Set of non-dominated fuzzy subsets and fuzzy set of non-dominated vertices in fuzzy graphs

A. Ibrahim*

P.G. and Research Department of Mathematics,
H.H. The Rajah's College,
Pudukkotai-622 001, Tamilnadu, India
Email: ibrahimaadhil@yahoo.com
*Corresponding author

Basheer Ahamed Mohideen

Department of Mathematics,
Faculty of Science,
University of Tabuk,
Tabuk, Kingdom of Saudi Arabia
Email: bahamed@ut.edu.sa
Email: dr.basheer09@gmail.com

Abstract: In this paper, the set of non-dominated vertices of crisp graphs is discussed. We consider the two logical operators, and a composition. We introduce the set of non-dominated fuzzy subsets *NDFS* and fuzzy set of non-dominated vertices of fuzzy graphs *FND*. Also, we obtain some properties of *NDFS* and *FND*.

Keywords: logical operators; fuzzy graphs; set of non-dominated fuzzy subset; non-dominated vertices; weak lattice; sub-weak lattice.

Reference to this paper should be made as follows: Ibrahim, A. and Mohideen, B.A. (2016) 'Set of non-dominated fuzzy subsets and fuzzy set of non-dominated vertices in fuzzy graphs', *Int. J. Fuzzy Computation and Modelling*, Vol. 2, No. 1, pp.50–60.

Biographical notes: A. Ibrahim is working as an Assistant Professor in the P.G. and Research Department of Mathematics, H.H. The Rajah's College, Pudukkottai, India. He worked as Head of the Department in Mathematics, Texcity Arts and Science College, Coimbatore, India. He has more than 19 years teaching experience at collegiate level. His research areas are fuzzy mathematics and theoretical computer science etc.

Basheer Ahamed Mohideen is working as an Associate Professor of Mathematics in Faculty of Science, University of Tabuk, Kingdom of Saudi Arabia. His research areas are discrete and fuzzy mathematics. He is a reviewer of many mathematics journals such as *Information Science*, *IJFS*, etc. He is a life member of Ramanujan Mathematical Society.

1 Introduction

The pictorial representation of graph consists of a set of points joined by arcs. To make use of computers to solve problems on graphs, they had to be stored in the memory of computers. This is done using matrices. Many kinds of matrices are associated with a graph. In the theory of crisp graphs, the set of non-dominated vertices is denoted by $CND(R)$ of graph $G = (X, R)$ has occupied a particular importance. In which the study of internally stable set $\text{Int}(R)$ and externally stable set $\text{Ext}(R)$ of the given graph $G = (X, R)$ are discussed.

Fuzzy graphs are encountered in fuzzy set theory. A fuzzy set was defined by Zadeh in 1965. His purpose was to develop a theory for sets which are ambiguous and imprecise in definition, a characteristic of most of the sets found in the real world. Every element in the universal set is assigned a grade of membership, a value in [0; 1] in a subset called fuzzy subset. It is a generalisation of usual subsets also called crisp sets. A grade of membership of 1 to some elements and 0 to all others in the universal set gives a crisp set. Fuzzy sets are representations of how a human brain perceives the objects in the world. Hence, fuzzy set theory has applications in those areas where machine replacements are sought for humans for instance control engineering, artificial intelligence, expert systems, robotics, pattern recognition and so on. As in case of crisp sets, relations are defined for fuzzy subsets called fuzzy relations. Graphs are representations of binary relations. Similarly, fuzzy binary relations are represented by graphs called fuzzy graphs. Rosenfeld (1975) introduced the theory for fuzzy graphs in 1975.

The concepts of not external domination have extended under some valued fuzzy operators by Kitainik (1987, 1993). Alaoui (1999) extended the concepts of internal stability, external stability, external domination and some of their combinations to fuzzy graphs. Basheer Ahamed and Ibrahim (2009, 2010) introduced the notions of weak lattice, sub-weak lattice, not external domination vertices set, internally stable vertex set and externally stable vertex set of fuzzy graphs. Further, we investigated some of their properties. Somasundaram and Somasundaram (1998) and Somasundaram (2004) introduced the concepts of domination and total domination in fuzzy graphs and determined the domination number for several classes of fuzzy graphs and obtained bounds for the same.

In this paper, we introduce the $(NDFS(\rho, \boxed{L}))$ and $(FND(\rho))$ of fuzzy graphs. Also, we investigate some of their properties.

2 Preliminaries

In this section, some basic definitions and results are discussed.

Definition 2.1. (Zadeh, 1965) A fuzzy subset μ of a non-empty set X is a function $\mu: X \rightarrow [0, 1]$.

Definition 2.2 (Rosenfeld, 1975) A fuzzy relation on the non-empty set X is a fuzzy subset of $X \times X$.

Definition 2.3 (Basheer Ahamed and Ibrahim, 2009) The set of all fuzzy relations defined on X is denoted by $\wp(X \times X)$. If $\rho, \sigma \in \wp(X \times X)$, then the $\max\text{-}\overline{L}$ composition of ρ and σ is defined as follows $(\rho \boxed{L} \sigma)(x, y) = \max_{z \in X} \max\{\rho(x, z) + \sigma(z, y) - 1, 0\}$ for all $x, y \in X$.

Definition 2.4 (Rosenfeld, 1975) A fuzzy graph $G = (\mu, \rho)$ is a set with pair of functions $\mu: X \rightarrow [0, 1]$ and $\rho: X \times X \rightarrow [0, 1]$ such that $\rho(x, y) \leq \mu(x) \wedge \mu(y)$ for all $x, y \in X$.

Definition 2.5 (Tremblay and Manohar, 1997) A lattice is an algebraic system $(L, \wedge, \vee)$ with two binary operations $\wedge$ and $\vee$ on a non-empty set L which are both idempotent, commutative, associative and satisfy the absorption laws.

Example 2.6 (Alaui, 1999) The algebraic system $(\wp(X), \wedge, \vee)$ is a lattice under Zadeh's inclusion $\mu_1 \subseteq \mu_2 \Leftrightarrow (\forall x \in X / \mu_1(x) \leq \mu_2(x))$.

Definition 2.7 (Tremblay and Manohar, 1997) Let $(L, \wedge, \vee)$ be a lattice then the subset $S \neq \phi \subseteq L$ is said to be a sub-lattice if it is closed under the operations $\wedge$ and $\vee$ of L that is, if $(a \wedge b) \in S$ and $(a \vee b) \in S$.

Definition 2.8 (Alaui, 1999) Let $G = (X, R)$ be a graph, where X is an arbitrary finite non-empty set, R is a relation on X. If $A \subseteq X$, the set elements of X are dominated by A then the composition of A and R such that $A \circ R = \{y \in X / (\exists\, x \in A)\, xR\, y\}$.

Definition 2.9 (Basheer Ahamed and Ibrahim, 2009) The logical operators $\overline{L}$, $\underline{L}$ and the complement N are defined as follows, let μ_1 and μ_2 be any two fuzzy sub sets of X, then for all $x \in X$,

1 $(\mu_1 \overline{L} \mu_2) = \max_{x \in X} \{\mu_1(x) + \mu_2(x) - 1, 0\}$

2 $(\mu_1 \underline{L} \mu_2) = \max_{x \in X} \{\mu_1(x) + \mu_2(x), 1\}$

3 $\mathrm{N}(\mu_1) = \overline{\mu_1(x)} = 1 - \mu_1(x)$.

Definition 2.10 (Basheer Ahamed and Ibrahim, 2009) Let μ be a fuzzy subset and ρ a fuzzy relation on a non-empty set X and the composition $\boxed{L}$, then the composition of μ and ρ, $(\mu \boxed{L} \rho)$ is defined as for each $x \in X$, $(\mu \boxed{L} \rho)(x) = \max_{y \in X} \max\{\mu(x) + \rho(x, y) - 1, 0\}$.

Note. In a fuzzy graph $G = (\mu, \rho)$, the composition of μ and ρ can be defined as, for each $a \in X$, $(\mu(a) \boxed{L} \rho(a, b)) = \max_{b \in X} \max\{\mu(a) + \rho(a, b) - 1, 0\}$.

Definition 2.11 (Basheer Ahamed and Ibrahim, 2009) A set $W \neq \phi$ is closed under $\overline{L}$ and $\underline{L}$ is called a weak lattice $(W, \overline{L}, \underline{L})$ then, the following axioms commutative and associative laws are satisfied.

1 commutative laws

$(a \overline{L} b) = (b \overline{L} a)$

$(a \underline{L} b) = (b \underline{L} a)$

2 associative laws

$$(a\,\overline{L}\,b)\overline{L}\,c = a\,\overline{L}(b\,\overline{L}\,c)$$

$$(a\,\underline{L}\,b)\underline{L}\,c = a\,\underline{L}(b\,\underline{L}\,c) \quad \text{for all } a, b \text{ and } c \in W.$$

Remark 2.12 (Basheer Ahamed and Ibrahim, 2009) In this paper, an algebraic system $(\wp(X), \overline{L}, \underline{L})$ is a weak lattice under Zadeh's inclusion $\mu_1 \subseteq \mu_2 \Leftrightarrow (\forall x(\mu_1(x) \leq \mu_2(x))$. Thus in a fuzzy graph $G = (\mu, \rho)$, we have $(\mu(X), \overline{L}, \underline{L})$ is a weak lattice under the condition for each $a, b \in X \Rightarrow \mu(a) \leq \mu(b)$ and the composition $\boxed{L}$.

Definition 2.13 (Basheer Ahamed and Ibrahim, 2009) Let $(W, \overline{L}, \underline{L})$ be a weak lattice then the subset $S \neq \phi \subseteq W$ is said to be a sub-weak lattice if it is closed under the operations $\overline{L}$ and $\underline{L}$ that is, if $(a\,\overline{L}\,b) \in S$ and $(a\,\underline{L}\,b) \in S$ for all $a, b \in S$.

Definition 2.14 (Basheer Ahamed and Ibrahim, 2009) Let $G = (\mu, \rho)$ be a fuzzy graph without loops and with underlying set X where μ: $X \rightarrow [0, 1]$ and ρ: $X \times X \rightarrow [0, 1]$, a vertex $a \in X$ is not externally dominated under the composition $\boxed{L} \Leftrightarrow ((\overline{\mu(a)}\boxed{L}\rho(a, b) \leq \overline{\mu(a)})$ and $((\mu(a)\boxed{L}\rho^{-1}(a,b)) \leq \mu(a))$ for some $b \in X$. The set of all vertices are not externally dominated in X is denoted by $\text{Ned}(\rho, \boxed{L})$.

Definition 2.15 (Basheer Ahamed and Ibrahim, 2010) Let $G = (\mu, \rho)$ be a fuzzy graph without loops and with underlying set X where μ: $X \rightarrow [0, 1]$ and ρ: $X \times X \rightarrow [0, 1]$, a vertex $a \in X$ is internally stable under the composition $\boxed{L} \Leftrightarrow X((\mu(a)\boxed{L}\rho(a, b)) \leq \overline{\mu(a)})$ and $((\mu(a)\boxed{L}\rho^{-1}(a,b)) \leq \overline{\mu(a)})$ for some $b \in X$. The set of all vertices are internally stable in X is denoted by $\text{Int}(\rho, \boxed{L})$.

Definition 2.16 (Basheer Ahamed and Ibrahim, 2010) Let $G = (\mu, \rho)$ be a fuzzy graph without loops and with underlying set X where μ: $X \rightarrow [0, 1]$ and ρ: $X \times X \rightarrow [0, 1]$, a vertex $a \in X$ is externally stable under the composition $\boxed{L} \Leftrightarrow X((\mu(a)\boxed{L}\rho(a, b)) \geq \overline{\mu(a)})$ and $((\mu(a)\boxed{L}\rho^{-1}(a,b)) \geq \overline{\mu(a)})$ for some $b \in X$. The set of all vertices are externally stable in X is denoted by $\text{Ext}(\rho, \boxed{L})$.

Note. The above condition is satisfied, only if $0.3 \leq (\mu\,(a)\,L\,\rho\,(a, b)) \leq 1$ for all $a, b \in X$.

Proposition 2.17 (Basheer Ahamed and Ibrahim, 2009) Let $G = (\mu, \rho)$ be a fuzzy graph without loops and with underlying set X where μ: $X \rightarrow [0, 1]$ and ρ: $X \times X \rightarrow [0, 1]$, the set $\text{Ned}(\rho, \boxed{L})$ is a sub-weak lattice of the weak lattice $(\mu(X), \overline{L}, \underline{L})$.

Proposition 2.18 (Basheer Ahamed and Ibrahim, 2010) Let $G = (\mu, \rho)$ be a fuzzy graph without loops and with underlying set X where μ: $X \rightarrow [0, 1]$ and ρ: $X \times X \rightarrow [0, 1]$ for all $a, b \in X$, $\text{Int}(\rho, \boxed{L})$ is a $\overline{L}$-sub-weak lattice of the weak lattice $(\mu(X), \overline{L}, \underline{L})$.

Definition 2.19 (Alaui, 1999) Let $G = (X, R)$ be a graph, where X is finite non-empty set of vertices and R is the edge set $(R \neq \phi)$, the set of non-dominated vertices is defined as the subset of X consisting of those elements which are not dominated by any element of X, which is denoted by $CND(R)$. That is, $CND(R) = \{x \in X / \forall y \in X, \text{Not}(xRy)\}$.

Proposition 2.20 (Alaui, 1999) Let $G = (X, R)$ be a graph. Then the following conditions are hold

1 $CND(R) = \overline{X \circ R}$

2 $CND(R)$ is the maximal subset of X which satisfies both the properties of internal and external stability.

3 Main results

In this section, we introduce the set of non-dominated fuzzy subsets and fuzzy set of non-dominated vertices of fuzzy graphs and discuss some properties of these sets with illustrations.

Definition 3.1. Let $G = (\mu, \rho)$ be a fuzzy graph, without loops and with underlying set X where $\mu: X \rightarrow [0, 1]$, $\rho: X \times X \rightarrow [0, 1]$ and a vertex $a \in X$ is called a non-dominated with respect to the composition $\boxed{L}$, then $a \in \mathrm{Ned}(\rho, \boxed{L})$ and $a \in \mathrm{Int}(\rho, \boxed{L})$. The set of non-dominated fuzzy subsets $\mathrm{Ned}(\rho, \boxed{L})$ and $\mathrm{Int}(\rho, \boxed{L})$ is denoted by $NDFS(\rho, \boxed{L})$.

Note. From the Definition 3.1, we get $a \in NDFS(\rho, \boxed{L}) \Leftrightarrow [(\overline{\mu(a)\boxed{L}\rho(a, b))} \leq \overline{\mu(a)}$, $(\mu(a)\boxed{L}\rho^{-1}(a, b)) \leq \mu(a)]$ and $[\mu(a)\boxed{L}\rho(a, b)) \leq \overline{\mu(a)}, (\mu(a)\boxed{L}\rho^{-1}(a, b)) \leq \overline{\mu(a)}]$, for some $b \in X$ and we have $NDFS(\rho, \boxed{L}) = (\mathrm{Ned}(\rho, \boxed{L} \cap \mathrm{Int}(\rho, \boxed{L}))$.

Proposition 3.2. The set $NDFS(\rho, \boxed{L})$ is a sub-weak lattice under the logical operator $\overline{L}$, but not a sub-weak lattice of a weak lattice $(\mu(X), \overline{L}, \underline{L})$.

Proof. By the results (Proposition 2.17 and Proposition 2.18). Let $G = (\mu, \rho)$ be a fuzzy graph without loops and with underlying set X where $\mu: X \rightarrow [0, 1]$ and $\rho: X \times X \rightarrow [0, 1]$ and $a \in X$. The sets $\mathrm{Ned}(\rho, \boxed{L})$ is a sub-weak lattice of $(\mu(X), \overline{L}, \underline{L})$. and the set $\mathrm{Int}(\rho, \boxed{L})$ is a sub-weak lattice of $(\mu(X), \overline{L}, \underline{L})$. Under the logical operator $\overline{L}$. Thus, we have $NDFS(\rho, \boxed{L})$ is a sub-weak lattice under the logical operator $\overline{L}$, but not a sub-weak lattice of $(\mu(X), \overline{L}, \underline{L})$.

Definition 3.3. Let $G = (\mu, \rho)$ be a fuzzy graph, without loops and with underlying set X, where $\mu: X \rightarrow [0, 1]$ and $\rho: X \times X \rightarrow [0, 1]$, a set $FND(\rho(a)) = 1 - \max_{b \in X}\{\rho(a, b)\}$ for all $a \in X$ is called a fuzzy set of non-dominated vertices of G. Which is denoted by $FND(\rho)$.

Example 3.4. Let$G = (\mu, \rho)$ be a fuzzy graph, where $X = \{a, b, c, d, e\}$, $\mu: X \rightarrow [0, 1]$ and $\rho: X \times X \rightarrow [0, 1]$ defined as $\mu(a) = 0.5$, $\mu(b) = 0.4$, $\mu(c) = 0.7$, $\mu(d) = 0.6$, $\mu(e) = 0.9$, $\rho(a, b) = 0.3$, $\rho(b, c) = 0.2$, $\rho(c, d) = 0.5$, $\rho(d, e) = 0.4$, $\rho(e, a) = 0.3$, $\rho(e, b) = 0.1$, $\rho(e, c) = 0.4$ and $\rho(d, b) = 0.2$. We have $FND(\rho) = \{0.7, 0.7, 0.5, 0.5, 0.6\}$.

Example 3.5. Let$G = (\mu, \rho)$ be a fuzzy graph, with $X = \{a, b, c, d\}$, $\mu: X \rightarrow [0, 1]$ and $\rho: X \times X \rightarrow [0, 1]$ defined as $\mu(a) = 0.6$, $\mu(b) = 0.3$, $\mu(c) = 0.7$, $\mu(d) = 0.8$, $\rho(a, b) = 0.3$, $\rho(b, c) = 0.2$, $\rho(c, d) = 0.7$, $\rho(d, a) = 0.5$ and $\rho(b, d) = 0.1$. We have $FND(\rho) = \{0.5, 0.8, 0.8, 0.6\}$.

Proposition 3.6. Let $G = (\mu, \rho)$ be a fuzzy graph without loops and with underlying set X where $\mu: X \rightarrow [0, 1]$ and $\rho: X \times X \rightarrow [0, 1]$, then for all $a, b \in X$, the following conditions are satisfied under the composition $\boxed{L}$.

1 $(\overline{FND(\rho\ (a))} \boxed{L} \rho(a, b)) \leq \overline{FND(\rho(a))}$

2 $(FND(\rho(a)) \boxed{L} \rho(a, b)) \leq \overline{FND(\rho(a))}$

3 If $a \in \text{Ext}(\rho, \boxed{L})$, then $FND(\rho(a)) \leq \mu(a)$ for all $a, b \in X$.

Proof

1 By the definition of $FND(\rho)$, we have $FND(\rho(a)) = 1 - \square \max\{\rho(a, b)\}$ then $\overline{FND(\rho(a))} = \max\{\rho(a, b)\}$ for all $a, b \in X$

$$\left(\left(\overline{FND(\rho(a))} \boxed{L} \rho(a, b)\right) = \max \max \left\{\max_{b \in X} \{\rho(a, b)\} + \rho(a, b) - 1, 0\right\} \tag{1}$$

We have $((\overline{FND(\rho(a))} \boxed{L} \rho(a, b))$ is either 0 or $\max_{b \in X}\{\rho(a, b)\} + \rho(a, b) - 1$

- *Case 1*: if $((\overline{FND(\rho(a))} \boxed{L} \rho(a, b)) = 0,$ then the result is trivial
- *Case 2:* $((\overline{FND(\rho(a))} \boxed{L} \rho(a, b)) = \max_{b \in X}\{\rho(a, b)\} + \rho(a, b) - 1 \leq \max\{\rho(a, b)\}$

  $= \overline{FND(\rho(a))}$ for some $b \in X$.

  Therefore, we have $(\overline{FND(\rho(a))} \boxed{L} \rho(a, b)) \leq \overline{FND(\rho(a))}.$

2 Now

$$(FND(\rho(a)) \boxed{L} \rho(a, b)) = \max \max \left\{1 - \max_{b \in X}(\rho(a, b)) + \rho(a, b) - 1, 0\right\} \tag{2}$$

We have $(FND(\rho(a)) \boxed{L} \rho(a, b))$ is either 0 or $[(1 - \max_{b \in x}\{\rho(a, b)\} + \rho(a, b) - 1].$

- *Case 1:* if $(FND(\rho(a)) \boxed{L} \rho(a, b)) = 0,$ then the result is trivial
- *Case 2:* if $(FND(\rho(a)) \boxed{L} \rho(a, b)) = [1 - \max_{b \in x}\{\rho(a, b)\} + \rho(a, b) - 1]$

  $\leq \max\{\rho(a, b)\} = \overline{FND(\rho(a))}$ for some $b \in X$.

  Thus, we have $(FND(\rho(a)) \boxed{L} \rho(a, b)) \leq FND(\rho(a)).$

3 Let

$$\begin{aligned} a \in \text{Ext}\left(\rho, \boxed{L}\right) &\Leftrightarrow \left(\mu(a) \boxed{L} \rho(a, b)\right) \geq \overline{\mu(a)} \\ &\Leftrightarrow \max \max\{\mu(a) + \rho(a, b) - 1, 0\ \} \leq 1 - \mu(a) \end{aligned} \tag{3}$$

We have either $0 \geq 1 - \mu(a)$ or $(\mu(a) + \rho(a, b) - 1) \geq 1 - \mu(a)$.

- *Case 1:* if $0 \geq 1\ \mu(a)$, then it is not possible

- *Case 2:* if $(\mu(a) + \rho(a, b) - 1) \geq 1\ \mu(a) \Leftrightarrow \mu(a) + \rho(a, b) - 1 \geq 1 - \mu(a). \Leftrightarrow \mu(a) + \mu(a) - 1 \geq 1 - \rho(a, b) \Leftrightarrow 2\mu(a) - 1 - 1 - \max\{\rho(a, b)\} \Leftrightarrow \mu(a) \geq FND(\rho(a))$. (since $(2\mu(a) - 1) \geq \mu(a)$).

  Thus, if $a \in \text{Ext}(\rho, \boxed{L})$, then we get $FND(\rho\ (a)) \leq \mu\ (a)$. □

Example 3.7. Let $G = (\mu, \rho)$ be a fuzzy graph, where $X = \{a, b, c, d\}$, $\mu: X \to [0, 1]$ and $\rho: X \times X \to [0, 1]$ with $\mu(a) = 0.5$, $\mu(b) = 0.3$, $\mu(c) = 0.6$, $\mu(d) = 0.9$, $\rho(a, b) = 0.3$, $\rho(b, c) = 0.2$, $\rho(c, d) = 0.6$, $\rho(d, a) = 0.2$, $\rho(a, c) = 0.3$ and $\rho(d, b) = 0.1$.

1 From the graph, we have $FND(\rho) = \{0.7, 0.7, 0.4, 0.4\}$

Then, $\overline{FND(\rho)} = \{0.3, 0.3, 0.6, 0.6\}$

$\overline{((FND(\rho(a))\boxed{L}\rho(a, b))} = \max\max\{\max\{\rho(a, b)\} + \rho(a, b) - 1, 0\} = \max\max\{0.3 + 0.3 - 1, 0\} = 0 \leq \overline{FND(\rho(a))} = 0.3$.

Similarly, to prove all the elements of $\overline{FND(p)}$.

Thus, we get $\overline{(FND(\rho(a))}\boxed{L}\rho(a, b)) \leq \overline{FND(\rho(a))}$.

2 $FND(\rho) = \{0.7, 0.7, 0.4, 0.4\}$.

Consider $FND(\rho(b))$, we have $(FND(\rho(b))\boxed{L}\rho(b, a)) = \max\max\{0.7 + 0.3 - 1, 0\} = 0 \leq \overline{FND(\rho(b))} = 0.3$. Similarly, we get same result from all the elements of $FND(\rho)$ Thus, $(FND(\rho(a))\boxed{L}\rho(a, b)) \leq \overline{FND(\rho(a))}$.

3 From the graph, if $d \in \text{Ext}(\rho, \boxed{L})$, then from the $FND(\rho)$ we get $FND(\rho(d)) = 0.4 \leq \mu(d) = 0.9$. Thus, if $d \in \text{Ext}(\rho, \boxed{L})$ then we have $FND(\rho(d)) \leq \mu(d)$.

Example 3.8. Let $G = (\mu, \rho)$ be a fuzzy graph, where $X = \{a, b, c\}$, $\mu: X \to [0, 1]$ and $\rho: X \times X \to [0, 1]$ with $\mu(a) = 0.6$, $\mu(b) = 0.8$, $\mu(c) = 1$, $\rho(a, b) = 0.5$, $\rho(b, c) = 0.8$ and $\rho(c, a) = 0.6$.

1 From the graph we have $FND(\rho) = \{0.4, 0.2, 0.2\}$.

Then, $\overline{FND(\rho)} = \{0.6, 0.8, 0.8\}$.
$\overline{((FND(\rho(a)}\boxed{L}\rho(a, b)) = \max\max\{\max\{\rho(a, b)\} + \rho(a, b) - 1, 0\}$
$= \max\max\{0.6 + 0.5 - 1, 0\} = 0.1 \leq \overline{FND(\rho(a))} = 0.6\}$.

Similarly, to prove all the elements of $FND(\rho)$.

Thus, we get $\overline{(FND(\rho(a))}\boxed{L}\rho(a, b)) \leq \overline{FND(\rho(a))}$.

2 $FND(\rho) = \{0.4, 0.2, 0.2\}$.

Then, $(FND(\rho(a))\boxed{L}\rho(a, b)) = \max\max\{0.4 + 0.5 - 1, 0\} = 0 \leq \overline{FND(\rho(b))} = 0.6$.

Similarly we prove that all the elements of $FND(\rho)$.

Thus, we get $(FND(\rho(a))\boxed{L}\rho(a, b)) \leq \overline{FND(\rho(a))}$.

3 Let $b \in \text{Ext}(\rho, \boxed{L})$, then from the $FND(\rho)$, we get $FND(\rho(b)) = 0.2 \le \mu(b) = 0.8$.

Thus, if $b \in \text{Ext}(\rho, \boxed{L})$, then $FND(\rho(b)) \le \mu(b)$.

Similarly, if $c \in \text{Ext}(\rho, \boxed{L})$ then we get $FND(\rho(c)) \le \mu(c)$.

Proposition 3.9. Let$G = (\mu, \rho)$ be a fuzzy graph without loops and with underlying set X where μ: $X \to [0, 1]$ and ρ: $X \times X \to [0, 1]$ then for all a, $b \in X$, the following conditions are satisfied under the composition $\boxed{L}$.

1 If $FND(\rho(a)) \ge 0.5$ then $0 \le \rho\ (a, b) \le 0.5$

2 If $FND(\rho(a)) \in \text{Int}(\rho, L)$ then $FND(\rho(a)) \in NDFS(\rho, \boxed{L})$

3 If $FND(\rho(a)) \le 0.5$ then $0.5 \le \rho\ (a, b) \le 1$

4 If $FND\ (\rho(a)) \in \text{Ned}(\rho, \boxed{L})$ then $FND(\rho(a)) \in NDFS(\rho, \boxed{L})$.

Proof.

1 Let $FND(\rho(a)) \ge 0.5$

$\Rightarrow \left(1 - \max_{b \in X}\right) \ge 0.5$ for all

$\Rightarrow (1 - \rho(a, b)) \ge 0.5$

$\Rightarrow 0.5 \ge \rho(a, b)$,

$\Rightarrow 0 \le \rho(a, b) \le 0.5$.

Thus, if $FND(\rho(a)) \ge 0.5$, then we have $0 \le \rho(a, b) \le 0.5$.

2 Let $FND(\rho(a)) \in \text{Int}(\rho, \boxed{L})$ then from (2) of Proposition 3.6 we have, $(FND(\rho(a))\boxed{L}\rho(a, b)) \le \overline{FND(\rho(a))}$. By the definition of $NDFS(\rho, \boxed{L})$.

We get, $FND(\rho(a)) \in NDFS(\rho, \boxed{L})$.

3 Let $FND(\rho(a)) \le 0.5$

$\Rightarrow \left(1 - \max_{b \in X}\{\rho(a, b)\}\right) \le 0.5$ for all

$\Rightarrow (1 - \rho(a, b)) \le 0.5$,

$\Rightarrow 0.5 \le \rho(a, b)$,

$\Rightarrow 0.5 \le \rho(a, b) \le 1$.

Thus, if $FND(\rho(a)) \le 0.5$, then we have $0.5 \le \rho(a, b) \le 1$.

4 Let $FND\ (\rho(a)) \in \text{Ned}(\rho, \boxed{L})$, then from (1) of Proposition 3.6 we have $\overline{(FND(\rho(a))\boxed{L}\rho(a, b))} \le FND(\rho(a))$. By the definition of $NDFS(\rho, \boxed{L})$.

We get $FND(\rho(a)) \in NDFS(\rho, \boxed{L})$.

Note. From (1) and (2) Proposition 3.9, we get for all $a \in X$, $FND(\rho(a))$ is the maximum element under composition $\boxed{L}$ and from (3) and (4) we have 0.5 is the maximum element under composition $\boxed{L}$.

Example 3.10. Let $G = (\mu, \rho)$ be a fuzzy graph, where $X = \{a, b, c\}$, $\mu: X \rightarrow [0, 1]$ and $\rho: X \times X \rightarrow [0, 1]$ with $\mu(a) = 0.5$, $\mu(b) = 0.6$, $\mu(c) = 0.4$, $\rho(a, b) = 0.3$, $\rho(b, c) = 0.4$ and $\rho(c, a) = 0.1$.

1 From the graph, we have, $FND(\rho) = \{0.7, 0.6, 0.6\}$.

Thus, the elements of $FND(\rho) \geq 0.5$ and $0 \leq \rho\ (a, b) \leq 0.5$.

2 If $FND(\rho) \in \text{Int}(\rho, \boxed{L})$.

Then, $(FND(\rho(a))\boxed{L}\rho(a, b)) = \max \max_{b \in X}\{FND(\rho(a)) + \rho(a, b) - 1, 0\}$, $\forall a \in X = \max$ max $\{0.7 + 0.3 - 1, 0\} = 0 \leq 0.3 = \overline{FND(\rho(a))}$.

Hence, we get $(FND(\rho(a))\boxed{L}\rho(a, b)) \leq FND(\rho(a))$.

Similarly we prove that all the vertices of the given graph.

Thus, we have $FND(\rho(a)) \in NDFS\ (\rho, \boxed{L})$.

Next example shows the axioms (3) and (4) of Proposition 3.9.

Example 3.11. Let $G = (\mu, \rho)$ be a fuzzy graph, where $X = \{a, b, c\}$, $\mu: X \rightarrow [0, 1]$, $\rho: X \times X \rightarrow [0, 1]$ with $\mu(a) = 0.6$, $\mu(b) = 0.8$, $\mu(c) = 0.9$, $\rho(a, b) = 0.5$, $\rho(b, c) = 0.7$ and $\rho(c, a) = 0.6$.

3 From the graph, we get, $FND(\rho) = \{0.4, 0.3, 0.3\}$.

Thus, the elements of $FND(\rho) \leq 0.5$ and $0.5 \leq \rho(a, b) \leq 1$.

4 If $FND(\rho) \in \text{Ned}(\rho, \boxed{L})$.

$\overline{(FND(\rho(a))}\boxed{L}\rho(a, b)) = \max \max_{b \in X}\{(1 - FND\{\rho(a)\}) + \rho(a, b) - 1, 0\}$, $\forall a \in X = \max$ max $\{(1 - 0.4)\ 0.5 - 1, 0\} = 0.1 \leq 0.4 = \overline{FND(\rho(a))}$.

Thus, we have $\overline{(FND(\rho(a))}\boxed{L}\rho(a, b)) \leq FND(\rho(a))$.

Similarly to prove all the vertices of the given graph.

Hence, we get $FND(\rho(a)) \in NDFS(\rho, \boxed{L})$.

Proposition 3.12. Let $G = (\mu, \rho)$ be a fuzzy graph without loops and with underlying set X where $\mu: X \rightarrow [0, 1]$ and $\rho: X \times X \rightarrow [0, 1]$, then the set $NDFS\ (\rho, \boxed{L})$ is a subset of the interval $[0, (0.5\boxed{L}\rho(a, b))]$ for all $a, b \in X$.

Proof. Let $a \in NDFS(\rho, \boxed{L})$.

Then, we get $(\mu(a)\boxed{L}\rho(a, b)) \le \overline{\mu(a)}$ and $(\mu(a)\boxed{L}\rho(a, b)) \le \overline{\mu(a)} \Rightarrow ((\mu(a)\underline{L}\overline{\mu(a)})$ $\boxed{L}\ \rho(a,b)) \le \overline{\mu(a)}$.

But, we have $(0.5\boxed{L}\rho(a, b)) \le ((\mu(a)\underline{L}\mu(a))\boxed{L}\rho(a, b)) \le \overline{\mu(a)} \Rightarrow \overline{(0.5\boxed{L}\rho(a, b))} \le \overline{\mu(a)} \Rightarrow \overline{(0.5\boxed{L}\rho(a, b))} \ge \mu(a)$. That is, $\mu(a) \le \overline{(0.5\boxed{L}\rho(a, b))}$.

Thus, we get $NDFS(\rho, \boxed{L})$ is a subset of the interval $[0, \overline{(0.5\boxed{L}\rho(a, b))}]$. □

Example 3.13. Let $G = (\mu, \rho)$ be a fuzzy graph, where $X = \{a, b, c, d\}$, $\mu: X \to [0, 1]$ and $\rho: X \times X \to [0, 1]$ with $\mu(a) = 0.5$, $\mu(b) = 0.6$, $\mu(c) = 0.9$, $\mu(d) = 1$, $\rho(a, b) = 0.4$, $\rho(b, c) = 0.6$, $\rho(c, d) = 0.8$, $\rho(d, a) = 0.4$ and $\rho(d, b) = 0.7$.

Consider the edge ab, if $a \in NDFS(\rho, \boxed{L})$.

Then, we get $(\overline{\mu(a)}\boxed{L}\rho(a, b)) = 0$ and $\mu(a) = 0.5$. Thus, $(\overline{\mu(a)}\boxed{L}\rho(a, b)) \le \overline{\mu(a)}$. Also, we have $(\mu(a)\boxed{L}\rho(a, b)) \le \overline{\mu(a)}$.

Similarly, we prove all the vertices of given graph.

Now,

$$\begin{aligned}\left((\mu(a)\underline{L}\overline{\mu(a)})\boxed{L}\rho(a, b)\right) &= \left[\min\{0.5 + 0.4, 1\}\boxed{L}0.4\right] \\ &= \left[0.9\boxed{L}0.4\right] = 0.3 \le 0.5 = \overline{\mu(a)}\end{aligned} \tag{4}$$

But, we have

$$\left(0.5\boxed{L}\rho(a, b)\right) = (0.5\boxed{L}0.4) = 0 \tag{5}$$

From (4) and (5), we have $(0.5\boxed{L}\rho(a, b)) \le ((\mu(a)\underline{L}\overline{\mu(a)})\boxed{L}\rho(a, b)) \le \overline{\mu(a)}$. Thus, we get $\mu(a) \le \overline{(0.5\boxed{L}\rho(a, b))}$. Hence, the set $NDFS(\rho, \boxed{L})$ is a subset of the interval $[0, \overline{(0.5\boxed{L}\rho(a, b))}]$.

4 Conclusions

We have discussed the set of non-dominated fuzzy subset $NDFS(\rho, \boxed{L})$ of fuzzy graphs. This set is the intersection of the two sets $\text{Ned}(\rho, \boxed{L})$ and $\text{Int}(\rho, \boxed{L})$. Further, we have shown that $NDFS(\rho, \boxed{L})$ is sub-weak lattice of the weak lattice $(\mu(X), \overline{L}, \underline{L})$ under $\overline{L}$, but not a sub-weak lattice of a weak lattice $(\mu(X), \overline{L}, \underline{L})$. Finally, we have introduced the fuzzy set of non-dominated vertices in fuzzy graphs $FND(\rho)$ and we obtain the relation between the sets $FND(\rho)$ and $NDFS(\rho, \boxed{L})$ by applying some conditions.

References

Alaui, A. (1999) 'On fuzzification of some concepts of graph', *Fuzzy Sets and Systems*, Vol. 101, No. 3, pp.363–389.

Basheer Ahamed, M. and Ibrahim, A. (2009) 'Internally stable vertex set of fuzzy graphs', *Proceedings of National Conference on Discrete and Fuzzy Mathematics*, Vol. 1, No. 1, pp.7–13.

Basheer Ahamed, M. and Ibrahim, A. (2010) 'Logical operators and weak lattice in fuzzygraphs', *International Journal of Algorithms, Computing and Mathematics*, Vol. 3, No. 2, pp.41–47.

Kitainik, L. (1987) 'Fuzzy inclusion and fuzzy dichotomous decision procedures', in Kacprzyk, J. and Orlovski, A. (Eds.): *Optimization Models Using Fuzzy Sets and Possibility Theory*, pp.154–170, Springer-Verlag, Berlin Heidelberg.

Kitainik, L. (1993) *Fuzzy Decision Procedures with Binary Relations – Theory and Applications*, Kluwer Academic Publishers, Dordrecht.

Rosenfeld, A. (1975) 'Fuzzy graphs', in Zadeh, L.A., Fu, K.S. and Shimura, M. (Eds.): *Fuzzy Sets and their Applications*, pp.77–95, Academic press, New York.

Somasundaram, A. (2004) 'Domination in fuzzy graph-II', *Journal of Fuzzy Maths*, Vol. 20.

Somasundaram, A. and Somasundaram, S. (1998) 'Domination in fuzzy graphs-I', *Pattern Recognition Letters*, Vol. 19, No. 9, pp.787–791.

Tremblay, J.P. and Manohar, R. (1997) *Discrete Mathematical Structures with Applications to Computer Science*, McGraw-Hill Publishing Company Limited, New Delhi.

Zadeh, L.A. (1965) 'Fuzzy sets', *Inform and Control*, Vol. 8, No. 3, pp.338–353.