The $b$-chromatic number of Mycielskian of some graphs

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Abstract: A $b$-colouring of a graph $G$ is a proper colouring of the vertices of $G$ such that there exists a vertex in each colour class joined to at least one vertex in each other colour classes. The $b$-chromatic number of a graph $G$, denoted by $\phi(G)$ is the largest integer $k$ such that $G$ has a $b$-colouring with $k$ colours. The Mycielskian or Mycielski graph $\mu(H)$ of a graph $H$ with vertex set $\{v_1, v_2, \ldots, v_n\}$ is a graph $G$ obtained from $H$ by adding $n + 1$ new vertices $\{u, u_1, u_2, \ldots, u_n\}$, joining $u$ to each vertex $u_i$ $(1 \leq i \leq n)$ and joining $u_i$ to each neighbour of $v_j$ in $H$. In this paper, we obtain the $b$-chromatic number of Mycielskian of paths, complete graphs, complete bipartite graphs and wheels.

Keywords: $b$-chromatic number; $b$-colouring; $b$-dominating set; Mycielskian.


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This paper is a revised and expanded version of a paper entitled ‘The $b$-chromatic number of Mycielskian of paths’ presented at International Conference on Mathematical and Computational Sciences (ICMACS-2015), Don Bosco College Angadidakavu, Kannur, Kerala, 22–24 January 2015.
1 Introduction

The concept of b-chromatic number of a graph $G$ is introduced by Irving and Manlove in 1999. They have defined the b-chromatic number $\phi(G)$ of a graph $G$ as the largest positive integer $k$ such that $G$ admits a proper $k$-colouring in which every colour class has a representative vertex which is adjacent to at least one vertex in each of the other colour classes and this representative vertex is known as the b-dominating vertex. It is clear that for a graph $G$ to have a b-colouring of $k$ colours, $G$ must contain at least $k$ vertices, each of degree at least $k - 1$. Bonomo et al. (2009) proved that $P_4$-sparse graphs (and, in particular, cographs) are b-continuous and b-monotonic. Besides, they described a dynamic programming algorithm to compute the b-chromatic number in polynomial time within these graph classes. El Sahili and Kouider (2006) studied the b-chromatic number of a d-regular graph of girth 5. Also, Sergio and Marcove (2011) discussed the b-chromatic number of regular graphs according to their girth and diameter. Javadi and Omoomi (2009) studied the b-colouring of Kneser graphs $K(n; k)$ and determine the b-chromatic number of $K(n; k)$ for some values of $n$ and $k$. Moreover, they proved that $K(n; 2)$ is b-continuous for $n \leq 17$. In 2007, Elghazel et al. discussed the applications of b-colouring in clustering. Vivin and Venkatachalam (2012) obtained the b-chromatic number of corona of two graphs with same number of vertices. In 2011, Eric et al. discussed the applications of corona of graphs in network design and analysis. In 2013, Venkatachalam and Vivin obtained the b-chromatic number of windmill graph. In 2014, Venkatachalam and Vivin discussed the b-chromatic number for the central graph, middle graph, total graph and line graph of double star graph $K_{1,n,n}$. Also, they have found a relationship between the b-chromatic number and three other colouring parameters, the equitable chromatic number, harmonious chromatic number and the achromatic number. In 2014, Vivin and Venkatachalam (2014a) obtained the b-chromatic number of middle and total graph of fan graph. In 2014, Vivin and Venkatachalam (2014b) discussed the b-chromatic number for the sun let graph $S_n$, line graph of sun let graph $L(S_n)$, middle graph of sun let graph $M(S_n)$, total graph of sun let graph $T(S_n)$, middle graph of wheel graph $M(W_n)$ and the total graph of wheel graph $T(W_n)$. In 2012, Vijayalakshmi and Thilagavathi (2012b) obtained the b-chromatic number of corona product of path, cycle and star graph with complete graph, the strong product of path with cycle and Cartesian product of cycles. In 2012, Vijayalakshmi and Thilagavathi (2012a) discussed the b-chromatic number of transformation graph $G^{\leftrightarrow}$ for cycle, path and star graph. Also, they have discussed the b-chromatic number of corona product of path graph with cycle and path graph with complete graph along with its structural properties. The Mycielskian or Mycielski graph $\mu(H)$ of a graph $H$ with vertex set $\{v_1, v_2, ..., v_n\}$ is a graph $G$ obtained from $H$ by adding $n + 1$ new vertices $\{u, u_1, u_2, ..., u_n\}$, joining $u$ to each vertex $u_i$ ($1 \leq i \leq n$) and joining $u_i$ to each neighbour of $v_i$ in $H$. In 2001, Massimiliano et al. discussed the applications of Mycielski graphs in multiprocessor task scheduling problem. In this paper, we obtain the b-chromatic number of Mycielskian of paths, complete graphs, complete bipartite graphs and wheels.
Example 1.1:

Figure 1  A b-colouring of a graph with three colours

The Mycielskian or Mycielski graph of a graph \( G \) is an important concept in the area of graph theory. It is denoted by \( \mu(G) \) and is defined as follows:

**Definition 1.2** (Chartrand and Zhang, 2009): The Mycielskian or Mycielski graph \( \mu(H) \) of a graph \( H \) with vertex set \( \{v_1, v_2, ..., v_n\} \) is a graph \( G \) obtained from \( H \) by adding \( n + 1 \) new vertices \( \{u, u_1, u_2, ..., u_n\} \), joining \( u \) to each vertex \( u_i \) \((1 \leq i \leq n)\) and joining \( u_i \) to each neighbour of \( v_i \) in \( H \).

The Mycielski graph has an application in multiprocessor task scheduling problem. If we let \( M_1 \) be the Mycielski graph with two nodes and one single edge, \( M_i \) is recursively defined as \( M_{i+1} = \mu(M_i) \). The minimum number of processors such that the intersection graph of the processor task requirements is the given Mycielski graph \( M_i \) is exactly \(|E_i|\), where \( E_i \) is the edge set of \( M_i \). Also, the minimum completion time \( C_{\text{max}} \) of the multiprocessor task max scheduling problem obtained from \( M_i \) is equal to \( \chi(M_i) \), where \( \chi(M_i) \) is the chromatic number of \( M_i \) (Massimiliano and Paolo, 2001).

Example 1.3:

Figure 2  Mycielskian of the path \( P_5 \)

2 Main results

In this section, we discuss the b-chromatic number of Mycielskian of paths, complete graphs, complete bipartite graphs and wheels. These results are proved using the
m-degree \( m(G) \) of a graph \( G \), which is an upper bound for the b-chromatic number \( \phi(G) \) as suggested in Irving and Manlove (1999).

**Definition 2.1** (Irving and Manlove, 1999): For a graph \( G = (V, E) \), suppose that the vertices of \( G \) are ordered \( v_1, v_2, \ldots, v_n \) so that \( d(v_1) \geq d(v_2) \geq \ldots = d(v_n) \). The m-degree, \( m(G) \), of \( G \) is defined by

\[
m(G) = \max \{ i : d(v_i) = i - 1 \}.
\]

**Theorem 2.2** (Irving and Manlove, 1999): For any graph \( G \), \( \phi(G) \leq m(G) \).

**Example 2.3**: 

**Figure 3** A b-colouring of Petersen graph with three colours

For a Petersen graph, \( m(G) = 4 \) and \( \phi(G) = 3 \) and thus \( \phi(G) < m(G) \).

Next, we discuss the b-chromatic number of Mycielskian of paths, complete graphs, complete bipartite graphs and wheels.

**Theorem 2.4**: The b-chromatic number of Mycielskian of a path \( P_n \) is

\[
\phi(\mu(P_n)) = \begin{cases} 
2; & \text{for } n = 1 \\
3; & \text{for } n = 2, 3, 4 \\
4; & \text{for } n = 5, 6, 7 \\
5; & \text{for } n \geq 8
\end{cases}
\]

**Proof**: Let the vertex set of \( P_n \) be \( \{v_1, v_2, \ldots, v_n\} \) and that of \( \mu(P_n) \) be \( \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\} \cup \{u\} \).

**Case 1**: \( 1 \leq n \leq 7 \)

- \( n = 1 \)
  
  Here, \( \mu(P_1) \) is a disconnected graph \( K_2 \cup K_1 \) with three vertices and one edge.
  
  Hence, it is clear that \( \phi(\mu(P_1)) = 2 \).

- \( n = 2 \)
  
  \( \mu(P_2) \) is a cycle with five vertices and \( \phi(C_5) = 3 \) (Sergio and Marcove, 2011). Thus \( \phi(\mu(P_2)) = 3 \).
• $n = 3$

In this case, a b-colouring with four colours is not possible, because here we have only three vertices with degree at least 3. A b-colouring with three colours is given in Figure 4.

**Figure 4** A b-colouring of $\mu(P_3)$ with three colours

• $n = 4$

In this case also, we prove that $\varphi(\mu(P_4)) = 3$. On the contrary, assume that $\varphi(\mu(P_4)) = 4$. Here, we have exactly five vertices, $v_2, v_3, u_2, u_3$ and $u$ each having degree at least 3. Among these five vertices, we can choose any of the four vertices as the b-dominating vertices. Suppose that the vertex $u$ is included in the set of b-dominating vertices and let the colour of $u$ be $c$. Since $u$ is adjacent to all the $u_i$'s, $1 \leq i \leq 4$, we cannot assign colour $c$ to any of the $u_i$'s, $1 \leq i \leq 4$. But to make the remaining three vertices b-dominating, we assign the colour $c$ to some $v_i$'s. But if we assign colour $c$ to any of the $v_i$'s; $1 \leq i \leq 4$, $u_2$ and $u_1$ will have two neighbours with same colour. But the degree of $u_2$ and $u_3$ is exactly 3. So to make these vertices b-dominating, the three neighbours should receive distinct colours. Thus if assign colour $c$ to any of the $v_i$'s; $1 \leq i \leq 4$, $u_2$ and $u_3$ will not become b-dominating. If $u$ is not included in the set of b-dominating vertices, then to make $u_2$ and $u_3$ b-dominating we assign colour of $v_2$ or colour of $v_3$ to $u$. Hence in this case, either $u_2$ or $u_3$ will have two neighbours with same colour. So they will not become b-dominating. Thus a b-colouring with four colours is not possible here. A b-colouring with three colours is given in Figure 5.

**Figure 5** A b-colouring of $\mu(P_4)$ with three colours
Here, we prove that \( \varphi(\mu(P_5)) = 4 \). On the contrary, assume that \( \varphi(\mu(P_5)) = 5 \). Then, there will be at least five vertices each with degree at least 4. But, here we have only four vertices each with degree at least 4. Hence, a b-colouring with five colours is not possible. A b-colouring with four colours is obtained in Figure 6.

**Figure 6** A b-colouring of \( \mu(P_5) \) with four colours

Here we prove that \( \varphi(\mu(P_6)) = 4 \). On the contrary assume that \( \varphi(\mu(P_6)) = 5 \). Here, we have exactly five vertices \( v_2, v_3, v_4, v_5 \) and \( u \) each with degree at least 4. So, we can assign five different colours to these vertices. Suppose that the colour of \( u = c_1 \) and colour of \( v_i = c_{i+1} \), \( 1 \leq i \leq 4 \). Since all the \( u_i \)'s are adjacent to \( u \), we cannot assign colour \( c_1 \) to any of the \( u_i \)'s. So the vertices \( v_1 \) and \( v_4 \) are not adjacent to a vertex with colour \( c_1 \). Hence, these vertices will not become b-dominating. Thus, a b-colouring with five colours is not possible here. A b-colouring with four colours is obtained in Figure 7.

**Figure 7** A b-colouring of \( \mu(P_6) \) with four colours

Here, we prove that \( \varphi(\mu(P_7)) = 4 \). On the contrary, assume that \( \varphi(\mu(P_7)) = 5 \). Here, we have six vertices \( v_2, v_3, v_4, v_5, v_6 \) and \( u \) each with degree at least 4. We suppose that \( u \) is not a b-dominating vertex. Then \( v_2, v_3, v_4, v_5 \) and \( v_6 \) will be the b-dominating vertices, each with degree exactly 4. We suppose that colours of these vertices are \( c_1, c_2, c_3, c_4 \) and \( c_5 \) respectively. Here, \( u_4 \) is a common neighbour of \( v_1 \) and \( v_4 \). So, we cannot assign colours \( c_2 \) and \( c_4 \) to \( u_4 \). If we assign any colour from the remaining colours, then either \( v_1 \) or \( v_4 \) will have two neighbours with same colour. Hence, they will not become b-dominating. Next, we suppose that \( u \) is included in the set of b-dominating vertices. Let \( c_1 \) be the colour of \( u \). Since each \( u_i \) is adjacent to \( u \), we
cannot assign colour $c_1$ to any of the $u_i$'s, $1 \leq i \leq 7$. The vertices $v_1$ and $v_7$ have degree 2. But, here we are constructing a $b$-colouring with five colours. So each $b$-dominating vertex should have degree at least 4. Hence the remaining four $b$-dominating vertices will be from the set $\{v_2, v_3, v_4, v_5, v_6\}$. To make these four vertices $b$-dominating, we should assign colour $c_1$ to at least one of their neighbour. If we assign colour $c_1$ to $v_1, v_3$ and $v_5$, the vertices $v_2, v_6, v_3$ and $v_3$ become adjacent to a vertex having colour $c_1$. Thus we will get four vertices, which are adjacent to the vertex with colour $c_1$. So, we select $v_2, v_3, v_5$ and $v_6$ as the remaining four $b$-dominating vertices and assign colours $c_2, c_3, c_4$ and $c_5$ respectively. Now, each $b$-dominating vertex is adjacent to a vertex with colour $c_1$. Now, we consider the vertex $v_3$. Colour of $v_3$ is $c_3$. We consider the neighbours of $v_3$. The colour of $v_2$ is $c_2$ and $v_4$ is $c_1$. Now, the options for the other neighbours $u_2$ and $u_4$ of $v_3$ are $c_4$ and $c_5$. In this case, $v_3$ will have two neighbours with colour $c_2$. Hence, $v_3$ will not become $b$-dominating. Thus, in either case, a $b$-colouring with five colours is not possible and a $b$-colouring with four colours is given in Figure 8.

Case 2: $n \geq 8$

In this case, we prove that $\phi(\mu(P_n)) = 5$, $n \geq 8$. A $b$-colouring with six colours is not possible, because here we have only one vertex with degree more than 4. A $b$-colouring with five colours is obtained in Figure 9 for $\mu(P_8)$.

For $n \geq 8$, we can choose the colouring which is used for $\mu(P_8)$ and colour the remaining uncoloured vertices can be coloured properly as follows.

1. If $n$ is an odd number

We assign colour $c_1$ to $\{v_9, v_{11}, v_{13}, ..., v_n\}$ and $\{u_9, u_{11}, u_{13}, ..., u_n\}$ and colour $c_2$ to $\{v_{10}, v_{12}, v_{14}, ..., v_{n-1}\}$ and $\{u_{10}, u_{12}, u_{14}, ..., u_{n-1}\}$.
• If \( n \) is an even number

We assign colour \( c_1 \) to \{\( v_9, v_{11}, v_{13}, \ldots, v_{n-1} \)\} and \{\( u_9, u_{11}, u_{13}, \ldots, u_{n-1} \)\} and colour \( c_2 \) to \{\( v_{10}, v_{12}, v_{14}, \ldots, v_n \)\} and \{\( u_{10}, u_{12}, u_{14}, \ldots, u_n \)\}.

Thus, we obtain a proper b-colouring with five colours. \( \square \)

**Theorem 2.5:** The b-chromatic number of Mycielskian of a complete bipartite graph \( Kn, m \) is 

\[
\phi(\mu(K_{n,m})) = 3, \quad \forall n, m \geq 1.
\]

**Proof:** Let the vertex set of \( K_{n,m} \) be \{\( v_1, v_2, v_3, \ldots, v_{n+m} \)\}, where each of the \( v_i \)'s are adjacent to all the \( v_j \)'s; \( 1 \leq i \leq n, n + 1 \leq j \leq n + m \). Let the vertex set of Mycielskian of \( K_{n,m} \) be \( V(\mu(K_{n,m})) = V(K_{n,m}) \cup \{u, u_i; 1 \leq i \leq n + m\} \). Here, each \( u \) is adjacent to all the neighbours of \( v_i; 1 \leq i \leq n + m \) and \( u \) is adjacent to all the \( u_i \)'s (\( 1 \leq i \leq n + m \)). Now, a b-colouring with three colours can be obtained as given in Figure 10 by assigning colour \( c_1 \) to \( v_1, v_2, \ldots, v_n \), colour \( c_2 \) to \( u_1, u_2, \ldots, u_{n+m} \) and colour \( c_3 \) to \( v_{n+1}, v_{n+2}, \ldots, v_{n+m} \) and \( u \).

Next, we prove that a b-colouring with more than four colours does not exist. On the contrary, we assume that there exists a b-colouring with four colours. Now, we partition the vertex set of \( \mu(K_{n,m}) \) into four parts, \( P_1 = \{v_i; 1 \leq i \leq n\}, P_2 = \{v_j; n + 1 \leq j \leq n + m\}, P_3 = \{u_i; 1 \leq i \leq n\} \) and \( P_4 = \{u_j; n + 1 \leq j \leq n + m\} \).

We assume that, we have selected two b-dominating vertices from a single partite set. Let it be \( v' \) and \( v'' \) with colours \( c_1 \) and \( c_2 \) respectively.

**Figure 10** A b-colouring of \( \mu(K_{n,m}) \) with three colours
Since $v'$ and $v''$ have same neighbours, we cannot assign colour $c_1$ to a neighbour of $v''$ and colour $c_2$ to a neighbour of $v'$. Thus, the vertex $v'$ will not be adjacent to the colour $c_2$ and similarly the vertex $v''$ will not be adjacent to the colour $c_1$. Thus, these two vertices will not become b-dominating vertices. Thus the maximum number of b-dominating vertices that can be selected from each partite set is one. Next, we assume that there exists a b-colouring with four colours. Let $B$ be the set of b-dominating vertices. Here either $u \in B$ or $u \notin B$. We assume that $u \in B$ and we assign colour $c_1$ to $u$. Thus, the remaining three b-dominating vertices will be from any of the four partite sets. But in this case, we can select only one b-dominating vertex from the partite sets $P_1$ and $P_2$. The reason is as follows. Suppose that we have selected two b-dominating vertices $v_i$ and $v_j$ from $P_1$ and $P_2$ respectively and assign colours $c_2$ and $c_3$ to $v_i$ and $v_j$. To become a b-dominating vertex, these two vertices should be adjacent to a vertex with colour $c_1$. Since $u$ is adjacent to all the vertices in $P_1$ and $P_2$, colour $c_1$ cannot be assigned to any of the vertices in these two partite sets. So, we assign colour $c_1$ to one of the vertices in $P_1$. Thus, the vertex $v_j$ will be adjacent to a vertex with colour $c_1$. Now, we cannot assign colour $c_1$ to any of the remaining uncoloured vertices in $\mu(K_{m,n})$. Thus the vertex $v_j$ will not be adjacent to a vertex with colour $c_1$ and hence it will not become a b-dominating vertex. Thus, from $P_1$ and $P_2$ we can select only one b-dominating vertex. Without loss of generality, we select the three b-dominating vertices from $P_1$, $P_2$ and $P_4$. Let it be $v_i$, $u_k$ and $u_l$ and assign colours $c_2$, $c_3$ and $c_4$ respectively to these vertices. We consider the vertex $v_i$ having colour $c_2$. This vertex is adjacent to $u_k$ having colour $c_4$. The vertex $v_i$ become a b-dominating vertex if it is adjacent to vertices having colour $c_1$ and $c_3$. Since we cannot assign colour $c_3$ to any of the vertices in $P_2$, we assign colour $c_3$ to an uncoloured vertex in $P_4$. But now we cannot assign colour $c_3$ to any of the neighbours of $u_l$. Thus $u_l$ will not become a b-dominating vertex. Hence in this case a b-colouring with four colours is not possible.

Next, we assume that $u \notin B$. If so, four b-dominating vertices will be from the four partite. Let $v_i$, $v_j$, $u_k$ and $u_l$ be the four b-dominating vertices from $P_1$, $P_2$, $P_3$ and $P_4$ respectively. We assign colours $c_1$, $c_2$, $c_3$ and $c_4$ to $v_i$, $v_j$, $u_k$ and $u_l$ respectively. We consider the vertex $u_i$. It is not adjacent to the vertex with colour $c_2$. Since all the vertices in $P_1$ is adjacent to $v_i$, we cannot assign colour $c_2$ to any of the vertices in $P_2$. So, we assign the colour $c_2$ to $u_i$. Next, we consider the vertex $u_k$. Here all the uncoloured neighbours of $u_k$ are adjacent to $v_j$. So, we cannot assign colour $c_1$ to any of its neighbours and hence the vertex $u_k$ is not adjacent to a vertex with colour $c_1$. Thus the vertex $u_k$ will not become a b-dominating vertex. So in this case a b-colouring with four colours is not possible.

**Corollary 2.6:** The b-chromatic number of the Mycielskian of a star graph $K_{1,n}$ is

$$\varphi(\mu(K_{1,n})) = 3$$

for all $n$.

**Theorem 2.7:** The b-chromatic number of a Mycielskian of a complete graph $K_n$ is $n + 1$.

**Proof:** Let the vertex set of the complete graph $K_n$ be $\{v_1, v_2, \ldots, v_n\}$ and that of $\mu(K_n)$ be $V(K_n) \cup \{u_1, u_2, \ldots, u_n, u\}$, where each $u_i$ neighbours of $v_i$, $1 \leq i \leq n$ and the vertex $u$ is adjacent to all the $u_i; 1 \leq i \leq n$. Thus, the Mycielskian of a complete graph $K_n$ consists of $n + 1$ vertices with degree $n$ and $n$ vertices with degree $2n - 2$. A b-colouring with $n + 1$ colours can be obtained by assigning colour $c_i$ to $v_i; 1 \leq i \leq n$, $c_{i+1}$ to $u_i; 1 \leq i \leq n$, and $c_1$
to $u$. By Theorem 2.2, $\phi(\mu(K_n)) \leq m(\mu(K_n)) \leq n + 1$. Since we have already established a $b$-colouring with $n + 1$ colours, $\phi(\mu(K_n)) = n + 1$. □

**Theorem 2.8:** The $b$-chromatic number of the mycielskian of a wheel graph $W_{n+1}$ is

$$
\phi(\mu(W_{n+1})) = \begin{cases} 
5; & \text{for } n = 3, 5 \\
4; & \text{for } n = 4 \\
6; & \text{for } n = 6, 8 \\
7; & \text{for } n = 7 \text{ and } n \geq 9
\end{cases}
$$

**Proof:** Let $W_{n+1}$ be a wheel graph with $n + 1$ vertices, say $v_1, v_2, \ldots, v_{n+1}$ with $v_{n+1}$ as the central vertex with $\deg(v_{n+1}) = n$. Let $\mu(W_{n+1})$ denote the Mycielskian of $W_{n+1}$ with vertex set $V(W_{n+1}) \cup \{u, u_1, u_2, \ldots, u_{n+1}\}$, where $u$ is adjacent to all the $u_i$'s; $1 \leq i \leq n + 1$. Hence, $\mu(W_{n+1})$ contains $n$ vertices with degree 6, $n$ vertices with degree 4, one vertex with degree 2 and two vertices with degree $n$. Thus by Theorem 2.2, $\phi(\mu(W_{n+1})) \leq 7$.

- **$n = 3$**

  Assign colour $i$ to $v_i$, $1 \leq i \leq 4$ and colour 5 to $u_i$, $1 \leq i \leq 4$ and colour 4 to $u$. By Theorem 2.2, $\phi(\mu(W_3)) = m(\mu(W_3)) = 5$. Since we have already established a $b$-colouring with five colours, $\phi(\mu(W_3)) = 5$.

**Figure 11** A $b$-colouring of $\mu(W_4)$ with five colours

- **$n = 4$**

  Here $\mu(W_4) \setminus \{v_4, u_4\} = \mu(C_4)$. So first we prove that $\phi(\mu(C_4)) = 3$ and after that we find $\phi(\mu(W_4))$. We consider $\mu(C_4)$. Let $B$ be a set of $b$-dominating vertices. Let $N(v)$ denote the neighbourhood of $v$. Here $N(v_1) = N(v_3), N(v_2) = N(v_4), N(u_1) = N(u_3)$ and $N(u_2) = N(u_4)$. Thus, from each set $\{v_1, v_3\}, \{v_2, v_4\}, \{u_1, u_3\}$ and $\{u_2, u_4\}$ we can select at most one vertex to the set $B$. Here first we prove that $\phi(\mu(C_4)) = 3$. On the contrary, we assume that $\phi(\mu(C_4)) = 4$. Let $u \notin B$. Here we select one vertex from each of the set $\{v_1, v_3\}, \{v_2, v_4\}, \{u_1, u_3\}$ and $\{u_2, u_4\}$. Let it be $v_i, v_{j+1}, u_j$ and $u_{j+1}$, where $i + k = i + k \mod 4$ and $j + k = j + k \mod 4$ for $i + k > 4$ and $j + k > 4$ $(1 \leq i, j, k \leq 4)$. Note that $v_i$ is adjacent to $u_j$ or $u_{j+1}$ but not to both. Also note that $v_i$ and $v_{j+1}$ are adjacent to each other but these two vertices does not have any common neighbour. Without loss of generality we assume that $v_i$ is adjacent to $u_j$ and $v_{j+1}$ is adjacent to $u_{j+1}$. Also we assign colours $c_1, c_2, c_3$ and $c_4$ to $v_i, v_{j+1}, u_j$ and $u_{j+1}$ respectively. Now we consider $u_j$ and $u_{j+1}$. $u_j$ is having colour $c_3$ and is adjacent to the vertex $v_i$ with colour $c_1$. To become a $b$-dominating vertex it should be adjacent to
the vertices having colour $c_2$ and $c_4$. The remaining uncoloured neighbours of $u_j$ are $v_{i+1}$ and $u$. Since $v_{i+1}$ is adjacent to $v_{i+2}$, we cannot assign colour $c_2$ to $v_{i+2}$. So the only choice is to assign colour $c_2$ to $u$. Since $u_{i+1}$ is having only three neighbours and we are constructing a $b$-colouring with four colours, the three neighbours of $u_{i+1}$ should receive distinct colours. But if we assign colour $c_2$ to $u$, the vertex $u_{i+1}$ will have two neighbours with colour $c_2$. Thus, the vertex $u_{i+1}$ will not become a $b$-dominating vertex. Next, we assume that $u \in B$. Assume that we have selected a vertex $u_j$ from the set $\{u_i; 1 \leq i \leq 4\}$ and two adjacent vertices $v_i$ and $v_{i+1}$ from the set $\{v_i; 1 \leq i \leq 4\}$. We assign colour $c_1$, $c_2$, $c_3$ and $c_4$ to $u$, $u_j$, $v_i$ and $v_{i+1}$ respectively. The vertex $u_j$ will be adjacent to either $v_i$ or $v_{i+1}$ but not to both. Without loss of generality we assume that $u_j$ is adjacent to $v_i$. Now, $u_j$ is adjacent to vertices having colour $c_1$ and $c_3$. To become a $b$-dominating vertex it should be adjacent to a vertex having colour $c_4$. Now the only uncoloured neighbour of $u_j$ is $v_{i+2}$. But since $v_{i+1}$ is adjacent to $v_{i+2}$, we cannot assign colour $c_4$ to this vertex. Thus, in this case $u_j$ will not become a $b$-dominating vertex. Next, we consider that we have selected two vertices $u_j$ and $u_{j+1}$ from the set $\{u_j; 1 \leq j \leq 4\}$ and one vertex $v_i$ from the set $\{v_i; 1 \leq i \leq 4\}$. As in the last case here also we cannot construct a $b$-colouring with four colours. A $b$-colouring with three colours can be obtained as shown in Figure 12(a). Thus the $b$-chromatic number of $\mu(C_4)$ is 3. Next, we consider $\mu(W_5) = \mu(C_4) \cup \{v_5, u_5\}$ such that the vertex $v_5$ is adjacent to $u_i$ and $v_i; 1 \leq i \leq 4$ and $u_5$ is adjacent to all the $v_i$’s and $u$. Here we prove that $\mu(W_5) = 4$. So, we colour the $u_i$’s, $v_i$’s and the vertex $u$ in the case of $\mu(C_4)$ for $1 \leq i \leq 4$ and select the $b$-dominating vertices as in $\mu(C_4)$. Since our aim is to construct a $b$-colouring with four colours, we assign a new colour $c_4$ to $v_5$. We cannot assign a new colour to the vertex $u_5$, because it is not adjacent to any of the $u_i$’s; $1 \leq i \leq 4$ and $v_5$. So we assign any colour from the list $\{c_1, c_2, c_3, c_4\}$ in a proper way. Thus we obtained a four $b$-colouring for $\mu(W_5)$.

Figure 12  A $b$-colouring of $\mu(C_4)$ and $\mu(W_5)$ with three and four colours

- $n = 5$

In this case we prove that the $b$-chromatic number of $\mu(W_{n+1}) = 5$. On the contrary, we assume that $\phi(\mu(W_{n+1})) = 7$. Here we have exactly eight vertices $\{u, u_5, v_i; 1 \leq i \leq 6\}$ with degree at least 6. From this set of eight vertices, we can select any of
the seven vertices as the b-dominating vertices. Out of these seven vertices, at least four of them will be from the set \( \{v_i; 1 \leq i \leq 5\} \). We assume that we have selected any four vertices from the set \( \{v_i; 1 \leq i \leq 5\} \) and the vertices \( v_6, u_6, u \) as the b-dominating vertices. We assign seven distinct colours to these vertices. Since \( u \) is not adjacent to \( v_6 \), any of the neighbours of \( u \) should receive colour of \( v_6 \). But all the neighbours of \( u \) is adjacent to \( v_6 \). So \( u \) will not be adjacent to a vertex having colour of \( v_6 \). Thus the vertex \( u \) will not become b-dominating. Next, we assume that we have selected all the five vertices from the set \( \{v_i; 1 \leq i \leq 5\} \) and any two vertices from the set \( \{v_6, u_6, u\} \) as the b-dominating vertices. We assign colour \( c_7 \) to \( v_i \) for \( 1 \leq i \leq 5 \). Since \( v_6 \) and \( u_6 \) common neighbours of \( \{v_i; 1 \leq i \leq 5\} \), we assign colour \( c_6 \) and \( c_5 \) to \( v_6 \) and \( u_6 \) respectively. The degree of each \( v_i; 1 \leq i \leq 5 \) is exactly 6. So to become a b-dominating vertex, the six neighbours of these five vertices should receive distinct colours. We consider the vertex \( v_3 \). \( v_3 \) is adjacent to vertices having colours \( c_2, c_4, c_6 \) and \( c_5 \). To become a b-dominating vertex, \( v_3 \) should be adjacent to vertices having colours \( c_1 \) and \( c_3 \). Here \( u_6 \) and \( u_4 \) are the only uncoloured neighbours of \( v_3 \). Since \( v_2 \) is adjacent to \( v_1 \) and \( u_4 \) is adjacent to \( v_5 \), the only choice is to assign colour \( c_1 \) to \( u_4 \) and \( c_3 \) to \( u_5 \). But if we colour like this, the vertices \( v_1 \) and \( v_5 \) will have two neighbours with same colour. Thus they will not become b-dominating. Thus a b-colouring with seven colours is not possible here.

Next, we will check the existence of a b-colouring with six colours. Let \( B \) be a set of b-dominating vertices. We consider the set \( \{u, u_6, v_6\} \). Out of these three vertices, only two belongs to \( B \). On the contrary we assume that all the three vertices, \( u, u_6 \) and \( v_6 \) are included in \( B \) with colours \( c_1, c_2 \) and \( c_3 \) respectively. We consider the neighbours of \( u \). We can see that all the neighbours of \( u \) are adjacent to \( v_6 \) except \( u_6 \). Since \( u_6 \in B \), we cannot assign colour \( c_5 \) to any of the neighbours of \( u \). Thus, \( u \) will not become a b-dominating vertex. Thus from the set \( \{u, u_6, v_6\} \) at most two vertices belongs to \( B \) and the remaining b-dominating vertices will be from the set \( \{v_i; 1 \leq i \leq 5\} \). We suppose that we have selected two vertices from the set \( \{u, u_6, v_6\} \) and four vertices from the set \( \{v_i; 1 \leq i \leq 5\} \). Without loss of generality we assume that the four b-dominating vertices from the set \( \{v_i; 1 \leq i \leq 5\} \) are \( v_1, v_2, v_3 \) and \( v_4 \). The two b-dominating vertices from the set \( \{u, u_6, v_6\} \) can be either \( u \) and \( u_6 \) or \( u \) and \( v_6 \) or \( u_6 \) and \( v_6 \). First, we assume that the two b-dominating vertices are \( v_6 \) and \( u_6 \) and we assign colours \( c_1 \) and \( c_2 \) respectively. For the remaining four b-dominating vertices \( v_1, v_2, v_3 \) and \( v_4 \), we assign colours \( c_3, c_4, c_5 \) and \( c_6 \) respectively. We consider the vertex \( v_1 \). \( v_1 \) is having colour \( c_3 \) and is adjacent to the colours \( c_1, c_2 \) and \( c_4 \). To become a b-dominating vertex, \( v_1 \) should be adjacent to the vertices having colours \( c_1 \) and \( c_4 \). So we assign colours \( c_3 \) and \( c_5 \) to \( v_1 \) and \( u_2 \) respectively. Next, we consider the vertex \( v_2 \). \( v_2 \) is having colour \( c_1 \) and is adjacent to the vertices with colours \( c_5 \) and \( c_6 \). So we assign colours \( c_2 \) and \( c_4 \) to \( v_3 \) and \( u_5 \) respectively. To become a b-dominating vertex, \( v_3 \) should be adjacent to a vertex having colour \( c_6 \). So we assign colour \( c_5 \) to \( u_4 \). Now we consider the vertex \( v_4 \). The vertex \( v_4 \) is having colour \( c_6 \) and is adjacent to the vertices with colours \( c_1, c_2 \) and \( c_5 \). To become a b-dominating vertex, \( v_4 \) should be adjacent to vertices having colours \( c_1 \) and \( c_4 \). So we assign colours \( c_3 \) and \( c_4 \) to \( u_1 \) and \( u_3 \) respectively. Now, we consider the vertex \( v_6 \). This vertex is not adjacent to the vertex with colour \( c_2 \). So we have to assign colour \( c_2 \) any of the uncoloured neighbours of \( v_6 \). But to make the vertices \( v_1, v_2, v_3 \) and \( v_4 \) b-dominating, we assign colours to all the
uncoloured neighbours of $v_6$. Thus we cannot assign colour $c_2$ to any of the neighbours of $v_6$. Thus the vertex $v_6$ will not become a b-dominating vertex. Next we assume that the two b-dominating vertices from the set \{u, u_6, v_6\} are $u$ and $v_6$. We assign colours to $v_i$’s and $u_i$’s; $1 \leq i \leq 5$ as mentioned above. Also we assign colours $c_1$ and $c_3$ to $v_6$ and $u$ respectively. Since $u_6$ is adjacent to $u$ we cannot assign colour $c_2$ to $u_6$. So the vertices $v_i$; $1 \leq i \leq 4$ will not be adjacent to the vertices with colour $c_2$. Hence, they will not become b-dominating. Next we assume that the two b-dominating vertices from the set \{u, u_6, v_6\} are $u$ and $v_6$. We assign colours to $v_i$’s and $u_i$’s; $1 \leq i \leq 5$ as mentioned above and we assign colour $c_1$ and $c_2$ to $u$ and $u_6$ respectively. But, here the vertex $u$ is not adjacent to the vertex with colour $c_6$. Thus the vertex $u$ will not become b-dominating. Next, we assume that we have selected all the vertices from \{v_i; 1 \leq i \leq 5\} and one vertex from the set \{u, u_6, v_6\} as the b-dominating vertices. We assign colour $c_i$ to $v_i$; $1 \leq i \leq 5$. We consider the vertex $v_1$ having colour $c_3$. It is adjacent to the vertices $v_2$ and $v_4$ having colours $c_2$ and $c_4$ respectively. To become a b-dominating vertex, $v_1$ should be adjacent to vertices having colours $c_1$ and $c_5$. So we assign colours $c_1$ and $c_5$ to $u_4$ and $u_5$ respectively. Next, we consider the vertex $v_3$ having colour $c_4$. $v_3$ is adjacent to vertices having colours $c_3$ and $c_5$. But to become a b-dominating vertex, it should be adjacent to vertices with colour $c_1$ and $c_2$. The vertex $u_5$ is adjacent to $v_2$. So we can assign colour $c_2$ to $u_5$. Now, the remaining uncoloured neighbours of $v_4$ are adjacent to the vertex $v_1$. So, we cannot assign colour $c_1$ to any of the neighbours of $v_4$. Thus, the vertex $v_4$ will not become b-dominating. Thus a b-colouring with six colours is also not possible. A b-colouring with five colours can be obtained as given in Figure 13.

Figure 13  A b-colouring of $\mu(W_6)$ with five colours

- $n = 6$

If possible, we assume that there exists a b-colouring with seven colours. Here, we have nine vertices \{u, u_7, v_i; 1 \leq i \leq 7\} with degree at least 6. Out of this nine vertices we can select any seven vertices as the b-dominating vertices and let $B$ be the set of seven b-dominating vertices.

We assume that $u \in B$. Note that, if $u \in B$, then either $v_7 \in B$ or $u_7 \in B$ but not both. On the contrary, we assume that $u_7$ and $v_7$ belongs to $B$. We assign colours $c_1$, $c_2$, $c_3$ to $u$, $u_7$, $v_7$ respectively. Here, the vertex $u$ is not adjacent to $v_7$. So, we have to assign colour $c_3$ to any of the uncoloured neighbours of $u$. But, all the uncoloured neighbours of $u$ are adjacent to $v_7$. Thus, the vertex $u$ will not become b-dominating.
Thus, if \( u \in B \), then either \( u_7 \in B \) or \( v_7 \in B \) but not both. Next, we assume that \( u \notin B \). So, either \( u_7 \notin B \) or \( v_7 \notin B \). Thus, the remaining five \( b \)-dominating vertices will be from the set \( \{v_i; 1 \leq i \leq 6\} \). Note that the set \( \{v_i; 1 \leq i \leq 6\} \) constitute a cycle with six vertices. But from this set of six vertices, we are selecting only five vertices as the \( b \)-dominating vertices. So there will be a vertex in this set, which is not in \( B \). Let it be \( v_k, k \in \{1, 2, \ldots, 6\} \). Now, \( \{v_i; 1 \leq i \leq 6\}\setminus v_k \) forms a path on five vertices, say \( P \). Let the vertex set of \( P \) be \( \{v_a, v_b, v_c, v_d, v_e\} \), where \( a, b, c, d, e \in \{1, 2, \ldots, 6\} \) and \( a \neq b \neq c \neq d \neq e \neq k \). We assign colours \( c_1, c_2, \ldots, c_5 \) to \( v_a, v_b, v_c, v_d, v_e \) respectively. Since \( v_7 \) and \( u_7 \) are common neighbours of \( v_i; 1 \leq i \leq 6 \), we assign colours \( c_6 \) and \( c_7 \) to them. Now, each \( v_i; 1 \leq i \leq 6 \) has exactly four uncoloured neighbours. Since we are constructing a \( b \)-colouring with seven colours, each of these four uncoloured neighbours should receive distinct colours. We consider the vertex \( v_k \), \( v_7 \) is with colour \( c_2 \) and is adjacent to vertices having colours \( c_1, c_3, c_6 \) and \( c_7 \). To become a \( b \)-dominating vertex, \( v_7 \) should be adjacent to the vertices with colours \( c_4 \) and \( c_5 \). The vertices \( u_t \) and \( u_s \) are the only uncoloured neighbours of \( v_k \). Since \( u_t \) is adjacent to \( v_k \), we cannot assign colour \( c_4 \) to \( u_t \). So we assign colour \( c_4 \) to \( u_s \) and \( c_5 \) to \( u_c \). Now the vertex \( v_7 \) will have two neighbours with colour \( c_5 \). Thus the vertex \( v_7 \) will not become a \( b \)-dominating vertex.

Next we assume that \( u \notin B \). If \( u \notin B \), then at least five \( b \)-dominating vertices will be from the set \( v_i; 1 \leq i \leq 6 \). Thus as in the above case, here also we cannot construct a \( b \)-colouring with seven colours. A \( b \)-colouring with six colours can be obtained as given in Figure 14.

**Figure 14** A \( b \)-colouring of \( \mu(W_7) \) with six colours

\[
\begin{array}{c}
\begin{array}{c}
\text{\begin{tikzpicture}
\node (u) at (0,0) {$u$};
\node (c_7) at (1,1) {$c_7$};
\node (c_5) at (2,2) {$c_5$};
\node (c_3) at (3,1) {$c_3$};
\node (c_1) at (4,0) {$c_1$};
\node (c_4) at (2,-1) {$c_4$};
\node (v_7) at (1,2) {$v_7$};
\draw (u) -- (v_7);
\draw (v_7) -- (c_1);
\draw (v_7) -- (c_2);
\draw (v_7) -- (c_3);
\draw (v_7) -- (c_4);
\draw (v_7) -- (c_5);
\end{tikzpicture}}
\end{array}
\end{array}
\]

- \( n = 8 \)

We consider the graph \( \mu(C_8) \backslash \{u, v_9, u_9\} = G' \).

First we prove that \( \varphi(G') = 4 \).

We assume that \( \varphi(G') = 5 \). Here the vertices with degree at least 4 are \( \{v_i; 1 \leq i \leq 8\} \).

All the other vertices have degree 2. So if the \( b \)-chromatic number of \( G' \) is 5, then all the five \( b \)-dominating vertices will be from the set \( \{v_i; 1 \leq i \leq 8\} \). Thus, from a set of eight vertices we are selecting five vertices and that can be done in \( \binom{8}{5} \) ways. In each
case there will be two b-dominating vertices $v_{i-1}$ and $v_i$ such that $v_{i-2}$ and $v_i$ are also b-dominating or three or four consecutive b-dominating vertices.

We consider the first case, that is there exist two b-dominating vertices $v_{i-1}$ and $v_i$ such that $v_{i-2}$ and $v_i$ are also b-dominating. Here the fifth b-dominating vertex will be $v_{i+4}$. Since the degree of each $v_i; 1 \leq i \leq 8$ is 4 and we are constructing a b-colouring with five colours, the four neighbours of each of these five b-dominating vertices should receive distinct colours. We assign colours $c_1, c_2, c_3$ and $c_4$ to $v_{i-2}, v_{i-1}, v_{i+1}$ and $v_{i+2}$ respectively. The only one colour that can be assigned to the vertex $v_i$ is $c_5$. So we assign colour $c_5$ to $v_i$. The vertex $v_{i-1}$ is assumed to a b-dominating vertex with colour $c_2$ and is adjacent to the vertices $v_{i-2}$ and $v_i$ with colours $c_1$ and $c_5$ respectively. To become a b-dominating vertex, $v_{i-1}$ should be adjacent to two vertices with colours $c_3$ and $c_4$. The only two uncoloured neighbours of $v_{i-1}$ are $u_i$ and $u_{i+1}$. Since the colour of $v_{i-1}$ is $c_5$, we cannot assign colour $c_3$ to $u_i$. So the only option is to assign colour $c_4$ to $u_i$ and $c_3$ to $u_{i+1}$. But if we do so, the vertex $v_{i+1}$ will have two neighbours with same colour $c_4$. Thus the vertex $v_{i+1}$ will not become b-dominating. Hence, in this case a b-colouring with five colours is not possible.

Next we consider the case that there are three consecutive b-dominating vertices. Let the three consecutive b-dominating vertices be $v_r, v_{r+1}$ and $v_{r+3}$. Now the remaining two b-dominating vertices will be $v_{i+4}$ and $v_{i+6}$ (if $i + k > 8, i + k = i + k \mod 8$ for $1 \leq k \leq 7, 1 \leq i \leq 8$). We assign colours $c_1, c_2, c_3, c_4$ and $c_5$ to $v_r, v_{r+1}, v_{r+2}, v_{r+4}$ and $v_{r+6}$ respectively. Here also the four neighbours of each of the b-dominating vertices should receive distinct colours. We consider the vertices $v_{r+5}$ and $u_{r+3}$. These vertices can receive either colour $c_1$ or colour $c_5$. So without loss of generality we assign colour $c_1$ to $v_{r+4}$ and $c_5$ to $u_{r+3}$. Similarly, the vertices $v_{r+4}$ and $u_i$ can receive either colour $c_2$ or colour $c_4$. So without loss of generality we assign the colour $c_2$ to $v_{r+5}$ and $c_4$ to $u_{r+4}$. Now, we consider the vertex $v_{r+6}$. To become a b-dominating vertex, it should be adjacent to two vertices with colours $c_3$ and $c_1$. But $v_{r+7}$ and $u_{r+7}$ are the only two uncoloured neighbours of $v_{r+6}$ and these two vertices are connected to the vertex $v_i$ with colour $c_1$. Thus, here also we cannot have a b-colouring with five colours.

Next, we consider the case that there exists four consecutive b-dominating vertices. Let it be $v_r, v_{r+1}, v_{r+2}$ and $v_{r+3}$. The remaining one b-dominating vertex can be either $v_{r+5}$ or $v_{r+6}$ (if $i + k > 8, i + k = i + k \mod 8$ for $1 \leq k \leq 7, 1 \leq i \leq 8$). Without loss of generality we assume that the fifth b-dominating vertex be $v_{r+5}$. We assign colours $c_1, c_2, c_3, c_4$ and $c_5$ to $v_r, v_{r+1}, v_{r+2}, v_{r+3}$ and $v_{r+5}$ respectively. Note that here all the four neighbours of each of the b-dominating vertices should receive distinct colours. Thus, the vertices $v_{r+4}$ and $u_{r+3}$ can receive either colour $c_1$ or colour $c_2$. So we assign colour $c_1$ to $v_{r+4}$ and $c_2$ to $u_{r+4}$. To become a b-dominating vertex, $v_{r+5}$ should be adjacent to two vertices with colour $c_3$ and $c_4$. So we assign colour $c_3$ to $v_{r+4}$ and $c_4$ to $u_{r+5}$. Now, the only one colour that can be assigned to $u_{r+4}$ is $c_5$. So we assign colour $c_5$ to $u_{r+4}$. Now, we consider the vertex $v_r$. To become a b-dominating vertex, $v_r$ should be adjacent to two vertices with colours $c_3$ and $c_4$. But here $v_{r+3}$ and $u_{r+7}$ are the only uncoloured neighbours of $v_r$ and the vertex $v_{r+5}$ is adjacent to two vertices with colours $c_1$ and $c_4$. Hence, we cannot assign $c_3$ or $c_4$ to $u_{r+7}$. Thus here a b-colouring with five colours is not possible. Thus $\phi(G) \neq 5$. A b-colouring with four
colours can be obtained by assigning colour $c_1$ to $v_0$, $v_{i+2}$, $v_{i+5}$ and $u_i$, colour $c_2$ to $v_{i+3}$, $u_{i+5}$, $u_{i+6}$ and $u_{i+7}$, colour $c_3$ to $v_{i+1}$, $v_{i+4}$ and $v_{i+6}$ and $c_4$ to $v_{i+7}$, $u_{i+1}$, $u_{i+2}$, $u_{i+3}$ and $u_{i+4}$.

Next, we consider $\mu(W_9) = G' \cup \{u, v_0, u_9\}$ such that $v_9$ is adjacent to all the $v_i$’s; $1 \leq i \leq 8$, $u$ is adjacent to all the $v_i$’s; $1 \leq i \leq 9$ and $u_9$ is adjacent to all the $v_i$’s; $1 \leq i \leq 8$. Assume that we have assigned three new colours $c_5$, $c_6$ and $c_7$ to $u$, $u_9$ and $v_9$ respectively. Here all the neighbours of $u$ are adjacent to $v_9$ but they are not adjacent to $u_9$. Since we have already assigned colour $c_9$ to $u_9$, we cannot assign colour $c_7$ to any of the neighbours of $u$ and thus $u$ will not become a b-dominating vertex. So we cannot assign three new colours to $u$, $v_9$ and $u_9$. Thus a b-colouring with seven colours is not possible. A b-colouring with six colours can be obtained as given in Figure 15.

**Figure 15**  A b-colouring of $\mu(W_9)$ with six colours

- $n = 7$ and $n = 9$

Consider the case when $n = 7$. A b-colouring with seven colours can be obtained by assigning colour $c_1$ to $v_2$, $u_5$ and $u_6$, colour $c_2$ to $v_3$ and $v_6$, colour $c_3$ to $v_1$ and $v_4$, colour $c_4$ to $v_5$, $u_1$ and $u_2$, colour $c_5$ to $v_7$, $u_3$ and $u_4$, colour $c_6$ to $v_8$ and $u$ and colour $c_7$ to $u_7$ and $u_6$.

Next, we consider the case when $n = 9$. A b-colouring with seven colours can be obtained by assigning colour $c_1$ to $v_2$, $u_5$ and $u_6$, colour $c_2$ to $v_3$, $v_6$ and $v_9$, colour $c_3$ to $v_1$, $v_4$ and $v_8$, colour $c_4$ to $v_5$, $u_1$, $u_2$ and $u_9$, colour $c_5$ to $v_7$, $u_3$, $u_4$ and $u_9$, colour $c_6$ to $v_{10}$ and $u$ and colour $c_7$ to $u_{10}$ and $u_9$. For all the remaining value of $n$, we can assign this colouring and the remaining uncoloured vertices can be coloured as follows

a  If $n$ is an even number

We assign colour $c_1$ to $v_{10}$, $v_{12}$, ..., $v_{n}$ and colour $c_2$ to $v_{11}$, $v_{13}$, ..., $v_{n+1}$.

b  If $n$ is an odd number

We assign colour $c_1$ to $v_{10}$, $v_{12}$, ..., $v_{n-1}$ and colour $c_2$ to $v_{11}$, $v_{13}$, ..., $v_{n}$.

And assign colour $c_6$ to $v_{n+1}$ and $u$ and colour $c_7$ to $u_{n+1}$.
The b-chromatic number of Mycielskian of some graphs

3 Conclusions

The b-colouring of a graph has application in clustering techniques (Elghazel et al., 2007). Mycielskian is a graph operation, which has application in multiprocessor task scheduling problem (Eric et al., 2011). In this paper, we obtained the b-chromatic number of Mycielskian of paths, complete graphs, complete bipartite graphs and wheels. The b-chromatic number of Mycielskian of a graph is used in clustering techniques.

References


