso-elastic multiplicative demand

To capture reference effects in an iso-elastic multiplicative setting, we propose the following new 'symmetric' model of second period demand.

$$D_2 = \begin{cases} \left[ap_2^{-b} - \theta\left(p_2 - p_1\right)^{-d}\right]\varepsilon_2, & p_2 > p_1 \\ \left[ap_2^{-b} + \theta\left(p_1 - p_2\right)^{-d}\right]\varepsilon_2, & p_2 < p_1, \end{cases}$$

where $a, \theta > 0$, $1 > b > d > 0$ (presumably, $a > \theta$), $\varepsilon_2 \sim H_2$, h_2, $\varepsilon_2 \geq 0$, $E(\varepsilon_2) = 1$.

Here θ and d determine the reference price effect, while a and b determine the direct price effect. Denote $ap_2^{-b} - \theta(p_2 - p_1)^{-d} \equiv M$, assumed > 0, i.e.,

$$\frac{p_2^b}{\left(p_2 - p_1\right)^d} \leq \frac{a}{\theta}.$$

Consider the case $p_2 > p_1$ (i.e., $p_2 < \left(\frac{M}{a}\right)^{-\frac{1}{b}}$).

Then

$$P(D_2 \geq x) = H_2\left(\frac{x}{M}\right),$$

So expected sales are

$$S_2(q_2, p_2 | p_1) = q_2 - \int_0^{q_2 \wedge 2M} x h_2\left(\frac{x}{M}\right) dx,$$

where '$\wedge$' indicates 'the smaller of'.

But

$$E\pi_2(q_2, p_2 | p_1) = p_2 S_2 - w q_2$$

$$\Rightarrow E\pi_2 = p_2\left[q_2 - \int_0^{q_2 \wedge 2M} x h\left(\frac{x}{M}\right) dx\right] - w q_2$$

$$\frac{dE\pi_2}{dq_2} = \begin{cases} p_1\left[1 - q_2 h\left(\frac{q_2}{M}\right)\right] - w, & q_2 < 2M \\ p_2 - w(>0), & q_2 > 2M \end{cases}$$

We shall first consider the former range, where, assuming *h* is monotone,

$$q_2 h\left(\frac{q_2}{M}\right) = 1 - \frac{w}{p_2}$$

$$\frac{dE\pi_2}{dp_2} = q_2 - \int_0^{q_2 \wedge 2M} x h\left(\frac{x}{M}\right) dx$$

$$+ p_2 \begin{cases} -\int_0^{q_2} x h'\left(\frac{x}{M}\right)\left(-\frac{x}{M^2}\right)\frac{dM}{dp_2} dx, & q_2 < 2M \\ -\int_0^{2M} x h'\left(\frac{x}{M}\right)\left(-\frac{x}{M^2}\right)\frac{dM}{dp_2} dx \\ -2\frac{dM}{dp_2} \cdot 2M h\left(\frac{2M}{M}\right), & q_2 - 2M. \end{cases}$$

Now,

$$\frac{dM}{dp_2} = -abp_2^{-b-1} + \theta d(p_2 - p_1)^{-d-1}.$$

Thus, for $q_2 < 2M$, assuming that $h(0) = 0$,

$$\frac{dE\pi_2}{dp_2} = q_2 - \int_0^{q_2} xh\left(\frac{x}{M}\right)dx + \frac{1}{M^2}\cdot\frac{dM}{dp_2}\left[Mx^2h\left(\frac{x}{M}\right)\Big|_0^{q_2} - 2\int_0^{q_2} xh\left(\frac{x}{M}\right)Mdx\right]$$

$$= q_2 + \frac{1}{M}\cdot\frac{dM}{dp_2}q_2^2h\left(\frac{q_2}{M}\right) - \left(1+\frac{2}{M}\cdot\frac{dM}{dq_2}\right)\int_0^{q_2} xh\left(\frac{x}{M}\right)dx = 0$$

Substituting $\frac{dE\pi_2}{dp_2} = 0,$ obtain

$$q_2 + \frac{1}{M}\cdot\frac{dM}{dp_2}q_2\left(1-\frac{w}{p_2}\right) - \left(1+\frac{2}{M}\cdot\frac{dM}{dq_2}\right)\int_0^{q_2} xh\left(\frac{x}{M}\right)dx = 0.$$

If $q_2 > 2M$

$$\frac{dE\pi_2}{dp_2} = q_2 - \int_0^{2M} xh\left(\frac{x}{M}\right)dx - \int_0^{2M} xh'\left(\frac{x}{M}\right)\left(-\frac{x}{M^2}\right)\cdot\frac{dM}{dp_2}dx - 2\frac{dM}{dp_2}\cdot 2Mh\left(\frac{2M}{M}\right)$$

$$= q_2 - \int_0^{2M} xh\left(\frac{x}{M}\right)dx + \frac{dM}{dp_2}\left[(2-M)h(2) - \frac{1}{M}H(2)\right] = 0.$$

3.1 Example

$$\varepsilon_2 \sim U[0, 2].$$

Then

$$q_2\left[1+\frac{1}{M}\cdot\frac{dM}{dq_2}\left(1-\frac{w}{p_2}\right) - \frac{1}{4}q_2\left(1+\frac{dM}{dp_2}\right)\right] = 0,$$

i.e.,

$$4p_2\left[ap_2^{-b} - \theta\left(p_2 - p_1\right)^{-d}\right] + 4p_2\left[-abp_2^{-b-1} + \theta d\left(p_2 - p_1\right)^{-d-1}\right]$$
$$-4w\left[a - bp_2^{-b-1} + \theta d\left(p_2 - p_1\right)^{d-1}\right]$$
$$+p_2\left[ap_2^{-b} - \theta\left(p_2 - p_1\right)^{-d}\right]\left[1 - abp_2^{-b-1} + \theta d\left(p_2 - p_1\right)^{-d-1}\right] = 0.$$

That is,

$$5ap_2^{-b+1} - 5\theta p_2\left(p_2 - p_1\right)^{-d} - 4abp_2^{-b-1} + 4\theta dp_2\left(p_2 - p_1\right)^{-d-1}$$
$$+4abwp_2^{-b-1} - 4gqdw\left(p_2 - p_1\right)^{-d-1} - a^2bp_2^{-2b}$$
$$+ab\theta p_2^{-b}\left(p_2 - p_1\right)^{-d} + a\theta dp_2^{-b+1}\left(p_2 - p_1\right)^{-d-1}$$
$$-\theta^2 dp_2\left(p_2 - p_1\right)^{-2d-1} = 0.$$

Multiplying by $p_2^{b+1}(p_2 - p_1)^{2b+1}$ and assuming that $d = b \equiv b$,

$$5ap_2^2(p_2-p_1)^{2b+1}-5\theta p_2^{b+2}(p_2-p_1)^{b+1}-4ab(p_2-p_1)^{2b+1}$$
$$-4\theta bp_2^{b+2}(p_2-p_1)^b-4abw(p_2-p_1)^{2b+1}$$
$$-4\theta bwp_2^{b+1}(p_2-p_1)^b-a^2bp_2^{-b+1}+ab\theta p_2(p_2-p_1)^{b+1}$$
$$+a\theta bp_2^2(p_2-p_1)^b-\theta bp_2^{b+2}=0.$$

Assuming now that $b = 1$ and $a = 0$, obtain

$$-5\theta p_2^3(p_2-p_1)^2+4\theta p_2^3(p_2-p_1)+4\theta wp_2^2(p_2-p_1)-\theta^2p_2^3=0$$

Dividing by $(-p_2^2)$ and denoting $p_2-p_1\equiv y$, obtain

$$5\theta p_2y^2-4\theta(p_2-w)y+\theta^2p_2=0.$$

So

$$y=\frac{2\theta(p_2-w)\pm\sqrt{\Delta}}{5\theta p_2}$$

$$\Delta=4\theta^2(p_2-w)^2-5\theta^3p_2^2$$
$$=\theta^2\left[(4-5\theta)p_2^2-8wp_2+4w^2\right]$$

Thus

$$p_2-p_1=\frac{2(p_2-w)\pm\sqrt{(4-5\theta)p_2^2-8wp_2+4w^2}}{5p_2}$$
$$\Rightarrow\sqrt{}=5p_2^2-5p_1p_2-2p_2+2w$$
$$\Rightarrow 25p_2^4-10(5p_1+2)p_2^3+5(5p_1^2+4w+\theta)p_2^2-20wp_1p_2=0$$
$$\Rightarrow 5p_2^3-2(5p_1+2)p_2^2+(5p_1^2+4w+\theta)p_2-4wp_1=0.$$

So if $p_2 = 1$, above holds if

$$w=\frac{10p_1-5p_1^2-\theta-1}{4(1-p_1)}.$$

If $p_2 = p_1$

$$5p_1^3-(10p_1+4)p_1^2+(5p_1^2+4w+\theta)p_1-4wp_1$$
$$=-4p_1^2+\theta p_1=p_1(\theta-4p_1)=0\quad\text{if }p_1=\frac{1}{4}\theta.$$

If $q_2 > 2M$, assuming that $d = b =1$, and denoting $p_2 - p_1$ by y,

$$\left[4ap_2^3+2ap_2^2-2a(a+1)p_2+a^3\right]y^4+\left(-2\theta p_2^4+6a^2\theta p_2^2\right)y^3$$
$$+\left[-2\theta p_2^4+2a\theta(1-3\theta)p_2^3+a\theta(\theta-a)p_2^2\right]y^2$$
$$+\left[2\theta^2(\theta-1)p_2^4+2\theta^2ap_2^3\right]y-\theta^3p_2^4=0.$$

i.e.,

$$a\left[a^2+2p_2\left(2p_2^2+p_2-a^2-a\right)\right]y^4+2\theta p_2^2\left(-p_2^2+3a^2\right)y^3 \\ +\theta p_2^2\left[-2p_2^2+2a(1-3\theta)p_2+a(\theta-a)\right]y^2 \\ +2\theta^2p_2^3\left[(\theta-1)p_2+a\right]y-\theta^3p_2^4=0.$$

Take $a = 1$. Then

$$\left[1+2p_2\left(2p_2^2+p_2-2\right)\right]y^4+2\theta p_2^2\left(-p_2^2+3\right)y^3 \\ +\theta p_2^2\left[-2p_2^2+2(1-3\theta)p_2+\theta-1\right]y^2 \\ +2\theta^2p_2^3\left[(\theta-1)p_2+1\right]y-\theta^3p_2^4=0.$$

E.g., if $\theta = 0^+$ then

$$\left[1+2p_2\left(2p_2^2+p_2-2\right)\right]y^4=0$$

So either $y = 0$ ($\Rightarrow p_2 = p_1$) or

$$4p_2^3+2p_2^2-4p_2+1=0 \quad \left(\Rightarrow p_2=\frac{1}{2}\right).$$

4 Concluding remarks

As purchase behaviour contingent on reference prices is reportedly quite prevalent, our intent in this Note was to embed such behaviour in the price-setting newsvendor inventory model. Our iso-elastic model of dependence on reference price is a novel one.

Two periods are needed for that, and the model becomes complex. We thus provide examples with uniform noise distributions.

Future research may consider an infinite horizon model, where in each period the demand depends on the price in last period, as well as on the current price.

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